On pronic gracefulness of graphs

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1 Introduction

Graph labeling is a prospective research area due to its vital applications that could challenge our mind for eventual solutions. A graph labeling is an assignment of integers(values) to the vertices(points) or edges(lines) or both under certain conditions. There are usually two types of labeling of graphs:

Quantitative Labeling is nothing but an assignment of some numbers to the elements of a graph and this labeling has persuaded research in a wide variety of applications in (synch-set codes)coding theory, radio-astronomy, spectral characterization of materials using crystallography etc., under certain constraints.

The assignment of qualitative nature to the vertices or edges of graph is called **Qualitative Labeling.** These labelings have influenced research in variant areas of human enquiry such as conflict resolutions in social psychology, electrical circuit theory, energy crises etc.,

1.1 Graceful labeling on graphs

A graph which can be labeled gracefully is said to be a graceful graph. It is done by investigating such a graph with the labeling exists or not. Few results due to Golomb(1972) and Rosa(1967),(1977) are as folllows:

- The essential condition for a complete graph K_n to be graceful is $n \leq 4$ and the cycle graph C_n of order is $n \equiv 0(mod 4)$.

Few results on graceful labeling are listed below:

- Vaidya et al.(2009,2010,2011) analysed the gracefulness on certain family of graph.
- Uma and Murugesan(2012) discussed the graceful labeling on graphs and its subgraphs.
- Elumalai(2014) showed that cycle C_n with parellel edge extension admitss graceful labeling.
- Kaneria et al.(2015) analysed the gracefulness of $C_n(C_n)$ and $C_n(K_{m,n})$. Also Elumalai et al.(2015) showed the gracefulness of cycle with chords.
- The Fibonacci gracefulness of the paths, squares of paths P_n^2, Caterpillars are Fibonacci graceful and the bistar $B_{n,n}$ for $n \geq 5$ are showed by Kathiresan et al.(2010).
- Vaidya and Vihol(2011) proved that trees, switching of a vertex in a cycle and other graph familes admits Fibonacci graceful labeling and *some are not Fibonacci graceful*.

After going through a number of research works[3],[4],[13],[14],[18],[19] related to graceful labeling, in this chapter a graceful labeling using pronic numbers is defined and discussed for different graph families.

2 Graceful labeling using pronic numbers

Definition 2.1. *Graceful Labeling*

Let G be a graph of order p and size q. A *graceful labeling* of G is an injection $f : V \rightarrow \{0,1,...,q\}$such that while each edge uv is assigned the label(absolute difference of the corresponding vertex labels), the induced edge labels are all distinct. Such a function g_f is called the induced edge function and a graph which admits such a labeling is called a graceful graph.

Definition 2.1. *Pronic Number:*

A number of the form $n(n+1)$ *is called a pronic number. These numbers are also called oblong numbers, heteromecic or rectangular numbers. The sum of the first n even integers is its* n^{th} *pronic number.Aall pronic numbers are even(by definition), and the only prime pronic number is 2. Also 2 is the only pronic number in the Fibonacci sequence.*

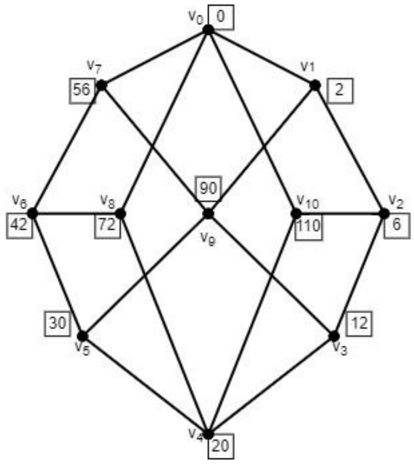

Figure 1: Herschel Graph-Pronic Graceful

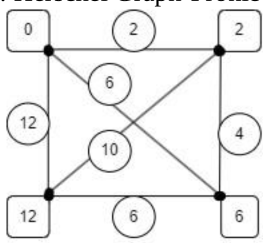

Figure 2: Not Pronic Graceful

Note 2.2. *A pronic number is squarefree if f if n and n + 1 are squarefree. 0, 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240, 272, 306, 342, 380, 420, 462 are few among them.*

