Banach Fixed Contraction Mapping Theorem in Vector S-metric Spaces

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Abstract

We demonstrate the Banach contraction mapping theorem on vector S-metric space. We also give an example to explain our results.
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1 Introduction

Banach Contraction Principle(BCP) was derived firstly by S. Banach [2] in 1922. It has a vital role in fixed point theory and became very famous due to iterations used in the theorem. Many researchers are proving new results in various generalizations of metric spaces. S-metric space is one of the generalizations in metric spaces. In 2012, S-metric space was defined by Sedghi et al.[7]. We start with some definitions and results for vector S-metric spaces.

Definition 1.[4] On a set $\mathtt{C}$, a relation $\preceq$ is a partial order if it follows the conditions stated below:

(a) $\eta_1 \preceq \eta_1$ (reflexive)

(b) $\eta_1 \preceq \eta_2$ and $\eta_2 \preceq \eta_1$ implies $\eta_1 = \eta_2$

$(anti - symmetry)$

(c) $\eta_1 \preceq \eta_2$ and $\eta_2 \preceq \eta_3$ implies $\eta_1 \preceq \eta_3$

$(transitivity)$

$\forall\ \eta_1, \eta_2, \eta_3 \in \mathsf{C}$.
The set C with partial order $\preceq$ is known as partially ordered set (poset).
A partially ordered set $(\mathsf{C}, \preceq)$ is called linearly ordered if for $\eta_1, \eta_2 \in \mathsf{C}$, we have either $\eta_1 \preceq \eta_2$ or $\eta_2 \preceq \eta_1$.
Definition 2.[4] Let C be linear space which is real and $(\mathsf{C}, \preceq)$ be a poset . Then the poset $(\mathsf{C}, \preceq)$ is said to be an ordered linear space if it follows the properties mentioned below:

(a) $p_1 \preceq p_2 \Longrightarrow p_1 + p_3 \preceq p_2 + p_3$

(b) $p_1 \preceq p_2 \Longrightarrow \omega p_1 \preceq \omega p_2$

$\forall p_1, p_2, p_3 \in \mathsf{C}$ and $\omega > 0$

Definition 3.[4] A poset is called lattice if each set with two elements has an infimum and a supremum.
Definition 4.[4] An ordered linear space where the ordering is lattice is called vector lattice.
Definition 5.[4] A vector lattice V is called Archimedean if $inf\{\frac{1}{m}\vartheta\} = 0$ for every $\vartheta \in V^+$ where

$$V^+ = \{\vartheta \in V : \vartheta \succeq 0\}.$$

Definition 6.[3] Let V be vector lattice and $\Re$ be a nonvoid set. A function $d : \Re \times \Re \to V$ is called vector metric on $\Re$ if it follows the conditions stated below:

(a) $d(\hbar_1, \hbar_2) = 0$ iff $\hbar_1 = \hbar_2$

(b) $d(\hbar_1, \hbar_2) \preceq d(\hbar_1, \hbar_3) + d(\hbar_3, \hbar_2)$

$\forall \hbar_1, \hbar_2, \hbar_3 \in \Re$

The triple $(\Re, d, V)$ is called vector metric space.
Definition 7.[8] Let $\Re$ be a nonvoid set. A function $S : \Re \times \Re \times \Re \to [0, \infty)$ is calles S-metric on $\Re$ if it follows the below conditions :

(a) $S(\flat_1, \flat_2, \flat_3) \succeq 0$,

(b) $S(\flat_1, \flat_2, \flat_3) = 0$ iff $\flat_1 = \flat_2 = \flat_3$,

(c) $S(\flat_1, \flat_2, \flat_3) \preceq S(\flat_1, \flat_2, \alpha) + S(\flat_2, \flat_2, \alpha) +$

$$S(\flat_3, \flat_3, \alpha),$$

for all $\flat_1, \flat_2, \flat_3, \alpha \in \Re$.
The pair $(\Re, S)$ is known as S-metric space .
Now, vector valued S-metric space is defined as follows:
Definition 8. Let V be vector lattice and $\Re$ be a nonvoid set. A function $S : \Re \times \Re \times \Re \to V$ is called vector S-metric on $\Re$ that satisfies the conditions mentioned below:

(a) $S(\flat_1, \flat_2, \flat_3) \succeq 0$,

(b) $S(\flat_1, \flat_2, \flat_3) = 0$ iff $\flat_1 = \flat_2 = \flat_3$,

(c) $S(\flat_1, \flat_2, \flat_3) \preceq S(\flat_1, \flat_2, \alpha) + S(\flat_2, \flat_2, \alpha)+$

$$S(\flat_3, \flat_3, \alpha),$$

for all $\flat_1, \flat_2, \flat_3, \alpha \in \Re$.
The triplet $(\Re, S, V)$ is called vector S-metric space.

