**SOME PROPERTIES OF MOBIUS TRANSFORMATION**

Anamika Rai1  and S. K. Tiwari2

1Department of Mathematics K. S. Saket P. G. College

Ayodhya – 224123, India

2Department of Mathematics K. S. Saket P. G. College

Ayodhya – 224123, India

E- mail : anamikarai2538@gmail.com

E- mail : sktiwarisaket@yahoo.com

**ABSTRACT**

In conformal mapping , we highlight mostly the topic of Möbius transformation . We identify how different areas and curves are transformed by this transformation. There are some elementary mappings which will be used frequently to explain the various concepts of conformal mapping. Here we describe “ Möbius transformation ’’ and its related properties .

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**1. Introduction**

The function w = f(z) = u + iv, involving complex variables, represents a mathematical rule that associates a complex number w on the w-plane with each complex number z on the z-plane. If a point z\_0 in the z-plane is transformed into a point w\_0 in the w-plane, we refer to w\_0 as the image of z\_0. When the points on the z-plane trace a curve C, the corresponding points on the w-plane trace a curve C^'. This means that for every point (x, y) in the z-plane, the function w = f(z) defines a corresponding point (u, v) in the w-plane. Therefore, the relationship w = f(z) establishes a mapping or transformation from the z-plane into the w-plane.

**2. conformal transformation :** Let two smooth curves curves and

intersecting at point in z- plane and their image curves and

in w- plane under the map w = f(z) intersect at the point

= f(). If the angle between the curves and at is same as

the angle between the image curves and at point both

in magnitude and sense , then the mapping w = f(z) is called

conformal transformation at .

**\* Necessary condition for** **transformation w = f(z) tobe a conformal**

**mapping :**

If a mapping w = f(z) is conformal at a point , then f(z) is analytic at

**\* Sufficient condition for transformation w = f(z) to be a conformal**

**mapping :**

If f(z) is analytic function of z of domain D of z- plane and (z) ≠ 0 inside

D then thetransformation w = f(z) is conformal at all points of D.

**\* Critical Points :**The points at which (z) = = 0 or ꝏ are called critical

points. In other words , we can say that at critical points the conformal

property does not hold good .

**\* Coefficient of Magnification :**Coefficient of magnification for the

conformal mapping w = f(z) at z = α+ iβ is .

**\* Angle of Rotation :**Angle of rotation for the conformal mapping w = f(z)

at z = α + iβ is arg .

**3 Möbius Transformation :**Möbius transformation is composition of the following four types of general transformations :

**3.1 Some General Transformations :**

**Translation :**The map w = z + β corresponds to a translation . By this transformation the figure in w – plane is same as figure in z – plane with different origin .

**Magnification or dilation :** Examine the map w = az, where a represents a real number. A map is said to be magnified if two figures in the z- and w-planes are identical and positioned similarly about their respective origins, but the figure in the w-plane is "a" times the figure in the z-plane.

**Rotation :**By the transformation w = z , ϴ ϵ R figures in z- plane are rotated through an angle ϴ . If ϴ >0 , the rotation is anti clockwise . If ϴ < 0 then rotation is clockwise .

**Inversion :**By means of a transformation w = , figures in z- plane are mapped upon the reciprocal figures in w- plane .

**3.2 Definition of Möbius Transformation :** Möbius transformation is of the form

w = T(z) = (3.1)

where a, b, c, d are complex constants such that ad – bc ≠ 0 . This transformation is called “ Bilinear Transformation ’’ or “ Linear Fractional Transformation’’ . From equation (3.1) , we have

cwz + dw – az – b = 0 .(3.2)

Above equation is linear in z and w so it is called bilinear transformation . Further

from (3.1), we obtained

z = (w) = , (3.3)

where (-d)(-a) – bc = ad – bc ≠ 0 . The transformation (3.3) is the inverse of (3.1).

Thus the inverse of bilinear transformation is also bilinear transformation with the same determinant .

**Note :**All the four general transformations mentioned above are bilinear transformation.

Further the transformations T and can be discussed in extended complex plane . We defined

T(ꝏ) = = =

and it’s inverse is () = ꝏ .

Furthermore, the value of (ꝏ) is givenas

(ꝏ) = (w) = -

and it’s inverse is T(- ) = ꝏ .

Thus, we deduce that the extended complex z-plane is mapped onto the extended complex w-plane in a one-to-one fashion by the transformation w = T(z).

**3.3 Properties of Möbius Transformation**

Möbius transformations have a lot of important properties . These properties are valid forwhole extended complex z- plane .In this section , we discussed the most significant properties of Möbius transformations*.*

**()** The extended z-plane is conformally mapped by bilinear transformations.

***Proof :***A bilinear transformation is given by

w = , ad – bc ≠ 0 .

Then, = = ≠ 0.

So, w(z) is a conformal mapping .