Definition 2.3. *Pronic Graceful Labeling:[23][24][25]*

Let $G(p,q)$ *be graph with* $p \geq 2$*. A pronic graceful labeling of G is a bijection* $f : V(G) \rightarrow \{0,2,6,12,...,P_{n-1}\}$ *such that the resulting edge labels obtained by* $|f(u) - f(v)|$ *on every edge uv are pairwise disjoint. A graph G is called pronic graceful if it admits pronic graceful labeling.*

Example 2.4. *An example for a graph which admits pronic graceful labeling is given in 1*

Example 2.5. *An example for a graph which does not admits pronic graceful labeling is given in 2*

2.1 Main theorems

Theorem 2.6. *Path graph* P_n*,* $n \geq 3$ *admits pronic graceful labeling.*

Theorem 2.7. *Cycle graph* C_n*,* $n \geq 3$ *admits pronic graceful labeling.*

Theorem 2.8. *Star graph* $K_{1,n}$*,* $n \geq 3$ *admits pronic graceful labeling.*

Theorem 2.9. *Path graph P_n, $n \geq 3$ admits pronic graceful labeling.*

Theorem 2.10. *Complete graph K_n, $n \geq 4$ does not admit pronic graceful labeling.*

Proof : If $n = 3$, the complete graph is nothing but the cycle graph of order 3 and it admits pronic graceful labeling is which is shown in previous theorem.

Assume the graph for $n \geq 4$.

Let $\{v_0,v_1,v_2...,v_{n-1}\}$ be the vertices of K_n, $n \geq 4$ and are assigned the pronic numbers $p_0,p_1,...p_{n-1}$. It is to be noted that the number "6" appears for the absolute difference of two pairs of pronic numbers (p_0,p_2) and (p_2,p_3).

Now as the given graph is complete, all edges of it are adjacent. Thus there exists two adjacent edges for which they are assigned by the label "6". Hence the complete graph does not admit pronic graceful labeling.

□

2.1.1 Wheel and shell related graphs

Theorem 2.11. *The wheel graph $K_1 + C_n$, $n \geq 4$ admits pronic graceful labeling.*

Proof : Let v_n be the apex vertex and $\{v_0,v_1,v_2...,v_{n-1}\}$ be the rim vertices of $K_1 + C_n$, $n \geq 4$. **Case (i):** n 6= 6,10

Define a bijection $f : V(G) \to \{p_0,p_1,...,p_n\}$ by

$$f(v_i) = p_i, i = 0,1,2,...,n-1 \qquad f(v_n) = p_n.$$

For the vertex labeling above, an induced edge function $f^* : E(G) \to N$ is given by

$$f^*(v_iv_{i+1}) = 2(i+1), i = 0,1,2,...,n-2;$$

$$f^*(v_nv_i) = n(n+1) - i(i+1), i = 0,1,2...,n-1;$$

$$f^*(v_0v_{n-1}) = (n-1)n.$$

The edges are hence labeled as follows:
(i)the labels of the edges $\{v_iv_{i+1},\ i = 0,1,2,..n-2, v_0n_{n-1}\}$ are $\{2,4,6,...,2(n-1),n(n-1)\}$.
(ii)the labels of the edges $\{v_nv_i, i = 0,1,2,...,n-1\}$ are $\{n(n+1),(n+2)(n-1),(n+3)(n-2),...,2n\}$.

Hence the wheel graph admits pronic graceful labeling in this case.

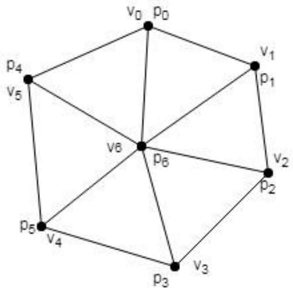

Figure 3: Pronic graceful labeling of Wheel graph $K_1 + C_6$

Case (ii)$n = 6,10$

Define a bijection $f : V(G) \to \{p_0,p_1,...,p_{n-1}\}$ by

$$fv_i = \begin{cases} p_i, & i = 0,1,\dots,n-3; \\ p_{\{i-1\}}, & i = n-1; \\ p_{\{i+1\}}, & i = n-2; \\ p_n, & i = n \end{cases}$$

For the vertex labeling above, an induced edge function $f^* : E(G) \to N$ is given by

$f^*(v_iv_{i+1}) = 2(i+1), i = 0,1,2,...,n-2;$ $\quad f^*(v_0v_{n-1}) = (n-1)(n-2); f^*(v_nv_{n-2}) = 2n;$

$f^*(v_nv_{n-1}) = 4n-2;$ $\quad f^*(v_nv_i) = n(n+1) - i(i+1), i = 0,1,2,...,n-3.$

The edges are hence labeled as follows:
(i)the labels of the edges $\{v_iv_{i+1}, i = 1,2,..n-2, v_0n_{n-1}\}$ are $\{2,4,6,...,2(n-1),(n-2)(n-1)\}$.

