DECOMPOSABLE OF SINGLE-TIME AND MULTI-TIME GEOMETRIC DYNAMICS ON RIEMANN-KAEHLERIAN MANIFOLDS

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Abstract:

Samuelson (1970), have studied a law of conservation of the capital-output ratio. After that, Isvoranu, and Udriste (2006), locate fluid flow versus Geometric Dynamics and achieved from flows and metrics to dynamics and winds. Also, Gay-Balmaz, Holm and Ratiu (2009) stumble on Geometric dynamics of optimization. In this paper, the author calculated decomposable single-time and multi-time dynamics on Riemann-Kaehlerian manifolds and disintegrate the second order partial differential equations (PDEs) like created by multi-time flows and pairs of multi-time dynamics.

Keywords: Decomposable dynamics, Dynamical systems, Single-time geometric dynamics, Multi-time geometric dynamics and partial differential equations.

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1. Introduction:

The single-time dynamics we identify with an ordinary differential equation related to Newton second law. A multi-time dynamics is explained by a second order elliptic partial differential equation. This is significant to highlight the subsistence of decomposable movements and the necessary and sufficient conditions in which the decomposition obtains position. Any ordinary differential equation is malformed addicted to a one-flow or any partial differential equation is malformed addicted to an m-flow in any satisfactorily huge dimension. The geometry of space converts the one-flow addicted to a geodesic motion in a gyroscopic field of forces. The geometry of two spaces (source, target) changes the m-flow (or integral manifolds of an m-distribution) into harmonic maps deformed by gyroscopic field of forces [Udriste (2005); Udriste and Bejenaru (2012)]./////

The Equations of mechanics may appear different in form: $\dot{x}(t) = X\big(x(t)\big)$, as they often involve higher time derivatives, but an equation that is second or higher order in time can always be rewritten as a set of first order equations. The ordinary differential equations of the form $F(x(t), \dot{x}(t), \ddot{x}(t), \dddot{x}(t)) = 0$ which contain third order derivatives in them are sometimes called jerk equations. It has been shown that a jerk equation is in a mathematically well-defined sense the minimal setting for solutions showing chaotic behaviour. A jerk equation is equivalent to a system of three first-order ordinary non-linear differential equations

$\dot{x}(t) = y(t), \dot{y}(t) = z(t), \dot{z}(t) = \emptyset\big(x(t), y(t), z(t)\big)$.

This motivates a least squares Lagrangian of interest in jerk systems, namely

$$2\mathrm{L}_1 = (\dot{x}(t) - y(t))^2 + (\dot{y}(t) - z(t))^2 + (\dot{z}(t) - \emptyset(x(t), y(t), z(t)))^2$$

on the jet space of coordinates $(t, x, y, z, \dot{x}, \dot{y}, \dot{z})$, and its associated geometric dynamics (Euler-Lagrange equations)

$$(\dot{z} - \emptyset)\emptyset_x + \dot{x} - \dot{y} = 0, (\dot{z} - \emptyset)\emptyset_y + \dot{y} - \dot{z} = 0,$$

$$(\dot{z} - \emptyset)\emptyset_z + D_t(\dot{z} - \emptyset)) = 0.$$

More generally, being given *n* Lagrangians:

$$L^i\big(t, x(t), \dot{x}(t)\big), \quad i = \overline{1, n}, \; x(t) = \big(x^1(t), \ldots., x^n(t)\big), \; t \in I \subset R,$$

the associated least squares lagrangian with respect to the Riemannian metric $g_{ij}\,(x)$ is

$$\mathcal{L} = \frac{1}{2} g_{ij}\,\big(x(t)\big) L^i\big(t, x(t), \dot{x}(t)\big) L^j\,\big(t, x(t), \dot{x}(t)\big).$$

The extremals are solutions of the Euler-Lagrange ordinary differential equations system.

$$\frac{1}{2}\frac{\partial g_{il}}{\partial x^k} L^i L^j \; + g_{ij}\, L^i \frac{\partial L^j}{\partial x^k} - D_t\left(g_{ij}\, L^i \frac{\partial L^j}{\partial x^k}\right) = 0.$$

If the Langrangian L^i is associated to ordinary differential equations $\mathrm{L}^i\big(t, x(t), x\dot{(t)}\big) = 0$, then the extremals contain the solutions of that equation and the dynamics is decomposable [Mihlin (1983); Stefanescu and Udriste (1993); Furi (1995); Treanta and Udriste (2013)].

