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ABSTRACT

In conformal mapping , we highlight mostly the topic of Möbius transformation and see how various regions and curves are transformed by this transformation. There are some elementary mappings which will be used frequently to explain the various concepts of conformal mapping. Here we describe “ Möbius transformation ’’ and its related properties .

CONTENTS

1. Introduction

The function w = f(z) = u + iv defines a correspondence between points of z- plane and w – plane . If the point z describes some curve C in the z – plane , then point w will move along a corresponding curve C' in the w – plane because to each (x, y) there corresponds a point (u, v) . Hence the function w = f (z) defines a mapping or transformation of z- plane into w-plane .

2. Conformal mapping or conformal transformation

Suppose two curves C_1, C_2 in the z- plane intersect at the point P and the corresponding curves C_1', C_2' in the w- plane intersect at P' under the transformation w = f(z).

If the angle of intersection of the curves at P is same as the angle of intersection of the curves at P', both in magnitude and sense, then the transformation is said to be conformal at P .

* **Necessary condition for w = f(z) to represents a conformal mapping :**

  If w = f(z) to represents a conformal mapping of a domain D in the z- plane into a domain D' of w – plane , then f(z) is an analytic function of z in D.

* **Sufficient condition for w = f(z) to represents a conformal mapping :**

  If f(z) is analytic and $f'(z) \neq 0$ in a region R of z- plane then the mapping w = f(z) is conformal at all points of R.

* **Critical Points :** The points at which $f'(z) = \frac{dw}{dz} = 0$ or ∞ are called critical points.

* **Coefficient of Magnification :** Coefficient of magnification for the conformal mapping w = f(z) at z = α + iβ is $|f'(\alpha + i\beta)|$.

* **Angle of Rotation :** Angle of rotation for the conformal mapping w = f(z) at z = α + i β is arg $[f'(\alpha + i\beta)]$.

3 Möbius Transformation : What is Möbius transformation ? They are simply a composition of one , some or all of the following special types of transformations .

Translation : It is a map of the form T(z) = z + α , α ∊ C/ {0} . If α = 0, then it is identity map.

Magnification : It is a map of the form T(z) = r z , r ∊ R/ {0} . Notice that for r=1, this is the identity map whereas for r = 0 it is a constant map . If r > 0 , then this is a " Magnification '' and if 0 < r < 1 , it is a " Shrinking '' depending on r< 1 or -1< r <0 .

Rotation : It is a map of the form T(z) = $e^{i\Theta}$z , Θ ∊ R . This map produces a rotation through an angle about the origin with positive sense if Θ > 0. The rotation coupled with magnification is referred to as " dilation '' T(z) = a z (a ≠0).

Inversion : It is a map of the form T(z) = $\frac{1}{z}$ which produces a geometric inversion (or reciprocal map or the inversion map) .

3.2 Definition of Möbius Transformation : Möbius transformation named in honour of the geometer A. F. Möbius (1790 – 1868) are rational function of the form

$$w = T(z) = \left(\frac{az+b}{cz+d}\right) \tag{3.1}$$

where a, b, c, d are complex constants such that ad – bc ≠ 0 . This map is called " Bilinear Transformation '' or sometimes " Linear Fractional Transformation'' . The relation (3.1) can also be written as

$$cwz + dw - az - b = 0 , \tag{3.2}$$

which is linear in w as well as in z : that is why the relation (3.1) is called bilinear transformation . Solving (3.1) for z , we get

$$z = T'(w) = \frac{-dw+b}{cw-a} , \tag{3.3}$$

where (-d)(-a) – bc = ad – bc ≠ 0 . The transformation (3.3) is the inverse of (3.1).

It follows that the inverse of a bilinear transformation is another bilinear transformation having the same determinant .

Note : All the elementary transformations discussed in (3.1) are bilinear transformation.

We can extend T and T' to mapping in the extended complex plane . The value T(ꝏ) should equal to the limit of T(z) as z → ꝏ . Therefore , we defined

$$T(\infty) = \lim_{z\to\infty} T(z) = \lim_{z\to\infty} \frac{a+\frac{b}{z}}{c+\frac{d}{z}} = \frac{a}{c}$$

and the inverse is $T^{-1}(\frac{a}{c}) = \infty$.