Example 1 *Let $\Re$ be a nonvoid set and V be a vector lattice. A function $S : \Re \times \Re \times \Re \to V$ is defined by*

$$S(\flat_1, \flat_2, \flat_3) = |(\flat_1, \flat_3)| + |(\flat_2, \flat_3)| \quad \forall \flat_1, \flat_2, \flat_3 \in \Re$$

then the triplet $(\Re, S, V)$ is vector S-metric space.

Definition 9. A sequence $\langle\vartheta_n\rangle$ in vector S-metric space $(\Re, S, V)$ is called V-convergent to some $\vartheta \in V$ if there is a sequence $\langle\mu_n\rangle$ in V satisfying $\mu_n \downarrow 0$ and $S(\vartheta_n, \vartheta_n, \vartheta) \leq \mu_n$ and denote it by $\mu_n \xrightarrow{S,V} \vartheta$.
Definition 10. A sequence $\langle\vartheta_n\rangle$ in vector S-metric space $(\Re, S, V)$ is known as V-Cauchy sequence if there is a sequence $\langle\mu_n\rangle$ in V satisfying $\mu_n \downarrow 0$ and $S(\vartheta_n, \vartheta_n, \vartheta_{n+q}) \leq \mu_n$ holds for all q and n.
Definition 11. If each V-Cauchy sequence in $\Re$ is V-converges to a limit in $\Re$ then vector S-metric space $(\Re, S, V)$ is called V-complete .
Lemma[8] For vector S-metric space $(\Re, S, V)$,

$$S(\vartheta, \vartheta, \mu) = S(\mu, \mu, \vartheta) \quad \forall \mu, \vartheta \in \Re.$$

2 Main Results

Theorem 1 *Let $(\Re, S, V)$ be a vector S-metric space which is complete and V be Archimedean. Suppose the mappings $f : Y \to Y$ satisfies*

$$S(f\hbar, f\hbar, f\vartheta) \preceq qS(\hbar, \hbar, \vartheta) \quad \forall \hbar, \vartheta \in \Re$$

where $q \in [0, 1)$ is constant. Then f has fixed point in $\Re$ which is unique and for any $\vartheta_0 \in \Re$, iterative sequence $\langle\vartheta_m\rangle$ defined by $\vartheta_m = f\vartheta_{m-1}$, for all $m \in \mathbb{N}$, V-converges to fixed point of f.
Proof *Let $\vartheta_0 \in \Re$ and sequence $\langle\vartheta_m\rangle$ defined by $\vartheta_m = f\vartheta_{m-1}$ for $m \in \mathbb{N}$. Then we have*

$$\begin{aligned} S(\vartheta_m, \vartheta_m, \vartheta_{m+1}) &= S(f\vartheta_{m-1}, f\vartheta_{m-1}, f\vartheta_m) \\ &\preceq qS(\vartheta_{m-1}, \vartheta_{m-1}, \vartheta_m) \preceq \\ &\ldots \preceq q^m S(\vartheta_0, \vartheta_0, \vartheta_1) \end{aligned}$$

