$\mathcal{M}$-Projective Curvature Tensor over (κ,μ)-Contact Riemannian Manifolds

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Abstract: -

In 1995, the concept of (κ,μ)-contact Riemannian manifolds was introduced by Blair, Koufogiorgos, and Papantoniou [5]. Subsequently, a comprehensive investigation into the classification of contact metric (κ,μ)-spaces was conducted by Boeckx, E. [7] in 2000. Blair explored the (κ,μ)-nullity condition in the context of contact Riemannian manifolds and provided various motivations for its study. The current paper focuses on the examination of flatness conditions concerning the $\mathcal{M}$-projective curvature tensor within the framework of (κ,μ)-contact Riemannian manifolds.

1. Introduction-

In 1958, Boothby and Wong first introduced the concept of odd-dimensional manifolds with contact and almost contact structures, primarily approaching it from a topological perspective. Subsequently, in 1961, Sasaki and Hatakeyama re-examined these structures using tensor calculus techniques.

Alternatively, in the work of Pokhariyal and Mishra, a tensor field W^* is introduced on a Riemannian manifold as

$$'W^*(S,T,U,V) = {'\mathcal{R}}(S,T,U,V) - \frac{1}{2(n-1)} \times [\rho(T,U)g(S,V) - \rho(S,U)g(T,V) + g(T,U)\rho(S,V) - g(S,U)\rho(T,V)]\,, \tag{1}$$

Where $'W^*(S,T,U,V) = g(W^*(S,T)U,V)$ and $'\mathcal{R}(S,T,U,V) = g(\mathcal{R}(S,T)U,V)$. The tensor field W^* is referred to as the $\mathcal{M}$-projective curvature tensor. Subsequently, Ojha conducted a comprehensive investigation of the properties of this tensor in both Sasakian and Kähler manifolds.

The category of (κ,μ)-contact Riemannian manifolds encompasses both Sasakian and non-Sasakian manifolds. Boeckx [7] provided a comprehensive categorization of (κ,μ)-contact Riemannian manifolds. These manifolds retain their properties under D-homothetic transformations.

In an earlier study [6], Blair, Kim, and Tripathi commenced an inquiry into the concircular curvature tensor of contact Riemannian manifolds. The examination of the pseudo-projective curvature tensor on a contact Riemannian manifold was recorded in [5]. More contemporarily, the investigations carried out by [14] and [15] delved into exploring the quasi-conformal curvature tensor and the E-Bochner curvature tensor on a (κ,μ)-contact Riemannian manifold, respectively. In addition to the well-known Riemannian curvature tensor, the Weyl conformal curvature tensor, and the concircular curvature tensor, the $\mathcal{M}$-projective curvature tensor emerges as a pivotal tensor within the realm of differential geometry. The curvature tensor serves as a unifying link between the conharmonic curvature tensor, the concircular curvature tensor and the conformal curvature tensor on the one hand while establishing a connection with the H-projective curvature tensor on the other.

Recently, the $\mathcal{M}$-projective curvature tensor has been a subject of study for various researchers, including Chaubey, Ojha [13], Singh [11], and others.

Expanding upon prior research, our current study investigates the symmetry and flatness characteristics of (κ,μ)-contact Riemannian manifolds in the context of the $\mathcal{M}$-projective curvature tensor. In Section 3, we review and deduce our initial findings. Subsequently, in Segment 4, we analyze $\mathcal{M}$-projectively flat (κ,μ)-contact Riemannian manifolds. Segment 5 centers on exploring ζ-$\mathcal{M}$-projectively Sasakian flat (κ,μ)-contact Riemannian manifolds, where we establish the requisite and sufficient conditions for the manifestation of ζ-$\mathcal{M}$-projective Sasakian flatness in an (κ,μ)-contact Riemannian manifold.