Let $u(x, t)$ be the density of the diffusing material at location $x \,\epsilon\, R^n$ and time $t \,\epsilon\, R$. Let $g^{ij}\,(u(x, t), x), i, j = \overline{1, n}$, be the collective diffusion coefficient (matrix) for density $\boldsymbol{u}$ at location $\boldsymbol{x}$. The diffusion partial differential equations is:

$$\frac{\partial u}{\partial t}\,(x, t) = \frac{\partial}{\partial x^i}\big(g^{ij}\,(u(x, t), x\big)\frac{\partial u}{\partial x^j}\,(x, t).$$

If the diffusion coefficient depends on the density, then the diffusion equation is nonlinear, otherwise it is linear. More generally, when $g^{ij}\,(u(x, t), x$ is a symmetric positive definite matrix (Riemannian metric), the equation describes anisotropic diffusion [Arnold (1969); Chorin and Marsden (2000); Udriste and Teleman (2004)].

The diffusion partial differential equations is equivalent to the first-order non-linear partial differential equations

$$\frac{\partial u}{\partial x^j} = v_j\,, \quad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x^i}\big(g^{ij}\, v_j\big),$$

Where the parameter of evolution (x, t) *is* $(n + 1)$ dimensional. A Riemannian metric $h^{ij}\,(u(x, t), x)$ produces a least squares Lagrangian

$$2L_2 = h^{ij}\left(\frac{\partial u}{\partial x^i} - v_i\right)\left(\frac{\partial u}{\partial x^j} - v_j\right) + \left(\frac{\partial u}{\partial t} - \frac{\partial}{\partial x^i}(g^{ij} v_i)\right)^2,$$

On the jet space of coordinates $(x, t, u, v, u_x, u_t, v_x, v_t)$. It appears the associated geometric dynamics (Euler-Lagrange equations)

$$\frac{1}{2}\frac{\partial h^{ij}}{\partial u}\left(\frac{\partial u}{\partial x^i} - v_i\right)\left(\frac{\partial u}{\partial x^j} - v_j\right) + \left(\frac{\partial u}{\partial t} - \frac{\partial}{\partial x^i}(g^{ij} v_j)\right)\left(-\frac{\partial}{\partial x^i}\left(\frac{\partial g^{ij}}{\partial u} v_j\right)\right)$$
$$- D_{x^i}\left(h^{ij}\left(\frac{\partial u}{\partial x^j} - v_j\right)\right) - D_t\left(\frac{\partial u}{\partial t} - \frac{\partial}{\partial x^i}(g^{ij} v_j)\right) = 0,$$

$Or\ g^{lm} D_{x^m}\left(\frac{\partial u}{\partial t} - \frac{\partial}{\partial x^i}(g^{ij} v_j)\right) = 0.$

Again, Let T be an orientable manifold with the coordinates $t = (t^1, \ldots., t^m)$ and M be a manifold with the coordinate $x = (x^1, \ldots., x^n)$. Using m vector field $X_\alpha(t, x)$ of class c^∞ on $T \times M$, we introduce the distribution described by pfaff equations

$dx^i(t) - X^i_\alpha(t, x)dt^\alpha = 0, i = \overline{1, n},\ \ \alpha = \overline{1, m}.$

Using some metric tensor $h_{\alpha\beta}(t), g_{ij}(t)$, and the components $\frac{\partial x^i}{\partial t^\alpha}(t) - X^i_\alpha(t, x)$ of the pullbacks, we build the least squares Lagrangian (non-decomposable dynamics).