Similarly, the value of $T^{-1}(\infty)$ is obtained by

$$T^{-1}(\infty) = \lim_{w\to\infty} T^{-1}(w) = -\frac{d}{c}$$

and the inverse is $T(-\frac{d}{c}) = \infty$.

With these extension we conclude that transformation w = T(z) is a one to one mapping of the extended complex w – plane .

3.3 Properties of Möbius Transformation

Möbius transformations have a number of remarkable properties . Furthermore , these properties are global . Namely , they are valid for any z including z = ꝏ, i.e., valid in the entire extended complex z plane . Through this section , we *desire the most important properties of Möbius transformations.*

(P_1) *Bilinear transformations are conformal mapping of the extended z- plane.*

Proof : Let $w = \frac{az+b}{cz+d}$, ad – bc ≠ 0 be a bilinear transformation. Then,

$$\frac{dw}{dz} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad - bc}{(cz+d)^2} \neq 0.$$

So, w(z) is a conformal mapping .

(P_2) The composition of two bilinear transformation is again a bilinear transformation .

Proof : T and S are two bilinear transformations given by

$$T(z) = \frac{az+b}{cz+d}\ ;\ ad - bc \neq 0\ ,$$

and $$S(z) = \frac{a'z+b'}{c'z+d'}\ ;\ a'd' - b'c' \neq 0.$$

Then the composition ToS is defined by

$$(ToS)(z) = T(S(z)) = \frac{a\left(\frac{a'z+b'}{c'z+d'}\right)+b}{c\left(\frac{a'z+b'}{c'z+d'}\right)+d} = \frac{(aa'+bc')z+ab'+bd'}{(ca'+dc')z+cb'+dd'} = \frac{Az+B}{Cz+D},$$

where $A = aa' + bc'$, $B = ab' + bd'$, $C = ca' + dc'$, $D = cb' + dd'$.

Also

$$AD - BC = ad\,(a'd' - b'd') - bc\,(a'd' - b'd')$$
$$= (ad - bc)(a'd' - b'c')$$
$$\neq 0 \text{ (since } ad - bc \neq 0 \text{ and } a'd' - b'c' \neq 0) .$$

Thus ToS is a bilinear transformation .

(P_3) Every bilinear transformation is a composition of translation , inversion , and dilation.

Proof: Let us consider a bilinear transformation

$$T(z) = \frac{az+b}{cz+d}, \text{ where } a, b, c, d \in C \text{ and } ad - bc \neq 0 .$$

Case -I : If $c = 0$ then $ad \neq 0$ i.e., $a \neq 0$, $d \neq 0$ and

$$T(z) = \frac{az+b}{d} = \frac{a}{d}z + \frac{b}{d}$$

$$= T_1(z) + \frac{b}{d} \text{ (where } T_1(z) = \frac{a}{d}z \text{ and } \frac{a}{d} \neq 0 \text{ i.e., } T_1 \text{ is a dilation)}$$

$$= T_2(T_1(z)) \text{ (where } T_2(z) = z + \frac{b}{d} \text{ is a translation) .}$$

Hence $T(z) = T_2 o T_1$.

Case – II $c \neq 0$

Now,

$$T(z) = \frac{az+b}{cz+d} - \frac{a}{c} + \frac{a}{c}$$

$$= \frac{bc-ad}{c^2}\frac{1}{z+\frac{d}{c}} + \frac{a}{c} \text{ (here } \frac{bc-ad}{c^2} \neq 0 \text{)}$$

$$= \frac{bc-ad}{c^2}\frac{1}{T_1(z)} + \frac{a}{c} \text{ (where } T_1(z) = z + \frac{d}{c}, \text{ is a translation)}$$

$$= \frac{bc-ad}{c^2}T_2(z) + \frac{a}{c} \text{ (where } T_2(z) = \frac{1}{z} \text{ is the inversion)}$$

$$= T_3(T_2(T_1(z)) + \frac{a}{c} \text{ (where } T_3(z) = \frac{bc-ad}{c^2}z \text{ is a dilation)}$$

= $T_4(T_3(T_2(T_1(z)))$ (where $T_4(z) = z + \frac{a}{c}$ is a translation).

Therefore T = $T_4 o T_3 o T_2 o T_1$.