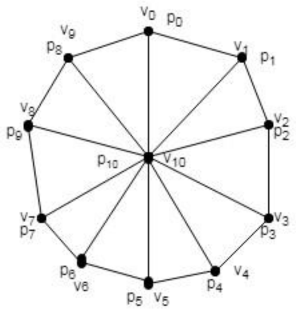

Figure 4: Pronic graceful labeling of Wheel graph $K_1 + C_{10}$

(ii)the labels of the edges $\{v_nv_i, i = 0,1,2,...n-3\}$ and $\{v_nv_{n-2}, v_nv_{n-1}\}$ are $\{n(n+1),(n+2)(n-1),(n+3)(n-2),...,6(n-1)\}$ and $\{2n, 4n-2\}$ and thus the graph admits pronic graceful labeling in this case. Hhus the wheel graph $K_1 + C_n$, for $n \geq 4$ admits pronic graceful labeling.

□

Theorem 2.12 (4)**.** *Gear graph G_n admits pronic graceful labeling*

Theorem 2.13 (4)**.** *Helm Graph HG_n, admits pronic graceful labeling*

Theorem 2.14 (12)**.** *A Shell Graph $C(n, n-3)$, for $n \geq 3$ admits pronic graceful labeling.*

Theorem 2.15 (12)**.** *A Shell Butterfly Graph G admits pronic graceful labeling.*

2.1.2 PGL on corona product and joint sum of graphs

Theorem 2.16 (5)**.** *Corona product $C_n \circ mK_1$ admits pronic graceful labeling.*

Theorem 2.17 (29)**.** *Barycentric subdivision of cycle $C_n(C_n)$ admits pronic graceful labeling.*

Theorem 2.18 (16)**.** *The joint sum of cycle C_m and C_n, $m,n \geq 3$ admits pronic graceful labeling.*

Proof : Let C_m and C_n, $m,n \geq 3$ be the cycles of order m and n.
Case(i):$m = n$**.**
Subcase(i):$m = n \geq 5$**.**

Let the vertices of the joint sum be $\{v_0,v_1,....v_{m-1},v_m,v_{m+1},v_{m+2},....v_{2m-1}\}$ and the edges be $\{v_iv_{i+1}, i = 0,1,2,...,m-1,m,m+1,m+2,...2m-1\} \cup \{v_0v_{m-1}, v_mv_{2m-1}\}$. Let us connect the two graphs by the new edge $v_{m-1}v_m$ so that $\{v_0,v_1,v_2...,v_{m-1},v_m,v_{m+1},v_{m+2},...,v_{2m-2}\}$ forms a spanning path in G.
Define a bijection $f : V(G) \rightarrow \{p_0,p_1,...,p_{2m-1}\}$ by

$$f(v_i) = p_i,\ i = 0,1,2,...,2m-1.$$

For the vertex labeling above, an induced edge function $f^* : E(G) \rightarrow N$ is given by

$$f^*(v_iv_{i+1}) = 2(i+1), i = 0,1,2,...,m-1,m,m+1,m+2,...2m-2;$$

$$f^*(v_0v_{m-1}) = m(m-1); f^*(v_mv_{m+n-1}) = 3m(m-1).$$

The edge labels are thus $\{2,4,6,...,2(m-1),2m,2(m+1),2(m+2),...,2(2m-1),m(m-1),3m(m-1)\}$ and hence in this case the joint sum of cycles admits pronic graceful labeling.

Subcase(ii): $m = n = 3,4$.