Thus for $m, p \in \mathbb{N}$

$$\begin{aligned}
S(\vartheta_m, \vartheta_m, \vartheta_{m+p}) &\preceq 2S(\vartheta_m, \vartheta_m, \vartheta_{m+1}) + \\
&\quad 2S(\vartheta_{m+1}, \vartheta_{m+1}, \vartheta_{m+2}) + \\
&\quad \cdots + \\
&\quad S(\vartheta_{m+p-1}, \vartheta_{m+p-1}, \vartheta_{m+p}) \\
&\preceq 2S(\vartheta_m, \vartheta_m, \vartheta_{m+1}) + \\
&\quad 2S(\vartheta_{m+1}, \vartheta_{m+1}, \vartheta_{m+2}) + \\
&\quad \cdots + \\
&\quad 2S(\vartheta_{m+p-1}, \vartheta_{m+p-1}, \vartheta_{m+p}) \\
&\preceq 2(q^m + q^{m+1} + \cdots + q^{m+p-1}) \\
&\quad S(\vartheta_0, \vartheta_0, \vartheta_1) \\
&\preceq 2q^{m+p-1}(1 + q + q^2 + \dots) \\
&\quad S(\vartheta_0, \vartheta_0, \vartheta_1) \\
&\preceq 2\frac{q^{m+p-1}}{1-q} S(\vartheta_0, \vartheta_0, \vartheta_1).
\end{aligned}$$

$\langle \vartheta_m \rangle$ is a V-Cauchy sequence because V be Archimedean. Then by V-completeness of $\Re$, there exist $\vartheta \in \Re$ such that $\vartheta_m \xrightarrow{S,V} \vartheta$. So there exist $\langle b_m \rangle$ in V such that $b_m \downarrow 0$ and $S(\vartheta_m, \vartheta_m, \vartheta) \preceq b_m$. Since

$$\begin{aligned}
S(f\vartheta, f\vartheta, \vartheta) &\preceq 2S(f\vartheta_m, f\vartheta_m, f\vartheta) + \\
&\quad S(f\vartheta_m, f\vartheta_m, \vartheta) \\
&\preceq 2qS(\vartheta_m, \vartheta_m, \vartheta) + \\
&\quad S(\vartheta_{m+1}, \vartheta_{m+1}, \vartheta) \\
&\preceq 2qb_m + b_{m+1} \\
&\preceq 2(q+1)b_m,
\end{aligned}$$

then $S(f\vartheta, f\vartheta, \vartheta) = 0$, i.e. $f\vartheta = \vartheta$.

We can also verify the following theorem as above.

Theorem 2 *Let $(\Re, S, V)$ be a vector S-metric space which is complete and V be Archimedean. Suppose the mappings $f : \Re \to \Re$ satisfies*

$$\begin{aligned}
S(f\hbar, f\hbar, f\vartheta) \preceq \{&a_1 S(\hbar, \hbar, f\hbar) + a_2 S(\vartheta, \vartheta, f\vartheta) \\
&+ a_3 S(\hbar, \hbar, f\vartheta) + a_4 S(\vartheta, \vartheta, f\hbar) \\
&+ a_5 S(\hbar, \hbar, \vartheta)\}
\end{aligned}$$

for all $\hbar, \vartheta \in \Re$*, where* a_1, a_2, a_3, a_4 *and* a_5 *are positive and* $a_1+a_2+a_3+a_4+a_5 < 1$*. Then* f *has fixed point in* $\Re$ *and for any* $\vartheta_0 \in \Re$*, iterative sequence* $\langle \vartheta_m \rangle$ *defined by* $y_m = f\vartheta_{m-1}$*,* $m \in \mathbb{N}$*,* V*-converges to fixed point of* f*.*

Example 2 *Let* $V = \mathbb{R}^2_+$ *with coordinatewise ordering and let*

$$\Re = \{(0, \vartheta) \in \mathbb{R}^2 : 0 \preceq \vartheta \preceq 1\} \cup$$

$$(\vartheta, 0) \in \mathbb{R}^2 : 0 \preceq \vartheta \preceq 1\}.$$

The mapping $S : \Re \times \Re \times \Re \to V$ *is defined by*

$$\begin{aligned} S((\hbar, 0), (\hbar, 0), (\vartheta, 0)) &= (\frac{4}{3}|\hbar - \vartheta|, |\hbar - \vartheta|) \\ S((0, \hbar), (0, \hbar), (0, \vartheta)) &= (|\hbar - \vartheta|, \frac{2}{3}|\hbar - \vartheta|) \\ S((\hbar, 0), (\hbar, 0), (0, \vartheta)) &= (\frac{4}{3}\hbar + \vartheta, \hbar + \frac{2}{3}\vartheta) \end{aligned}$$

Then $\Re$ *is vector* S*-metric space which is complete.*

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