Hence proved .

(P_4) The inverse of a bilinear transformation is also a bilinear transformation.

Proof : The proof is already done (see section 3.2) .

(P_5) The identity mapping w = z is trivially a bilinear transformation .

(P_6) The associative law for composition of bilinear transformation holds .

(P_7) Every bilinear transformation maps circles and straight lines into circles and lines (a line is a circle with infinite radius i.e., line is a circle through the point of infinity).

Proof : `Under each of the elementary transformations the family of circles and straight lines are transformed into the family of circles and lines .

Hence the result follows .

Therefore from the above properties (P_2), (P_4), (P_5) and (P_6) we can state the following :

Theorem 3.1 The set of all bilinear transformations form a group with respect to the composition of bilinear transformations .

3.4 Invariants/ Fixed Points :

Definition : A bilinear transformation T(z) has a fixed point (invariant point) z_0 if T(z_0) = z_0 .

Proposition 1 : Every bilinear transformation (except the identity map) has at most two fixed point .

Proof : If T(z) has a fixed point z, then T(z) = z of

$\frac{az+b}{cz+d}$ = z $\Leftrightarrow$ cz^2+ dz = az + b $\Leftrightarrow$ cz^2 – (a-d)z – b = 0.

The last equation is quadratic in z and hence can have at most two roots .

For the identity map, I(z) = z for all points in the domain of definition . Hence every point of the domain is a fixed point .

This completes the proof .

Proposition 2 : If a bilinear transformation w = f(z) has exactly two fixed points z_1 and z_2 , then for some non- zero constant k they satisfy the equation

$$\frac{w-z_1}{w-z_2} = k\,\frac{z-z_1}{z-z_2}\,. \tag{3.4}$$

Moreover, if T(z) has only one fixed point z_1 , then it can be written as

$$\frac{1}{w-z_1} = k' + \frac{1}{z-z_1}\,,\ k' \neq 0\,. \tag{3.5}$$

Proof : First Part : Let z_1 and z_2 be the given fixed points of the bilinear transformation $w = \frac{az+b}{cz+d}$ and these are the roots of the equation

$cz^2 - (a-d)z - b = 0$.

This means

$$cz_1^2 - (a-d)z_1 - b = 0 \Leftrightarrow cz_1^2 - a\,z_1 = b - dz_1\,, \tag{3.6}$$

$$cz_2^2 - (a-d)z_2 - b = 0 \Leftrightarrow cz_2^2 - a\,z_2 = b - dz_2\,. \tag{3.7}$$

Utilizing (3.6), we get

$$\begin{aligned} w - z_1 &= \frac{az+b}{cz+d} - z_1 \\ &= \frac{az + b - z_1(cz+d)}{cz+d} \\ &= \frac{(a - z_1c)z + b - dz_1}{cz+d} \\ &= \frac{(a - z_1c)z + cz_1^2 - a\,z_1}{cz+d} \\ &= \frac{(a-z_1c)\,(z-z_1)}{cz+d}\,. \end{aligned}$$

Similarly using equation (3.7), we have

$$w - z_2 = \frac{(a-cz_2)\,(z-z_2)}{cz+d}$$

Hence , $$\frac{w-z_1}{w-z_2} = \frac{a-cz_1}{a-cz_2}\cdot\frac{z-z_1}{z-z_2} = k\,\frac{z-z_1}{z-z_2}$$

where $$k = \frac{a-cz_1}{a-cz_2}.$$

Second Part : For the second part , z_1 is the only fixed point . Then the equation $cz^2 - (a\text{-}d)z - b = 0$ has one root z_1 , say . So

$$cz_1^2 - (a\text{-}d)z_1 - b = 0 \Leftrightarrow cz_1^2 - az_1 = b - dz_1$$

and z_1 (being the repeated root) is given by

$$z_1 = \frac{a-d}{2c} \Leftrightarrow a - cz_1 = d + cz_1 . \tag{3.8}$$

From previous analysis , we obtained

$$\frac{1}{w-z_1} = \frac{cz + a - cz_1 - cz_1}{(a-cz_1)(z-z_1)}$$

$$= \frac{c(z-z_1) + a - cz_1}{(a-cz_1)(z-z_1)}$$

$$= \frac{c}{a-cz_1} + \frac{1}{z-z_1} .$$

Therefore , $\frac{1}{w-z_1} = k' + \frac{1}{z-z_1}$, $k' = \frac{c}{a-cz_1} = \frac{2c}{a+d}$.