2. Contact Riemannian Manifold-

An almost contact structure on an $(2n+1)$-dimensional differentiable manifold M is defined by the existence of a tensor field $\mathcal{F}$ of type (1, 1), a vector field ζ, and a 1-form η such that

$$\mathcal{F}^2 = -I + \eta \otimes \zeta\,, \qquad \eta(\zeta) = 1 \tag{2}$$

$$\mathcal{F}\zeta = 0, \qquad \eta \circ \mathcal{F} = 0 \tag{3}$$

Take into account a consistent Riemannian metric g in conjunction with an almost contact structure $(\mathcal{F},\zeta,\eta)$

$$g(\mathcal{F}S,\mathcal{F}T) = g(S,T) - \eta(S)\eta(T) \tag{4}$$

Subsequently, when M^{2n+1} undergoes a transformation, it transforms into an almost contact Riemannian manifold by acquiring an almost contact metric structure represented as $(\mathcal{F},\zeta,\eta,g)$. By observing equations (2) and (4), it becomes evident that

$$g(S,\mathcal{F}T) = -g(\mathcal{F}S,T), \quad g(S,\zeta) = \eta(S), \tag{5}$$

for all vector fields S and T.

The fact that the tangent sphere bundle of a Euclidean Riemannian manifold possesses a contact metric structure with the property $\mathcal{R}(S,T)\zeta = 0$ is widely acknowledged. Conversely, in the context of a Sasakian manifold, the subsequent assertion is valid:

$$\mathcal{R}(S,T)\,\zeta = \eta(T)S - \eta(S)T. \tag{6}$$

Blair et al. extended the concepts of $\mathcal{R}(S,T)\zeta = 0$ and the Sasakian case by investigating the (κ,μ)-nullity condition on a contact Riemannian manifold. They introduced the (κ,μ)-nullity distribution $N(\kappa,\mu)$ ([3,5]) to characterize this condition on the contact Riemannian manifold.

$$N(\kappa,\mu){:}\,\mathcal{P} \rightarrow N_{\mathcal{P}}(\kappa,\mu) = \{U \in \mathcal{T}_{\mathcal{P}}M{:}\,\mathcal{R}(S,T)U = (\kappa I + \mu h)[g(T,U)S - g(S,U)T]\}$$

…(7)

For any pair of vectors S and T belonging to the tangent space $\mathcal{T}M$, where (κ,μ) are elements of the R^2, a Riemannian manifold M^{2n+1} possessing ζ in the set $N(\kappa,\mu)$ is referred to as a manifold with (κ, μ) characteristic. Specifically, on a manifold with (κ, μ) attributes, the following holds true

$$\mathcal{R}(S,T)\zeta = \kappa[\eta(T)S - \eta(S)T] + \mu\,[\eta(T)hS - \eta(S)hT]\,. \tag{8}$$

On a (κ, μ)-manifold, where $\kappa \leq 1$, the structure becomes Sasakian with $h = 0$ and μ remaining indeterminate when $\kappa = 1$. When $\kappa < 1$, the (κ, μ)-nullity condition uniquely prescribes the curvature of M^{2n+1} Essentially, for a (κ, μ)-manifold, the properties of being a Sasakian manifold, a K-contact manifold, $\kappa = 1$, and $h = 0$ are all interchangeable and equivalent.

In a (κ, μ)-manifold, the following relations hold:

$$h^2 = (\kappa - 1)^2\,\mathcal{F}^2\,,\ \kappa \leq\ 1,$$

$$\mathcal{R}(\zeta, S)T = \kappa[g(S,T)\zeta - \eta(T)S] + \mu[g(hS,T)\zeta - \eta(T)hS]\,,$$

$$\rho(S,\zeta) = 2n\kappa\eta\,(S), \tag{9}$$

$$\rho(S,T) = [2(n-1) - n\mu]g(S,T) + [2(n-1) + \mu]g(hS,T)$$

$$+\,[2(1-n) +\ n(\,2\kappa + \mu)]\,\eta(S)\eta(T),\ n \geq 1,$$

$$\rho(\mathcal{F}S, \mathcal{F}T) = \rho(S,T) - 2n\kappa\eta(S)\eta(T) - 2(2n - 2 + \mu)g(hS,T),$$

Where ρ is the Ricci tensor of type (0, 2), Q is the Ricci operator, that is, $g(QS,T) = \rho(S,T)$.

