Analyzing Fractional Differential Equations Using the Pourreza Transform

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ABSTRACT

This research investigates specific classes of fractional differential equations using a straightforward fractional calculus technique. The employed methodology yields various fascinating results, including a broader adaptation of the widely recognized classical Frobenius method. The approach outlined in this study primarily relies on fundamental theorems concerning the specific solutions of fractional differential equations, making use of the Pourreza transform and binomial series extension coefficients. Additionally, the study presents advanced techniques for solving fractional differential equations effectively, illustrated through practical examples.

Keywords— fractional-order differential equation; Riemann-Liouville (RL) fractional integrals; Mittag-Leffler function; gamma function; Pourreza transform of the fractional derivative.

#  INTRODUCTION

 In recent years, fractional differential equations have garnered substantial interest because of their capacity to capture complex phenomena in diverse fields of science and engineering. These equations go beyond traditional integer-order differentiation, enabling a more precise description of processes that display memory and non-local behaviors. This research delves into the realm of fractional differential equations, aiming to advance the understanding and solution techniques for specific classes of these equations. The foundation of our investigation draws inspiration from a rich history of mathematical contributions, as evidenced by the works of renowned scholars in this field. Caputo's seminal work [1] on elasticity and anelastic dissipation laid the groundwork for understanding the fundamentals of fractional calculus. Podlubny's comprehensive treatise [2] on fractional differential equations has served as a cornerstone reference for researchers and practitioners alike. Our research also draws upon innovative methods and techniques that have emerged in recent years. Notably, Zhang's Sumudu-based algorithm [3] provides a valuable computational tool for solving differential equations, while Aboodh's transformative work [4] introduced the Aboodh transform, offering a novel approach to tackling fractional differential equations. The study of Laplace transforms in the context of fractional differential equations has been a focus of research, as demonstrated by the contributions of Lin and Lu [5]. Mohamed's Elzaki transformation [6] and Kashuri, Fundo, and Liko's new integral transform [7] represent additional methodologies that have expanded the arsenal of techniques available for solving these equations. Furthermore, the New Integral Transform Mohand Transform [8], as proposed by Abdelrahim Mahgoub, has brought new perspectives to the field. Silva, Moreira, and Moret's work on conformable Laplace transforms [9] adds to the evolving landscape of fractional calculus techniques. Lastly, the Aboodh transform continues to be explored, as evidenced by Aruldoss and Anusuya Devi [10], who have employed it for solving fractional differential equations. Additionally, the study by Raghavendran et al. [11] explores the use of Aboodh transform for fractional integro-differential equations, showcasing its versatility.

 In this investigation, we utilize the Pourreza transform of fractional derivatives and the coefficients from binomial series extensions to address multiple fractional differential equations. Moreover, we unveil various properties that are relevant to our main focus. To illustrate our findings, we present practical examples.

# PRELIMINARIES

## In this section, we are listing some preliminaries that are useful throughout the paper [11].

1. The definition of the RL fractional integral with order for a function can be expressed as follows:

2. The Caputo fractional derivative of the function is defined as follows:

The Euler gamma function, denoted as , is defined as follows:

3. The Pourreza transform of a function , is defined by

4. The Mittag-Leffler function is defined by

5. The Simplest Wright function is defined by

6. The general Wright function ij is characterized by the following conditions , ,

 and real , as determined by the provided series.

 ij ij

7. The inverse Pourreza transform is defined by

## **Remark 2.1:**

**Note:** Fubini's theorem is employed to rearrange the order of integration in the preceding derivative.

# SOLUTIONS OF THE FRACTIONAL DIFFERENTIAL EQUATIONS

In this section, there are strong indications that the function alone may be adequate to enable the Pourreza transform to operate successfully at a certain value of the parameter .

**Theorem 3.1.** Let and and . Then thefractional differential equation

|  |  |
| --- | --- |
|  |  |

 with initial conditions and has the unique solution

|  |  |
| --- | --- |
|  |  |

**Proof:** Utilizing the Pourreza transform in and taking into consideration, we have

|  |  |
| --- | --- |
|  |  |

Since

|  |  |
| --- | --- |
|  |  |

Substituting the above equation in , we get

|  |  |
| --- | --- |
|  |  |

Thus, the inverse Pourreza transform to equation yields the solution

|  |  |
| --- | --- |
|  |  |

which is . This completes the proof of the theorem.

**Example 3.1.** The fractional differential equation is

with initial conditions and has the unique solution

Figure 1 illustrates the solution behavior of the fractional differential equation of Example 3.1 at various values of with the initial conditions and .



**Theorem 3.2.** Let and and . Then thefractional differential equation

|  |  |
| --- | --- |
|  |  |

 with initial conditions and has the unique solution

|  |  |
| --- | --- |
|  |  |

**Proof:** Utilizing the Pourreza transform in and taking into consideration, we have

Since

Substituting the above equation in and taking the inverse, yields the solution

|  |  |
| --- | --- |
|  |  |

which is . This completes the proof of the theorem. Also, the Wright function can express this solution as

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11

11

**Example 3.2.** Thefractional differential equation

With initial conditions and has the unique solution

Figure 2 illustrates the solution behavior of the fractional differential equation of Example 3.2 at various values of with the initial conditions and .



**Proposition 3.1.** Let and . Then thefractional differential equation

with initial conditions its proposal is provided by

**Proof:** The proof of this proposition as like as previous theorem.

**Remark 3.1.** Accordingly, in , then the derivative is

with initial conditions and its proposal is provided by

**Proposition 3.2.** A nearly simple harmonic vibration differential equation

with initial conditions and its proposal is provided by

**Proof:** The above proof is accomplished by implanting in equation.

# CONCLUSION

 The article utilized the Pourreza transform to address certain fractional differential equations. The connection between the Pourreza transform and the Laplace transform was explored in greater detail, revealing additional instances of the Pourreza transform's applicability. A unique methodology for tackling fractional differential equations was introduced, involving the application of the Pourreza transform alongside binomial series extension coefficients. The focus also encompassed the examination of various properties and illustrated examples.

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