Hence proved .

Remarks :

1. Equations (3.4) and (3.5) are known as the normal form or canonical form of a bilinear transformation .
2. A Möbius transformation which has a unique fixed point is parabolic .
3. If a Möbius transformation has exactly two fixed points , then it is called loxodromic .

3.5 Cross Ratio :

In this section , we develop the specific bilinear transformation which maps three distinct points in the extended z- plane onto three distinct points in the extended w- plane. For this purpose we introduce the concept of cross ratio .

Definition : For three distinct complex numbers z_1, z_2, z_3 in c_∞ , the cross ratio of four points z, z_1 , z_2 , z_3 is defined to be

$$(z, z_1, z_2, z_3) = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} . \tag{3.9}$$

If one of the numbers in (3.9) is replaced by infinity, say z_3 , then

$$(z, z_1, z_2, \infty) = \lim_{z_3 \to \infty} \frac{(z-z_1)(\frac{z_2}{z_1} - 1)}{(z_1-z_2)(1-\frac{z}{z_3})} = \frac{z-z_1}{z_1-z_2} .$$

This means that the factors involving z_3 are replaced by 1. To recollect cross ratio (3.9), we write differences of z, z_1, z_2 , z_3 in cyclic order z- z_1,

z_1- z_2 , z_2 - z_3, z_3 – z and put them in the numerator and the denominator alternatively .

Theorem 3.2 : The cross ratio is invariant under bilinear transformation .

Proof : Let the bilinear transformation be defined by

$$w = T(z)= \frac{az+b}{cz+d}, \; ad - bc = 1 ,$$

such that , $w_k = T(z_k)$, k= 1, 2, 3 then we have to show that

$$(T(z), T(z_1), T(z_2), T(z_3)) = (z, z_1, z_2, z_3) . \qquad (3.10)$$

Since z_k corresponds to w_k ; assuming ad – b c = 1 , we have

$$w - w_k = \frac{z - z_k}{(cz_k + d)\,(cz + d)},$$

so that for any pair of (z, w)

$$w - w_1 = \frac{z - z_1}{(cz_1 + d)\,(cz + d)},\; w - w_2 = \frac{z - z_2}{(cz_2 + d)\,(cz + d)},$$

$$w - w_3 = \frac{z - z_3}{(cz_3 + d)(cz + d)} . \qquad (3.11)$$

Replace w by w_2 , and z by z_2 , we get

$$w_2 - w_1 = \frac{z_2 - z_1}{(cz_1 + d)(cz_2 + d)},$$

$$w_2 - w_3 = \frac{z_2 - z_3}{(cz_3 + d)(cz_2 + d)}. \qquad (3.12)$$

Equations (3.11) and (3.12) yield

$$\frac{w - w_1}{w_1 - w_2} \cdot \frac{w_2 - w_3}{w_3 - w} = \frac{z - z_1}{z_1 - z_2} \cdot \frac{z_2 - z_3}{z_3 - z}$$

and this is nothing but

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3) .$$

Thus, equation (3.10) established .

4. Solved Problems :

Problem 1 Show that the transformation $w = \frac{2z+3}{z-4}$ changes the circle $x^2 + y^2 - 4x = 0$ into the straight line $4u + 3 = 0$.

Solution : The given transformation is

$$w = \frac{2z+3}{z-4} . \qquad (4.1)$$

Solving for z, we have

$$z = \frac{4w+3}{w-2} .$$

therefore $$\bar{z} = \frac{4\bar{w}+3}{\bar{w}-2} . \qquad (4.2)$$

The given equation of circle is

$$x^2 + y^2 - 4x = 0 ,$$

i.e., $$(x+iy)(x-iy) - 4x = 0 ,$$

or, $$z\bar{z} - 2(z + \bar{z}) = 0 \quad (\text{ since } 2x = z + \bar{z}) .$$

With the help of equation (4.2) , above equation can be written as

$$\frac{4w+3}{w-2} = \frac{4\bar{w}+3}{\bar{w}-2} - 2\left(\frac{4w+3}{w-2} + \frac{4\bar{w}+3}{\bar{w}-2}\right) = 0 ,$$

or $$2(w + \bar{w}) + 3 = 0 ,$$

or $$4u + 3 = 0 \quad (\text{ since } w + \bar{w} = 2u) ,$$

which is a straight line in u – plane .