$$L = \frac{1}{2} g_{ij} h^{\alpha\beta}\left(\frac{\partial x^i}{\partial t^\alpha}(t) - X^i_\alpha(t, x)\right)\left(\frac{\partial x^j}{\partial t^\beta}(t) - X^j_\beta(t, x)\right)\sqrt{det(h_{\alpha\beta})} > 0$$

Suppose the integral manifolds of the distribution have the dimension $1 \leq p < m$. Then we can introduce another least squares lagrangian constructed from ODEs/PDEs that describes the integral manifolds and the action is an integral with the volume element on the ***p*** parameters which define the integral manifold (decomposable dynamics) [Schubert and Counselman et al. (1977) ; Udriste and Udriste (2006)],

More generally being given n m Lagrangians

$$L^i_\alpha\left(t, x(t), x_\gamma(t)\right), i = \overline{1, n},\ \ \alpha = \overline{1, m}.\ x(t) = \left(x^i(t), \ldots. x^n(t)\right), t = (t^1, \ldots., t^m) \in I \subset T,$$

then the associated least squares Lagrangian density with respect to the Riemannian metrics $g_{ij}(x)h^{\alpha\beta}(t),\ h^{\alpha\beta}(t)$ *is*

$$\mathcal{L} = \frac{1}{2} g_{ij}\left(x(t)\right)h^{\alpha\beta}(t)L^i_\alpha\left(t, x(t), x_\gamma(t)\right)L^j_\beta\left(t, x(t), x_\gamma(t)\right).$$

If $T \subset R^m$, the extremals are solutions of the Euler-Lagrange *PDE* system

$$\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k} h^{\alpha\beta} L^i_\alpha L^j_\beta + g_{ij} h^{\alpha\beta} L^i \frac{\partial L^j_\beta}{\partial x^k} - D_\gamma\left(g_{ij} h^{\alpha\beta} L^i_\alpha \frac{\partial L^j}{\partial x^k_\gamma}\right) = 0.$$

If the Lagrangian L_α^i is associated to the PDE $L_\alpha^i(t, x(t), x_\gamma(t)) = 0$, then the extremals contain the solutions of that equation and the dynamics is decomposable. The ingredients needed to solve these problems are the Riemannian metrics, techniques of least squares Lagrangians and the idea of dynamics transversal decomposition [Lovelock and Rund (1975)].

2. SINGLE-TIME GEOMETRIC DYNAMICS ON RIEMANN-KAEHLERIAN MANIFOLDS:

Let M be a differentiable manifold and $I \subset R$ be a nontrivial interval. A (time dependent) non-autonomous first order differential equation on a manifold M is given by assigning, on an open subset V of $R \times M$, a non-autonomous C^∞ vector field $X: V \to R^n$, which is tangent to M for all $t \in R$. That is, for any all $t \in R$, the map $X_t: V_t \to R^n$, given by $X_t(x) = X(t, x)$, is a tangent vector field on the (possibly empty) open subset . $V_t = \{x \in M \mid (t, x) \in V\}\ of\ M$. In other words, $X(t, x) \in T_x M$ for each $(t, x) \in V$ and the first order differential equation associated to X is defined by [Furi (1995)]:

$$\dot{x} = X(t, x), (t, x) \in V. \tag{2.1}$$

A solution of the differential Equation (2.1) is a C^1 map $x: I \to M$, such that, for all $t \in I, (t, x(t)) \in V$ and $\dot{x}(t) = X(t, x(t))$ identically on I. In the case of Cauchy problem, a solution of the ODE (2.1), which satisfies the initial condition $x(t_0) = x_0$. Then the solution of this Cauchy problem exists and it is unique.