Furthermore, the (κ, μ)-manifold exhibits the following property:

$$\eta(\mathcal{R}(S,T)U) = \kappa[g(T,U)\eta(S) - g(S,U)\eta(T)]$$

$$+\mu[g(hT,U)\eta(S) - g(hS,U)\eta(T)] \tag{10}$$

In the context of Riemannian manifold, the $\mathcal{M}$-projective curvature tensor W^* can be stated as follows [8].

$$W^*(S,T)U = \mathcal{R}(S,T)U - \frac{1}{2(n-1)} \times [\rho(T,U)S - \rho(S,U)T + g(T,U)QS - g(S,U)QT],$$

…(11)

Given arbitrary vector fields S, T, and U, where ρ represents the Ricci tensor of type (0, 2) and Q denotes the Ricci operator, we have the relation $g(QS,T) =\ \rho(S,T)$.

Lemma 2.1. [1] In (κ, μ)-contact Riemannian manifolds that are not Sasakian, the conditions that follow are mutually equivalent:

(i) η-Einstein manifold,

(ii) $Q\mathcal{F} = \mathcal{F}Q$

Definition 2.1. An M manifold with a (k, μ)-contact metric structure is referred to as η-Einstein when the Ricci operator Q fulfills the conditions

$$Q = aI + b\eta \otimes \zeta \tag{12}$$

Where a and b represent smooth functions defined on the manifold. Notably, when b is set to zero, M qualifies as an Einstein manifold.

In the case where an $(2n+1)$-dimensional non-Sasakian (k, μ)-contact Riemannian manifold (M^{2n+1}, g) is η-Einstein, the expression for the non-zero Ricci tensor ρ takes the following form:

$$\rho(S,T) = ag(S,T) + b\eta(S)\eta(T). \tag{13}$$

Lemma 2.2. On a non-Sasakian (k, μ)-contact Riemannian manifold (M^{2n+1}, g), $a + b = 2n\kappa$

Proof. In view of (2)-(5) and (13), we have

$$QS = aS + b\eta(S)\zeta, \tag{14}$$

such that Ricci operator Q is defined by

$$\rho(S,T) = g(QS,T). \tag{15}$$

Again, contracting (14) with respect to S and using (2)-(5), we have

$$r = (2n+1)a + b. \tag{16}$$

Now, putting ζ instead of S and T in (13) and then using the equations in (2)-(5) and (9) we get

$$a + b = 2n\kappa. \tag{17}$$

Equations (16) and (17) give

$$a = \left(\frac{r}{2n} - \kappa\right) \text{ and } b = \left((2n+1)\kappa - \frac{r}{2n}\right). \tag{18}$$

Equation (18) prove the statement of the Lemma 2.2.

3. The $\mathcal{M}$-Projective Curvature Tensor W^* for an (κ, μ)-Contact Riemannian Manifolds-

The curvature tensor W^* associated with $\mathcal{M}$-projective geometry on a (κ, μ)-contact Riemannian manifold is expressed as

$$W^*(S,T)\zeta = -\frac{\kappa}{(n-1)}[\eta(T)S - \eta(S)T] + \mu[\eta(T)hS - \eta(S)hT] - \frac{1}{2(n-1)}[\eta(T)QS - \eta(S)QT], \tag{19}$$

$$\eta(W^*(S,T)\,\zeta) \;=\; 0, \tag{20}$$

$$W^*(\zeta,T)U = -W^*(T,\zeta)U = -\frac{\kappa}{(n-1)}[g\,(T,U)\zeta - \eta(U)T] + \mu[g(hT,U)\zeta - \eta(U)hT]$$

$$-\tfrac{1}{2(n-1)}[\rho(T,U)\zeta - \eta(U)QT], \tag{21}$$

$$\eta(W^*(\zeta,T)U) = -\eta(W^*(T,\zeta)U)$$
$$= -\frac{\kappa}{(n-1)}[g(T,U) - \eta(T)\eta(U)] + \mu[g(hT,U) - \eta(U)\eta(hT)]$$

$$-\tfrac{1}{2(n-1)}[\rho(T,U) - 2n\kappa\eta(T)\eta(U)], \tag{22}$$

$$\eta(W^*(S,T)U)$$
$$= -\frac{\kappa}{(n-1)}[g(T,U)\eta(S) - g(S,U)\eta(T)] + \mu[g(hT,U) - \eta(U)\eta(hT)]$$

$$-\tfrac{1}{2(n-1)}[\rho(T,U)\eta(S) - \rho(S,U)\eta(T)]. \tag{23}$$

4. $\mathcal{M}$-Projectively Flat (κ,μ)-Contact Riemannian Manifolds-

The class of (κ,μ)-contact Riemannian manifolds known as $\mathcal{M}$-projectively flat manifolds is a distinctive category within contact Riemannian manifold where the geometry is such that the curvature tensor satisfies certain conditions related to the $\mathcal{M}$-projective flatness property. The parameters κ and μ are involved in the definition of the curvature conditions and can affect the geometry of the manifold.