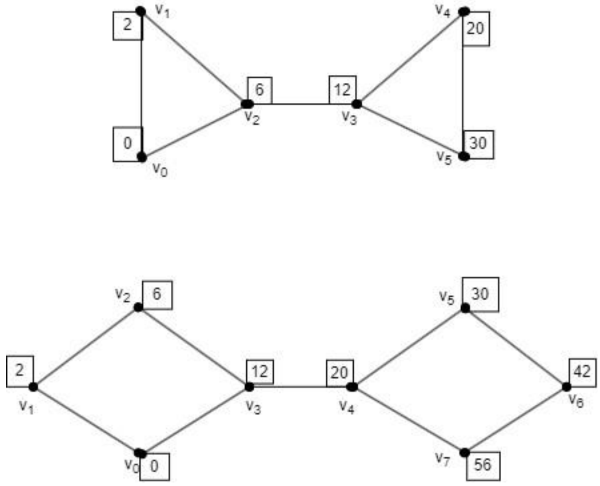

Figure 5: PGL of joint sum of two copies of C_3 and C_4

Case(ii) $m \neq n$.

Let the vertices of C_m be $\{v_0,v_1,....v_{m-1}\}$ and C_n be $\{v_m,v_{m+1},v_{m+2},....v_{m+n-1}\}$ and the edges of the C_m and C_n are $\{v_iv_{i+1}, i = 0,1,2,...,m-1,m,m+1,m+2,...,m+n-2\} \cup \{v_0v_{m-1},v_mv_{m+n-1}\}$. Let us connect the two graphs by the new edge $v_{m-1}v_m$ so that $\{v_0,v_1,v_2...,v_{m-1},v_m,v_{m+1},v_{m+2},...,v_{m+n-2}\}$ forms a spanning path in G.

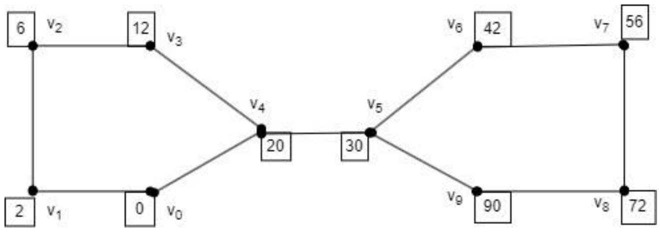

Figure 5: PGL of joint sum of two copies of C_3 and C_4

Define a bijection $f : V(G) \to \{p_0,p_1,...,p_{m+n-1}\}$ by $f(v_i) = p_i$, $i = 0,1,2,...,m+n-1$.
For the vertex labeling above, an induced edge function $f^* : E(G) \to N$ is given by

$$f^*(v_iv_{i+1}) = 2(i+1), i = 0,1,2,...,m-1,m,m+1,m+2,...m+n-2;$$

$$f^*(v_0v_{m-1}) = m(m-1);\ f^*(v_mv_{m+n-1}) = (n-1)(2m+n).$$

The edge labels are thus $\{2,4,6,...,2(m-1),2m,2(m+1),2(m+2),...,2(m+n-1),m(m-1),(n-1)(2m+n)\}$ and hence in this case the joint sum of cycles admits pronic graceful labeling. Hence the joint sum of C_m and C_n, $m,n \geq 3$ admits pronic graceful labeling. □

Note 2.19. *(m,n)-tadpole and n-pan graphs admit pronic graceful labeling.*

2.2 Pronic Graceful Labeling of Bipartite Graphs

In this section, the labeling for complete bipartite graphs have been investigated.

Theorem 2.20. *The complete bipartite graph $K_{2,n}$ admits pronic graceful labeling.*

Proof : Let X and Y be the partition of vertices of $K_{2,n}$ and let $V(X) = \{u_0,u_1\}$ and $V(Y) = \{v_0,v_1,v_2,...,v_{n-1}\}$. Hence $|V(X)| = 2;|V(Y)| = n \Rightarrow |V(K_{2,n})| = n + 2$.