Let $F: R \times TM \to R^n$ be a C^∞ map. An equality of the type:

$$\dot{x} = F(t, x, \dot{x}), (t, x, \dot{x}) \in R \times TM. \tag{2.2}$$

is called a (time dependent) second order differential equation on M, provided that the associated vector field [Furi (1995)]:

$$G: R \times TM \to R^n \times R^n, G(t, x, y) = (y, F(t, x, y))$$

is tangent to TM, i.e., $(y, F(t, x, y)) \in T_{(x,y)}TM$ for all $(t, x, y) \in R \times TM$. A solution of the differential Equation (2.2) is a C^2 curve $x: I \to R^n$, in such a way that $x(t) \in M$ and $\ddot{x}(t) = F(t, x(t), \dot{x}(t))$, identically on I. In the case of The Cauchy problem, a solution of the ODE (2.2) which satisfies the initial conditions $x(t_0) = x_0$, $\dot{x}(t_0) = v$. Then the solution of this Cauchy problem exists and it is unique. If we use the components, the relations (2.1) and (2.2) are called respectively first order and second order ODE systems.

Now, We start with the triple *(M, g, X)*, where M is a manifold of dimension $n, g(x) = \left(g_{ij}(x)\right), i, j = 1, \dots, n,$ is a Riemannian metric and $X(t, x) = (X^i(t, x))$ a time dependent C^∞ vector field, on the manifold M. Suppose the Levy-Civita connection ∇ of (M, g) has the components $G_{jk}^i, i, j, k = 1, \dots, n$.

Definition 2.1. We use the notations:

$$F_j = \left(F_j^i\right),\ F_j^i = \nabla_j X^i - g^{ih} g_{kj} \nabla_h X^k,\ \ f = \frac{1}{2} g(X, X).$$

A function $F: R \times TM \to R^n$ is said to be generated by the pair (X, g) if it is of the form:

$$F = -G_{jk}\dot{x}^j\dot{x}^k + F_j\dot{x}^j + \nabla \mathrm{f} + \frac{\partial}{\partial t}X.$$

If F is generated by X and g, then the ODE (2.2) represents a single-time geometric dynamics or a geodesic motion in a gyroscopic field of forces. By analogy with the reduction of the force system in mechanics, resultant and momentum, the decomposition of the set of solutions returns to the flow and the movement in the gyroscopic field of forces [Udriste (2000); Udriste (2004); Udriste (2005); Isvoranu, and Udriste (2006) and Udriste and Bejenaru (2012)].

Theorem 2.1. If $F: R \times TM \to R^n$ is generated by the pair (X, g), then the set of maximal solutions of ODE (2.2) is decomposable into a subset corresponding to the initial values

$$x(t_0) = x_0, \ \ \dot{x}(t_0) = X(\, t_0, x(t_0)),$$

solutions which are reducible to solutions of the ODE (2.1), and a subset of solutions corresponding to the initial values

$$x(t_0) = x_0, \qquad \dot{x}(t_0) = W \neq \lambda X\big(t_0, x(t_0)\big), \qquad \lambda > 0,$$

transversal to the solutions of the ODE (2.1). The converse is also true.

Proof. We have from existence and uniqueness theorem, each solution $x = x(t)$ of any second order continuance of first order ODE system has the property:

$$\dot{x}(t_0) = X\big(t_0, x(t_0)\big) \ \Rightarrow \ \dot{x}(t) = X\big(t, x(t)\big), \forall\, t \in I.$$

A flow X and a Riemannian metric g determine a least squares Lagrangian:

$$L(t, x, \dot{x}) = \frac{1}{2} g\big(\dot{x} - X(t, x), \dot{x} - X(t, x)\big).$$

The Euler-Lagrange ODEs represent a geometric continuance of the flow. The Euler-Lagrange ODEs constitute just a decomposable dynamic geodesic motion in gyroscopic fields of forces equivalent to set of flow trajectories with set of transversal trajectories imposed by the geometry of the space.

Theorem 2.2. Suppose that X is an autonomous vector field. If the function $F: TM \to R^n$ is generated by X and g, then the set of maximal solutions of ODE (2.2) divides into three parts i.e. Curves $[x(t), H\big(x(t)\big)] = const = 0; > 0; < 0.$

Proof. We have from Hamiltonian:

$$H(t, x, \dot{x}) = \frac{1}{2} g\big(\dot{x} - X(t, x), \qquad \dot{x} - X(t, x)\big) = \frac{1}{2}\Big(g(\dot{x}, \dot{x}) - g(X, X)\Big) = H(x, \dot{x}),$$

and the connected Hamilton ODEs. The curves $x(t)$ with $H\big(x(t)\big) = const = 0$ are solutions of ODE (2.1). The solutions with $H\big(x(t)\big) = const \neq 0$, are transversal to solutions of ODE (2.1).