Theorem 4.1. A (κ,μ)-contact Riemannian manifold M^{2n+1}that is $\mathcal{M}$-projectively flat exhibits the property of being an Einstein manifold.

Proof. Let $W^*(S,T,U,V) = 0$. Subsequently, utilizing equation (11), we derive the following outcome:

$$'\mathcal{R}(S,T,U,V) = \tfrac{1}{2(n-1)}\;[\rho(T,U)g(S,V) - \rho(S,U)g(T,V) + g(T,U)\rho(S,V)$$

$$-g(S,U)g(T,V)] \tag{24}$$

Considering e_i as an orthonormal basis of the tangent space at any point, if we set $T = U = e_i$ in the given equation and then sum up over i, where $1 \le i \le 2n+1$, we arrive at the same result,

$$\rho(S,T) = -rg\,(S,T), \tag{25}$$

Where r-Scalar curvature of the manifold and $r = 2n(2n-2+\kappa-n\mu)$.

This indicates that M^{2n+1} is a manifold that satisfies the Einstein condition. This completes the proof.

5. ζ-$\mathcal{M}$-Projectively Sasakian Flat (κ, μ)-Contact Riemannian Manifolds-

ζ-$\mathcal{M}$-Projectively Sasakian flat (κ, μ)-contact Riemannian manifolds likely refer to a specific class of contact Riemannian manifolds that satisfy curvature conditions related to $\mathcal{M}$-projective flatness and these manifolds also have a distinguished Reeb vector field (ζ) and Sasakian geometry. This indicates a very specialized and intricate geometric structure where various curvature conditions, contact structures, and vector fields are intertwined.

Definition 5.1. An (2n+1) (with $n > 1$)-dimensional (κ, μ)-contact Riemannian manifold is classified as ζ-$\mathcal{M}$-projectively Sasakian flat when the condition $W^*(S,T)\zeta = 0$ holds for all S and T belonging to the tangent space $\mathcal{T}M$.

Theorem 5.1. An (2n+1)-dimensional (n>1) (κ, μ)-contact Riemannian manifold exhibits ζ-$\mathcal{M}$-projective Sasakian flatness iff it possesses the characteristic of being an η-Einstein manifold.

Proof. Let $W^*(S,T)\zeta = 0$. Then, in view of (11), we have

$$\mathcal{R}(S,T)\zeta = \frac{1}{2(n-1)}[\rho(T,\zeta)S - \rho(S,\zeta)T + g(T,\zeta)QS - g(S,\zeta)QT] \quad (26)$$

Due to the presence of (5), (8), and (9), the equation above can be simplified to

$$\kappa[\eta(T)S - \eta(S)T] + \mu[\eta(T)hS - \eta(S)hT]$$

$$= \frac{n\kappa}{n-1}[\eta(T)S - \eta(S)T] + \frac{1}{2(n-1)}[\eta(T)QS - \eta(S)QT] \quad (27)$$

which by putting $T = \zeta$, gives

$$QS = 2\kappa\,[-S + (n+1)\eta(S)\zeta] + 2(n-1)\mu(hS) \quad (28)$$

In the case of Sasakian manifolds, $\kappa = 1$, (and hence $h = 0$)

Now, taking the inner product of above equation with V, we get

$$\rho(S,V) = 2[-g(S,V) + (n+1)\,\eta(S)\eta(V)] \quad (29)$$

Furthermore, it can be proved that a (κ, μ)-contact Riemannian manifold represents an η-Einstein manifold. Conversely, assume that condition (29) is fulfilled. As a result of the implications of (28) and (19), we can deduce $W^*(S,T)\zeta = 0$. Thus, the proof is concluded.

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