Problem 2 : Find the fixed point and the normal for the bilinear transformation $w = \frac{3z-4}{z-1}$.

Solution : Putting w = z in $w = \frac{3z-4}{z-1}$ for fixed points , we get $(z-2)^2 = 0$.

Thus z = 2 is the only fixed point so that transformation is parabolic .

For normal form of the given bilinear transformation , we proceed as follows

$$w - 2 = \frac{3z-4}{z-1} - 2 = \frac{z-2}{z-1} ,$$

i.e., $$\frac{1}{w-2} = \frac{z-1}{z-2} = \frac{z-2+1}{z-2} = 1 + \frac{1}{z-2} ,$$

which is the required normal form .

Problem 3 : Find the bilinear transformation which transforms the points z= 2, 1, 0 into w = 1, 0, i .

Solution : We know that the bilinear transformation which transforms z = z_1, z_2, z_3 respectively into w = w_1, w_2, w_3 is

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)} .$$

Substituting the points in this equation , we get

$$\frac{(w-1)\,(0-i)}{(1-0)\,(i-w)} = \frac{(\,z-2)\,(\,1-0)}{(2-1)\,(0-2)} ,$$

or $$\frac{i - iw}{i - w} = \frac{2 - z}{z} ,$$

or $$(i - iw)\, z = (i - w)\,(2 - z)$$

or $$w = \frac{2iz - 2i}{(1 - i)z - 2} .$$

This is the required transformation .

Problem 4 : Find the bilinear transformation which maps the points z = ∞, i , 0 into the points 0 , i , ∞ respectively .

Solution : We know the bilinear transformation, mapping z = z_1, z_2, z_3 onto

w = w_1, w_2, w_3 respectively is given by

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)} .$$

Substituting points in above equation , we have

$$\frac{(w - 0)(i - w_3)}{(0 - i)(w_3 - w)} = \frac{(z - z_1)(i - 0)}{(\,z_1 - i)\,(0 - z)} ,$$

where $z_1 \to \infty$, $w_3 \to \infty$.

Since , $z_1 = \infty$ and $w_3 = \infty$, we take the quotient involving z_1 and w_3 in above equation to be (-1) .

Thus $$\frac{w - 0}{0 - i} = \frac{(\,i - 0)}{(\,0 - z)} ,$$

or $$\frac{w}{-i} = \frac{i}{-z} ,$$

or $$w = -\frac{1}{z} .$$

This is the required transformation .

Problem 5 : For the conformal mapping w = z^2, show that

(a) The coefficient of magnification at z = 2 +I is $2\sqrt{5}$.

(b) The angle of rotation at z = 2 + i is $\tan^{-1} 0.5$.

Solution : Let w = f(z) = z^2

therefore f'(z) = 2z.

Hence f'(2 +i) = 2(2 + i) = 4 + 2i .

(a) Coefficient of magnification at z = 2+i is

$$= \left|f''(2+i)\right| = |4+2i| = \sqrt{4^2+2^2} = 2\sqrt{5}\ .$$

(b) Angle of rotation at z = 2+ i is

arg $[f'(2+i)]$ = arg (4+2i) = $\tan^{-1}\frac{2}{4} = \tan^{-1} 0.5.$

Problem 6: What is the image of the rectangular region of the z- plane bounded by the lines x= 0, y = 0, x = 1, y =2 under the transformation w = z+(2+i) in the w – plane .

Solution : Given transformation is

$$w = z + (2- i) \quad . \tag{4.3}$$

Here, z = x +iy and w = u +iv .

Using above in equation (4.3) , we have

$$u + iv = x + iy + (2 – i) = (x+ 2) + i(y -1) .$$

Comparing real and imaginary parts on both sides , we obtain

$$u = x +2 \text{ and } v = y – 1 . \tag{4.4}$$

(a) x = 0, u = 2 ,

(b) y = 0 , v = -1 ,

(c) x = 1, u = 3,

(d) y = 2 , v = 1 .