If any normal ODE generates in the phase space a flow, which together with the phase space geometry gives a geometric dynamic. This statement is true for any ODE, but then appears a flow with constraint.

Let us consider the operators (M, X, g, Γ), where M is a Riemannian manifold, X is a flow on M, g is a fundamental tensor field and Γ is a symmetric connection (derivation). The operators (X, g, Γ) generates an extended geometric dynamic on M determined by ODEs.

$$\ddot{x}^i(t) = \left(\delta^i_k \delta^l_j - g_{kj} g^{li}\right) X^k_{,\,l} \dot{x}^j(t) + \frac{\partial X^i}{\partial t} g_{kj} g^{li} X^k_{,\,l} X^j.$$

On the Riemannian manifold $\left((0,\infty),\ g(x) = 1\right)$, let us take the flow $\dot{x} = 1$. We attach the least squares lagrangian $L_1 = (\dot{x} - 1)^2$, with Euler-Lagrange equation $\ddot{x} = 0$. On any other Riemannian manifold $\left((0,\infty),\ g(x)\right)$, we find the least squares Lagrangian $L_2 = g(x)(\dot{x} - 1)^2$, with Euler-Lagrange equation $\ddot{x} = \frac{g'(x)}{2g(x)}(1 - \dot{x})(1 + \dot{x})$. Here, $\Gamma(x) = \frac{g'(x)}{2g(x)}$ is a linear connection. We can extend the previous ODE to the ODE system:

$$\ddot{x}^i(t) = a^i_0\left(x(t)\right) + a^i_j\left(x(t)\right)\dot{x}^j(t) + b^i_{jk}\left(x(t)\right)\dot{x}^j(t)\dot{x}^k(t), i, j, \qquad k = 1, \dots, n,$$

with possible disorder in velocities.

Now, Let ***M*** be a differentiable manifold of dimension ***n*** and $I \subset R$ be a nontrivial interval. If the ODE system (2.2) is an Euler-Lagrange system on M for a regular Lagrangian $L(t, x, \dot{x})$, then there exists a fundamental tensor field $g = \left(g_{ij}\right)$ on TM such that:

$$g_{ij}(t, x, \dot{x}) = \frac{1}{2} \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}(t, x, \dot{x}), i, j = 1, \dots., n.$$

Conversely, given $g_{ij}(t, x, \dot{x})$, to determine $L(t, x, \dot{x})$, we need complete integrability conditions. In these conditions, using two successive curvilinear integrals of the second type, we can write

$$L(t, x, \dot{x}) = \int_{\gamma \dot{x}_0 \dot{x}} \int_{\gamma \dot{x}_0 \dot{x}} g_{ij}(t, x, \dot{x})\, d\dot{x}^i d\dot{x}^j + a_i(t, x)\dot{x}^i + b(t, x).$$

The pair (M, g) is called a Lagrangian manifold.

Theorem 2.3. If u is positive, then the motion described by the Hamiltonian $H = \frac{1}{2} g(\dot{x}, \dot{x}) - u(x)$ is generatedby a flow and a Riemannian metric.

Proof. Let us show that the movement of planets and motion in closed Newmann economical systems are generated by flows and Riemannian metrics. Given a function $u(x),\ x \in M$ and a Riemannian metric g on M, let us consider the Hamiltonian $H = \frac{1}{2} g(\dot{x}, \dot{x}) - u(x)$ [Udriste (2004)]. If $u(x) > 0$, then the vector field $X(x) = =\sqrt{2u(x)}\, E(x)$ (Galileiformula), where E is an arbitrary unit vector field with respect to the metric g, satisfies $g(X, X) = 2u(x)$. Consequently such a Hamiltonian, equal to the difference between the kinetic energy and a positive function, is coming from a vector field (flow) and a Riemannian metric, corresponding to a perfect square Lagrangian.