Define a bijection $f : V(G) \to \{p_0,p_1,...,p_{n+1}\}$ by

$$f(u_i) = p_i, i = 0,1;\ f(v_i) = p_{i+2}, i = 0,1,2,...,n-1.$$

For the vertex labeling above, the induced edge labeling $f^* : E(G) \to N$ is given by

$$f^*(u_0v_i) = (i+2)(i+3), i = 0,1,2,3,...,n-1;$$

$$f^*(u_1v_i) = (i+1)(i+4), i = 0,1,2,3,...,n-1.$$

The distinct labels thus obtained for the edges u_0v_i and u_1v_i for $i = 0,1,2,...,n-1$ are $\{p_2,p_3,p_4,...,p_{n+1}\}$ and $\{p_2-2,p_3-2,p_4-2,...,p_{n+1}-2\}$ which results the graph$K_{2,n}$ admits pronic graceful labeling. □

Theorem 2.21. *The complete bipartite graph $K_{3,n}$ admits pronic graceful labeling.*

Proof : Let X and Y be the partition of vertices of $K_{3,n}$ and let $V(X) = \{u_0,u_1,u_2\}$ and $V(Y) = \{v_0,v_1,v_2,...,v_{n-1}\}$. Hence $|V(X)| = 3;|V(Y)| = n \Rightarrow |V(K_{3,n})| = n + 3$.

Define a bijection $f : V(G) \to \{p_0,p_1,...,p_{n+2}\}$ by

$$f(u_i) = p_i, i = 0,1,2;\ f(v_i) = p_{i+3}, i = 0,1,2,...,n-1.$$

For the vertex labeling above, the induced edge labeling $f^* : E(G) \to N$ is given by

$$f^*(u_0v_i) = (i+3)(i+4), i = 0,1,2,3,...,n-1;$$

$$f^*(u_1v_i) = (i+2)(i+5), i = 0,1,2,3,...,n-1;$$

$$f^*(u_1v_i) = (i+1)(i+6), i = 0,1,2,3,...,n-1.$$

The distinct labels thus obtained for the edges $\{u_0v_i,\ u_1,v_i\}$ and $\{u_2v_i, i = 0,1,2,...,n-1$ are $\{p_3,p_4,...,p_{n+2}\}$, $\{p_3-2,p_4-2,...,p_{n+2}-2\}$ and $\{p_3-6,p_4-6,...,p_{n+2}-6\}$ which results that $K_{3,n}$ admits pronic graceful labeling.□

Observation 2.22. The complete bipartite graph $K_{4,4}$ does not admit pronic graceful labeling.

For, the pronic number p_3, while commutes with the pronic numbers $\{p_0,p_1,p_2,p_5\}$ induces a label "k" which occurs twice for two different edges.i.e., the same label is assigned for different edges. The pairs are listed below:

$$(12,6) = (0,6);(12,2) = (20,30);\ (12,0) = (30,42);\quad (12,30) = (20,2).$$

Hence the above mentioned pronic numbers including p_3 must be assigned to same partition of vertices. Such a labeling is not possible since only 4 vertices are in one partition. Hence the $K_{4,4}$ does not admit pronic graceful labeling.

Problem 2.23. *Does there exist any n other than $n = 2,3$ for which the complete bipartite graph $K_{n,n}$ admits pronic graceful labeling?*

2.3 Pronic Graceful labeling of Generalized Peterson Graph $P(n,1)$

In graph theory, the **generalized Petersen graphs** are a family of cubic graphs formed by connecting the vertices of a regular polygon to the corresponding vertices of a star polygon. The Peterson Graph is the complement of the line graph of the complete graph K_5.

Alice Steimle and William Staton(2009) analysed the isomorphism classes of the generalized Petersen graphs. Zehui Shao et al(2017) proposed a backtracking algorithm with a specific static variable ordering and dynamic value ordering to find graceful labeling for generalized Petersen graphs and that algorithm is able to find gracefulness of generalized $P(n,k)$ with the number of vertices greater than or equal to 75 within several seconds.