Thus , the image of the rectangular region in the z – plane bounded by the lines x = 0 , y = 0, x = 1 and y = 2 under the transformation w = z +(2- i) is the rectangular region bounded by u = 2 , v = -1, u = 3 and v = 1 in the w – plane .

Problem 7 : What is the image of the rectangular region of the z- plane bounded by the lines x = 0, y= 0, x = 1, y= 2 under the transformation w = 2z in the w – plane .

Solution : The given transformation is

$$w = 2z . \tag{4.5}$$

Here, z = x +iy and w = u +iv .

Putting above in equation (4.5), we obtained

$$u + iv = 2\ (x + iy)\ ,$$

or $$u + iv = 2x + 2iy .$$

Comparing real and imaginary parts on both sides , we get

u = 2x and v = 2y .

(a) x = 0, u = 0,

(b) y = 0, v = 0 ‘

(c) x = 1 , u = 2 ,

(d) y = 2 , v = 4 .

Hence, the image of the rectangular region of the z – plane bounded by the lines x = 0, y = 0, x = 1, y = 2 under the transformation w = 2z is the rectangular region bounded by u = 0, v = 0, u =2, v = 4 in the w – plane .

Problem 8 : What is the image of triangular region of the z – plane bounded by the lines x = 0, y = 0, $\sqrt{3}$ x + y = 1 under the transformation w = $e^{i\pi/3}$z in the z- plane .

Solution : Given transformation is

$$w = e^{i\pi/3} z . \tag{4.6}$$

Here, z = x +iy and w = u + iv .

Substituting above in equation (4.6) , we get

$$u + iv = \left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3} \right) (x + iy) \ , \ (\text{ since } e^{i\theta} = \cos\theta + i\sin\theta\ ) ,$$

or $$u + iv = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) (x + iy) ,$$

or $\quad u + iv = \frac{(1 + i\sqrt{3})}{2}(x + iy)$,

or $\quad 2(u + iv) = x + iy + i\sqrt{3}\,x - \sqrt{3}\,y$,

or $\quad 2(u + iv) = (x - \sqrt{3}\,y) + i(\sqrt{3}\,x + y)$.

Comparing real and imaginary parts on both sides, we have

$$2u = x - \sqrt{3}\,y , \tag{4.7}$$

$$2v = \sqrt{3}\,x + y . \tag{4.8}$$

Multiplying both sides of equation (4.8) by $\sqrt{3}$, we have

$$2\sqrt{3}\,v = 3x + \sqrt{3}\,y . \tag{4.9}$$

Adding equations (4.7) and (4.9) , we obtain

$$2u + 2\sqrt{3}\,v = x + 3x = 4x ,$$

or $$u + \sqrt{3}\,v = 2x . \tag{4.10}$$

Further multiplying both sides of equation (4.7) by $\sqrt{3}$, we get

$$2\sqrt{3}\,u = \sqrt{3}\,x - 3y . \tag{4.11}$$

Again subtracting equation (4.8) from equation (4.11), we have

$$2\sqrt{3}\,u - 2v = -3y - y ,$$

or $\quad 2(\sqrt{3}\,u - v) = -4y$,

or $\quad \sqrt{3}\,u - v = -2y$,

or $$v - \sqrt{3}\,u = 2y . \tag{4.12}$$

(a) $\quad x = 0,\ v = -\frac{1}{\sqrt{3}}u$,

(b) $\quad y = 0 ,\ v = \sqrt{3}\,u$,

(c) $\quad \sqrt{3}\,x + y = 1 ,\ v = \frac{1}{2}$.

Hence, the image of triangular region of the z- plane bounded by the lines x=0, y= 0, $\sqrt{3}\,x + y = 1$ under the transformation $w = e^{i\pi/3}z$ is the triangular region bounded by $v = -\frac{1}{\sqrt{3}}u$, $v = \sqrt{3}\,u$ and $v = \frac{1}{2}$ in w – plane .

Problem 9 : Find the image of the line $y - x + 1 = 0$ under the transformation $w = \frac{1}{z}$ in w – plane .

Solution : Given transformation is

$$w = \frac{1}{z} \ . \tag{4.13}$$

Here, $z = x + iy$ and $w = u + iv$.