3. MULTI-TIME GEOMETRIC DYNAMICS ON RIEMANN-KAEHLERIAN MANIFOLDS.

We start with an operator $\big((T,h),(M,g),\ X_n\big)$, where:

(i) (T,h) is an oriented Riemannian manifold (source space) of dimension m, with local coordinates $t=(t^\alpha),\ \alpha=1,\dots,m$, metric tensor $h_{\alpha\beta}$ and Christoffel symbols $H^\alpha_{\beta\gamma}$.

(ii) (M,g) is a Riemannian manifold (target space) of dimension n, withlocal coordinates $x=\left(x^i\right),\ i=1,\dots,n$, metric tensor g_{ij} and Christoffel symbols G^i_{jk} .

(iii) $X_\alpha(t,x)=\left(X^i_\alpha(t,x)\right),\alpha=1,\dots,m; i=1,\dots,n$ are C^∞ vector fields on M, dependent on (t,x)which define the first order PDE system:

$$\frac{\partial x}{\partial t^\alpha}(t)=X_\alpha\big(t,x(t)\big). \tag{3.1}$$

Theorem 3.1. The Cauchy problem consisting in the PDE system (3.1) and the initial condition $x(t_0)=x_0$ has a unique solution (existence and uniqueness), if and only if the system is completely integrable. An equality of the type:

$$h^{\alpha\beta}\frac{\partial^2 x}{\partial t^\alpha\partial t^\beta}(t)=F\left(t,x(t),x_\gamma(t)\right),(t,x,x_\gamma)J^1\in(T,M) \tag{3.2}$$

is called a (time dependent) second order elliptical PDE (system) on ***M***.

Proof: Let $\Gamma\!:G(t)=0$ be a hypersurface in T, containing the point t_0 and $\Lambda(t)$ be a unit vector field along Γ, transversal (non-tangent) to Γ . Denote $\varphi_0(t)$ and $\varphi_1(t)$ as vector functions with n componentson Γ, the first being of class C^1 and the second of class C^0. The Cauchy problem attached to PDE (3.2) [Mihlin(1983)] and find in an unilateral neighbourhood of Γ, the solution of the PDE (3.2) satisfying the Cauchy conditions:

$$x(t)|\Gamma=\varphi_0(t),D_\Lambda x(t)|\Gamma=\varphi_1(t). \tag{3.3}$$

Hence the solution of this Cauchy problem exists and it is unique.

We know the Cauchy conditions, one can find the values of all first order partial derivatives of the function $x(t)$ on the Cauchy surface Γ, firstly,

$$\left.\frac{\partial x}{\partial t^\alpha}\right|_\Gamma=\frac{\partial\varphi_0}{\partial t^\alpha}(t),\alpha=1,\dots,m-1$$

and then the equalities:

$$\varphi_1(t)=D_\Lambda x(t)|_\Gamma=\frac{\partial x}{\partial t^\alpha}(t)\Lambda^\alpha(t),$$

Together with $\Lambda^m\neq 0, give$

$$\frac{\partial x}{\partial t^m}(t)|_\Gamma=\frac{1}{\Lambda^m(t)}\Big[\varphi_1(t)-\sum_{\alpha=1}^{m-1}\frac{\partial\varphi_0}{\partial t^\alpha}(t)\,\Lambda^\alpha(t)\Big].$$

The initial conditions (3.3) are equivalent either to the initial conditions:

$x(t)|_\Gamma = \varphi_0(t), \frac{\partial x}{\partial t^m}(t)|_\Gamma = W_m(t)$ and

$$x(t)|_\Gamma = \varphi_0(t), \frac{\partial x}{\partial t^\alpha}(t)|_\Gamma = W_\alpha(t), \alpha = 1, \dots, m.$$

with the complete integrability conditions and the compatibility condition to φ_0.