**()** One bilinear transformation plus one bilinear transformation equals another bilinear transformation.

**Proof :** Suppose T and S are two bilinear transformations defined by

T(z) = ; ad – bc ≠ 0 ,

and S(z) = ; - ≠ 0.

Then the productToS is given by

(ToS)(z) = T(S(z)) = = = ,

where A = a +b, B = a+ b , C = c+ d, D = +d .

Also

AD – BC = ad () – bc (-)

= ( ad – bc) ( - )

≠ 0 ( since ad- bc ≠ 0 and - ≠ 0) .

Thus ToS is a bilinear transformation .

**()** Translation, inversion, and dilation combine to produce every bilinear transformation.

**Proof:** Let us consider a bilinear transformation

T(z) = , where a, b, c, d ϵ C and ad- bc ≠ 0 .

**Case -I :**If c = 0 then ad ≠ 0 i.e., a ≠ 0, d ≠ 0 and

T(z) = = z +

= (z) + ( where(z) = z and ≠ 0 i.e., is a dilation )

= ( (z)) ( where(z) = (z) + is a translation ) .

Hence T(z) = o .

**Case – II** c ≠ 0

Now,

T(z) = - +

= + ( here ≠ 0 )

= + ( where(z) = z + , is a translation )

= (z) + ( where(z) = is the inversion )

= ( ( (z)) + ( where(( (z)) = (( (z)) is a dilation )

= (( ( (z))) ( where(z) = (( ( (z)))) + is a translation ).

Therefore T = ooo .

Hence proved .

**()** A bilinear transformation's inverse is likewise a bilinear transformation.

**Proof :**The proof is already done ( see section 3.2) .

**()**The identity mapping w = z is trivially a bilinear transformation .

**()** The associative law for composition of bilinear transformation holds .

**()** A bilinear transformation transfers circles and straight lines into other circles and straight lines because a line is a circle with an infinite radius or a circle that passes through the point of infinity.

**Proof :**` The family of circles and straight lines becomes the family of circles and lines under each of the elementary transformations.

Hence the result follows .

Therefore from the above properties (), (), () and () we can state the following :

**Theorem 3.1** In terms of the composition of bilinear transformations, the set of all bilinear transformations forms a group.

**3.4 Invariant or Fixed Points :**

**Definition :** The point which coinsides with their transformation is called invariant point of the transformation i.e., fixed point of a transformation is obtained by the equation z = f(z) .

**Proposition 1 :**Every bilinear transformation ( except the identity map ) has at most two fixed point .

**Proof :**If T(z) has a fixed point z, then T(z) = z of

= z c+ dz = az + b ⇔ c – (a-d)z – b = 0.

The last equation is quadratic in z and hence can have at most two roots .

For the identity map, I(z) = z ,every point of the domain is a fixed point .

This completes the proof .

**Proposition 2 :**If a bilinear transformation w = f(z) has exactly two fixed points and , then they satisfy the equation

= k . (3.4)

where k is non- zero constant .

Further, if T(z) has only one fixed point , then it can be written as

= + , ≠ 0 . (3.5)

**Proof :**First Part : Let and be the given fixed points of the bilinear transformation w = and these are the roots of the equation

c – (a – d)z – b = 0 .

This means

c – (a – d) – b = 0 ⇔ c – a = b - d , (3.6)

c – (a – d) – b = 0 ⇔ c – a = b - d . (3.7)

Using (3.6), we get

w - = -

=

=

=

= .

Similarly using equation (3.7), we have

w - =

Hence , = . = k

where k = .

**Second Part :** For the second part , is the only fixed point . Then the equation c – (a-d)z – b = 0 has one root , say . So

c – (a- d) – b = 0 ⇔ c - a = b- d

and ( being the repeated root) is given by

= ⇔ a - c = d + c . (3.8)

From previous analysis , we obtained

=

=

= + .

Therefore , = + , = = .

Hence proved .

**Remarks :**

1. The normal form, often called the canonical form, of a bilinear transformation is represented by equations (3.4) and (3.5).
2. 2. A parabolic Möbius transformation has a single fixed point.
3. A Möbius transformation is referred to as loxodromic if it has exactly two fixed points.
   1. **Cross Ratio :**

This section introduces the concept of cross ratio before looking at the specific bilinear transformation that maps three different locations in the extended z-plane onto three different positions in the extended w-plane.

**Definition :**If z, , , are distinct points , then cross ratio of z, , , is denoted by (z, , , ) and defined by

(z, , , ) = . (3.9)

**Theorem 3.2 :**A bilinear transformation preserve cross ratio i.e., if z, , , are transform to w, , , respectively then (z, , , ) =

(w, , ,) .