Theorem 2.24. *Peterson graph* $P(5,2)$ *admits pronic graceful labeling.*

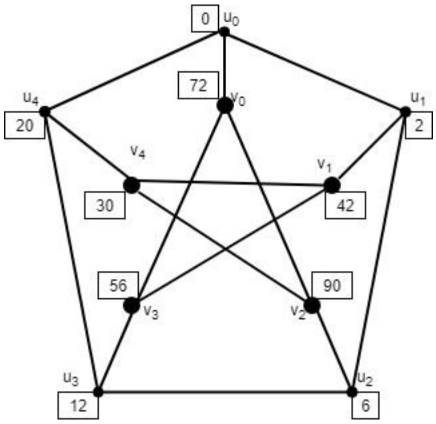

Figure 7: Pronic graceful labeling of Peterson graph $P(5,2)$

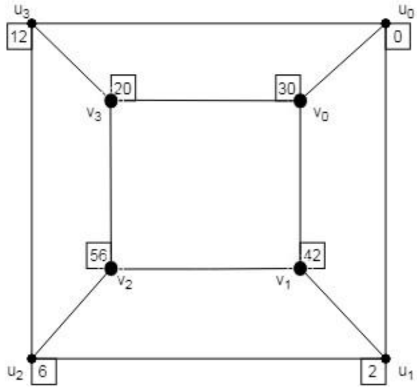

Figure 8: Pronic Graceful Labeling of Cubical Graph $P(4,1)$

Proof : Let $\{v_0,v_1,v_2,v_3,v_4\}$ be the inner vertices and $\{u_0,u_1,u_2,u_3,u_4\}$ be the outer vertices of $P(5,2)$. Define a bijection $f : V(G) \rightarrow \{p_0,p_1,p_2,...,p_9\}$ by

$$f(u_i) = p_i, i = 0,1,2,3,4; \quad f(v_{2i+1}) = p_{i+6}, i = 0,1; f(v_4) = p_5; \quad f(v_{2i}) = p_{i+8}, i = 0,1.$$

Clearly f is a bijection.

For the vertex labeling above, the induced edge labels are as follows:

(i) Consider the path in $\{u_0,u_1,u_2,u_3,u_4,v_4,v_1,v_3,v_0,v_2\}$ in $P(5,2)$. The edges of the path $\{u_iu_{i+1},(0 \leq i \leq 3),u_4v_4,v_4v_1,v_1v_3,v_3v_0,v_0v_2\}$ are consecutively labeled by the numbers $\{2,4,6,...,2(2n-1)\}$.

(ii) the remaining edges are labeled as follows:

$$f^*(u_0u_{n-1}) = (n-1)n; \quad f^*(u_{2i}v_{2i}) = 72 + 12i, i = 0,1;$$

$$f^*(u_{2i+1}v_{2i+1}) = 40 + 4i, i = 0,1; \quad f^*(v_{n-3}v_{n-1}) = 20(n-2).$$

Hence the labels are {20,72,84,40,44,60} respectively. Thus the edge labels are distinct which results that the Peterson graph admits pronic graceful labeling. □

Theorem 2.25. *n-prism $P(n,1)$ for $n > 3$ admits pronic graceful labeling.*

Proof : Let $\{v_0,v_1,v_2,v_3,...,v_{n-1}\}$ be the inner vertices and $\{u_0,u_1,u_2,u_3,...,u_{n-1}\}$ be the outer vertices of $P(n,1)$. Define a function $f : V(G) \to \{p_0,p_1,p_2,...,p_{2n-1}\}$ as follows:

$$f(u_i) = p_i, i = 0,1,2,...n-1;\ f(v_i) = p_{i+1}, i = n-1;$$

$$f(v_i) = p_{i+(n+1)}, i = 0,1,2,...,n-2.$$

Clearly f is a bijection.

Let A_1, A_2 and A_3 denote the set of edge labels of $\{u_iu_{i+1}(0 \leq i \leq n-2), u_{n-1}v_{n-1}, v_{n-1}v_0, v_iv_{i+1}, (0 \leq i \leq n-3\}$, $\{u_iv_i, (0 \leq i \leq n-2)\}$ and $\{u_{n-1}u_0, v_{n-2}v_{n-1}\}$. Clearly the labels of the edges for the above sets are as follows:

$A_1 = \{2,4,6,...4n-2\}$;

$A_2 = \{p_{n+1}, p_{n+1} + 2(n+1), p_{n+1} + 4(n+1),..., p_{n+1} + 2(n-2)(n+1)\}$;

$A_3 = \{n(n-1), n[3(n-4)+9]\}$.

In the view of above defined labeling, it is observed that $A_1 \cap A_2 \cap A_3 = \varphi$ and hence the Peterson graph $P(n,1)$ admits its pronic gracefulness. □

Example 2.26. The pronic graceful labeling for Peterson graphs $P(10,15)$ and $P(4,1)$ are given in Figure 7 and Figure 8. The graph $P(4,1)$ is also called as the cubical graph.

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