The multi-time geometric dynamics was introduced in our papers [Udriste (2004), (2012)] like Multi-time World Force Law involving field potentials (components of the d-tensor), gravitational potentials (components of the two Riemannian metrics), and the Yang-Mills potentials (components of the Riemannian connections and the nonlinear connection). This evolution can be called also harmonic maps deformation in a gyroscopic field of forces.

Definition 3.2. Using the vector field X_α, the metric tensor $h_{\alpha\beta}, g_{ij}$, and the Christoffel symbols $H^\alpha_{\beta\gamma}$, G^i_{jk}, we define:

$$F^i_{j\alpha} = \nabla_j X^i_\alpha - g^{ih} g_{kj} \nabla_h X^k_\alpha, \quad f = \frac{1}{2} h^{\alpha\beta} g_{ij} X^i_\alpha X^j_\beta$$

and

$$\nabla_j X^i_\alpha = \frac{\partial X^i_\alpha}{\partial X^j} + G^i_{jk} X^k_\alpha, \quad D_\beta X^i_\alpha = \frac{\partial X^i_\alpha}{\partial X^j} - H^\gamma_{\alpha\beta} X^i_\gamma.$$

The function $F: J^1(T, M) \to R^n$ is said to be generated by the operator (X_α, h, g) if it is of the form:

$$F = h^{\alpha\beta} \left(-G_{jk} x^j_\alpha x^k_\beta + H^\gamma_{\alpha\beta} x_\gamma + F_{j\alpha} x^j_\beta + g_{kj} (\nabla X^k_\alpha) X^j_\beta + D_\beta X_\alpha \right).$$

Theorem 3.2. If $F: J^1(T, M) \to R^n$ is generated by the triplet (X_α, h, g) then the set of maximal solutions ofPDE (3.2) is decomposable into a subset corresponding to the initial values

$$x(t)|_\Gamma = \varphi_0(t), \frac{\partial x}{\partial t^\alpha}(t)|_\Gamma = X_\alpha\big(t, x(t)\big),$$

solutions which are reducible to solutions of PDE (3.1), and a subset of solutions corresponding to the initial values:

$$x(t)|_\Gamma = \varphi_0(t), \frac{\partial x}{\partial t^\alpha}(t)|_\Gamma = W_\alpha(t) \notin K^+\{X_\alpha(t, x(t))\},$$

transversal to the solutions of PDE (3.1). The converse is also true.

Proof. Each solution $x = x(t)$ of any second order prolongation of the first order PDE system has the property: $X_\alpha(t_0) = X_\alpha(t_0, x(t_0))$ implies $X_\alpha(t) = X_\alpha\big(t, x(t)\big), \forall t \in T$.

Any m-flow X_α and two Riemannian metrics h and g determine a least squares Lagrangian density:

$$\mathrm{L}(\mathrm{t}, \mathrm{x}, \mathrm{x}_\gamma) = \frac{1}{2} \mathrm{h}^{\alpha\beta} \mathrm{g}_{\mathrm{ij}} \left(\mathrm{x}^{\mathrm{i}}_\alpha - \mathrm{x}^{\mathrm{i}}_\alpha(\mathrm{t}, \mathrm{x})\right) \left(\mathrm{x}^{\mathrm{j}}_\beta - \mathrm{x}^{\mathrm{j}}_\beta(\mathrm{t}, \mathrm{x})\right).$$

The Euler-Lagrange PDEs represent a prolongation of the m-flow and just a decomposable dynamics.

Again, normal PDE generates in the phase space a multidimensional flow, which together with the phase space geometry gives a geometric dynamic. This statement is true for any PDE, but then appears a multidimensional flow with constraints.

Let us consider the operator (T, h, H), where T is a Kaehlerian manifold, h is a fundamental tensor field and H is a symmetric connection (derivation). We add the operator (M, X_α, g, G), where M is a Kaehlerian manifold, X_α is a m-flow on M, g is a fundamental tensor field and G is a symmetric connection (derivation). The quintuple $(X_\alpha; h, H; g, G)$ generates an extended geometric dynamic on $T \times M$.

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