**Proof :**The bilinear transformation is given by

w = T(z)= , ad – bc = 1 ,

such that , = T(), k= 1, 2, 3 then we have to show that

(w, , ,) =(T(z),T(),T(),T()) =(z, , , ) . (3.10)

Since corresponds to therefore we have

w - = ,

where we have used ad- bc = 1 .

From above equation , we have

w - = , w - = ,

w - = . (3.11)

Replace w by , and z by in equation (3.11) , we get

- = ,

- (3.12)

Equations (3.11) and (3.12) yield

. = .

or (w, ) = (z, ) .

Hence proved the theorem .

**4. Solved Problems :**

**Problem 1 :**Prove that the transformation w = maps the circle +-2x = 0 into the straight line 2u+ 3 = 0.

**Solution :**The given transformation is

w = . (4.1)

From equation (4.1), we have

z = .

therefore = . (4.2)

The given equation of circle is

+ – 2x = 0 ,

i.e., (x +iy) (x – iy) – 2x = 0 ,

or, z – ( z + ) = 0 ( since 2x = z + ) .

With the help of equation (4.2) , above equation can be written as

= – ( + ) = 0 ,

or 5 (w + ) + 15 = 0 ,

or 2u + 3 = 0 ( since w + = 2u ) .

This is theequation of straight line in u – plane .

**Problem 2 :**Find out the fixed point and the normal for the bilinear transformation w = .

**Solution :**For fixed ( invariant) point, put w = z in the given bilinear transformation w = for fixed points , we get = 0.

Thus z = 2 is the only fixed point .

To obtain normal form of the given bilinear transformation , we proceed as follows

w – 2 = – 2 = ,

i.e., = = = 1 + ,

which is the required normal form .

**Problem 3 :**Find out the bilinear transformation that maps the points 1, 2, 0 into points 1, 0, i .

**Solution :**We know that the bilinear transformation thatmaps the points into points respectively is

= .

Substituting the points in above equation , we obtain

= ,

or = ,

or (i – iw) z = 2(i – w) (z - 1)

or w = .

This is the required bilinear transformation .

**Problem 4 :**Find out the bilinear transformation which maps the points i ,ꝏ, 0 into the points 0 , i , ꝏ respectively .

**Solution :**We know the bilinear transformation that maps onto

respectively is given by

= ,

or =

Substituting points in above equation , we have

= ,

or = - )

or wz – iz = - iz– 1

orwz = -1,

or w = - ,

which is the required transformation .

**Problem 5 :**For the conformal mapping w = , show that

**(a)** The coefficient of magnification at z = 3 +i is 2 .

**(b)** The angle of rotation at z = 3 + i is .

**Solution : (a)** For conformal mapping w = f(z) , the coefficient of magnification at z = is .

Here w = f(z) = therefore (z) = 2z.

Hence (3 +i) = 2(3+ i) = 6+ 2i .

Thereforecoefficient of magnification at z = 3+i is

= = = = 2 .

**(b)**For conformal mapping w= f(z),angle of rotation at z = isarg {} .

Here f(z) = , therefore (z) = 2z.

Therefore angle of rotation at z = 3+ i

= arg = arg (6+2i) = = .

**Problem 6:** Find the image of the rectangular region of the z- plane bounded by the lines x= 0, y = 0, x = 2, y =1 under the transformation w = z+(3-i) in the w – plane .

**Solution :**Thegiven transformation is

w = z + (3- i) . (4.3)

Using z= x+ iy and w = u+ iv in above equation , we get

u + iv = x + iy +( 3 – i) = (x+3) + i(y – 1) .

Comparing real and imaginary parts on both sides in above equation, we obtain

u = x +3 and v = y – 1 . (4.4)

Put

(a) x = 0 in (4.4), we get u = 3 ,

(b) y = 0 in (4.4) , we get v = -1 ,

(c) x = 2 in (4.4), we get u = 5,

(d) y = 1 in (4.4) ,we get v = 0 .

Thus , the image of the rectangular region of the z – plane bounded by the lines x = 0 , y = 0, x = 2 and y = 1 under the transformation w = z +( 3- i) is the rectangular region bounded by u = 3 , v = -1, u = 5 and v = 0 in the w – plane .

**Problem 7 :**Find out the image of the rectangular region bounded by the lines x = 0, y= 0, x = 1, y= 2 in z- plane under the transformation w = 2z in the w – plane .

**Solution :**The given transformation is

w = 2z . (4.5)

Using z= x +iy and w = u + iv in above equation (4.5), we have

u + iv = 2 (x + iy) ,

or u + iv = 2x + 2iy .

Comparing real and imaginary parts on both sides in above equation , we get

u = 2x and v = 2y .

Put

(a) x = 0 in , we get u = 0,

(b) y = 0, we get v = 0 ,

(c) x = 2 , we get u = 4 ,

(d) y = 3 in , we get v = 6 .

Hence, the image of the rectangular region of the z – plane bounded by the lines x = 0, y = 0, x = 2, y = 3 under the transformation w = 2z is the rectangular region bounded by u = 0, v = 0, u =4, v = 6in w – plane .

**Problem 8 :** What is the image of triangular region of z – plane bounded by the lines x = 0, y = 0, x + y = 1 under the transformation w = z in the w- plane .

**Solution :**Thegiven transformation is

w = z . (4.6)

Substituting z = x +iy and w = u + iv in above equation (4.6) , we get

u + iv = ( + i ) ( x +iy) , ( since = + i ) ,

or u + iv = ( x + iy) ,

or 2(u +iv) = x + iy + i x - y ,

or 2u + 2iv = ( x - y) + i ( x + y) .

Comparing real and imaginary parts on both sides in above equation, we have

2u = x - y , (4.7)

2v = x + y . (4.8)

Multiplying both sides of equation (4.8) by , we have

2 v = 3x + y . (4.9)

Adding equations (4.7) and (4.9) , we obtain

2u + 2 v = x + 3x = 4x ,

or u + v = 2x .(4.10)

Further multiplying both sides of equation (4.7) by , we get

2 u = x – 3y . (4.11)

Again subtracting equation (4.8) from equation (4.11), we have

2 u – 2v = -3y – y ,

or 2( u – v) = - 4y ,

or u – v = - 2y ,

or v - u = 2y . (4.12)

Put

(a) x = 0 in (4.10) ,we get v = - u ,

(b) y = 0 in (4.12) , we get v = u,

Further from equation (4.8), we have

2v = 1 ( since x + y = 1),

or v = .

Hence, the image of triangular region of z- plane bounded by the lines x=0, y= 0, x + y = 1 under the transformation w = z is the triangular region bounded by v = - u , v = u and v = in w – plane .

**Problem 9 :**What is the the image of the line y – x + 1 = 0 in z- plane under the transformation w = in w – plane .

**Solution :**Thegiven transformation is

w = . (4.13)

Here, z = x +iy and w = u +iv .

Putting above in equation (4.13), we get

u + iv = ,

or x + iy = . = ,

or x + iy = - i .

Comparing real and imaginary parts on both sides in above equation , we have

x = , (4.14)

and y = . (4.15)

With the help of equations (4.14) and (4.15) , equation y – x + 1 = 0 can be written as

- + 1 = 0 ,

or - v – u + + = 0 ,

or – u ++ – v + = 0 ,

or + = + ,

or + = .

This is equation of the circle with centre ( ) and radius .

Thus the image of the line y – x + 1 = 0 in z- plane under the transformation w = is a circle in w – plane .

**Problem 10 :**Show that the transformation w = maps the circle in z – plane to a circle in w – plane or a straight line (if the former passes through the origin) .

**Solution :**The given transformation is

w = , (4.16)

Here, w = u + iv and z = x +iy .

Putting above in equation (4.16), we get

u + iv = ,

or x + iy = ,

or x + iy = . = ,

or x + iy = - i .

Comparing real and imaginary parts on both sides , we have

x = , (4.17)

and y = . (4.18)

Equation of circle in z- plane is

+ + 2gx + 2fy + c = 0 . (4.19)

Using equations (4.17) and (4.18) in above , we obtain

+ + + + c = 0 ,

or + + c = 0 ,

or 1 + 2gu – 2fv + c (+ ) = 0 ,

or c(+ ) + 2gu – 2fv + 1 = 0 . (4.20)

Case(i) If c 0 , equation (4.20) is a equation of circle .

Case(ii) If c = 0, 2gu – 2fv + 1 = 0 ,

is equation of a straight line .

**Problem 11 :**Show that the transformation w = maps a line in z – plane to a circle or staright line in w – plane .

**Solution :**Given transformation is

w = . (4.21)

Putting w = u +iv and z = x + iy inabove equation (4.21) , we obtain

u + iv = ,

or x + iy = ,

or x + iy = . = ,

or x + iy = - i .

Comparing real and imaginary parts on both sides , we have

x = , y = . (4.22)

The equation of line in z – plane is

ax + by +c = 0 . (4.23)

Using equation (4.22) in above equation , we have

- + c = 0 ,

Or au – bv + c ( +) = 0 ,

or c( +) + au – bv = 0 . (4.24)

**Case (i)** If c ≠ 0 i.e. , line in z- plane does not pass through origin then equation (4.24) becomes a circle in w – plane .

**Case (ii)** If c = 0 i.e., line in z – plane passes through origin then equation (4.24) becomes a straight line .

**5 .Conclusions :**

In our book chapter, we basically highlight about the topic “ Möbius transformation ’’ and see how it transforms different curves and regions from one complex plane to the other complex plane . We have also discussed some of its remarkable properties like conformality ( angle preserving property), circle preserving property and what a Möbius transformation is composed of .

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