Putting above in equation (4.13), we get

$$u + iv = \frac{1}{x+iy} ,$$

or $$x + iy = \frac{1}{u+iv} \cdot \frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2} ,$$

or $$x + iy = \frac{u}{u^2+v^2} - i\,\frac{v}{u^2+v^2} \ .$$

Comparing real and imaginary parts on both sides , we have

$$x = \frac{u}{u^2+v^2} , \tag{4.14}$$

and $$y = -\frac{v}{u^2+v^2} . \tag{4.15}$$

With the help of equations (4.14) and (4.15) , equation $y - x + 1 = 0$ can be written as

$$-\frac{v}{u^2+v^2} - \frac{u}{u^2+v^2} + 1 = 0 ,$$

or $$-v - u + u^2 + v^2 = 0 ,$$

or $$u^2 - u + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + v^2 - v + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = 0 ,$$

or $$\left(u - \frac{1}{2}\right)^2 + \left(v - \frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} ,$$

or $$\left(u - \frac{1}{2}\right)^2 + \left(v - \frac{1}{2}\right)^2 = \frac{1}{2} .$$

This is equation of the circle with centre $(\frac{1}{2}, \frac{1}{2})$ and radius $\frac{1}{\sqrt{2}}$.

Thus the image of the line $y - x + 1 = 0$ under the transformation $w = \frac{1}{z}$ is a circle in w – plane .

Problem 10 : Show that the transformation $w = \frac{1}{z}$ maps the circle in z – plane to a circle in w – plane or a straight line if the former passes through the origin .

Solution : The given transformation is

$$w = \frac{1}{z}, \tag{4.16}$$

Here, w = u + iv and z = x +iy .

Putting above in equation (4.16), we get

$$u + iv = \frac{1}{x+iy},$$

or $$x + iy = \frac{1}{u+iv},$$

or $$x + iy = \frac{1}{u+iv} \cdot \frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2},$$

or $$x + iy = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}.$$

Comparing real and imaginary parts on both sides , we have

$$x = \frac{u}{u^2+v^2}, \tag{4.17}$$

and $$y = -\frac{v}{u^2+v^2}. \tag{4.18}$$

Equation of circle in z- plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 . \tag{4.19}$$

Using equations (4.17) and (4.18) in above , we obtain

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{2gu}{u^2+v^2} + \frac{2f(-v)}{u^2+v^2} + c = 0,$$

or $$\frac{1}{u^2+v^2} + \frac{2gu-2fv}{u^2+v^2} + c = 0,$$

or $$1 + 2gu - 2fv + c(u^2+v^2) = 0,$$

or $$c(u^2+v^2) + 2gu - 2fu + 1 = 0 . \tag{4.20}$$

Case(i) If $c \neq 0$, equation (4.20) is a equation of circle .

Case(ii) If c = 0, 2gu – 2fv + 1 = 0 ,

is equation of a straight line .

Problem 11 : Show that the transformation $w = \frac{1}{z}$ maps a line in z – plane to a circle or starigh line in w – plane .

Solution : Given transformation is

$$w = \frac{1}{z} . \tag{4.21}$$

Here, w = u +iv and z = x + iy .

Putting above in equation (4.21) , we obtain

$$u + iv = \frac{1}{x + iy} ,$$

or $$x + iy = \frac{1}{u + iv} ,$$

or $$x + iy = \frac{1}{u + iv} . \frac{u - iv}{u - iv} = \frac{u - iv}{u^2 + v^2} ,$$

or $$x + iy = \frac{u}{u^2 + v^2} - i \frac{v}{u^2 + v^2} .$$

Comparing real and imaginary parts on both sides , we have

$$x = \frac{u}{u^2 + v^2} , \quad y = - \frac{v}{u^2 + v^2} . \tag{4.22}$$

The equation of line in z – plane is

$$ax + by + c = 0 . \tag{4.23}$$

Using equation (4.22) in above equation , we have

$$\frac{au}{u^2+v^2} - \frac{bv}{u^2+v^2} + c = 0 ,$$

Or $$au - bv + c (u^2 + v^2) = 0 ,$$

or $$c(u^2 + v^2) + au - bv = 0 . \tag{4.24}$$

Case (i) If c ≠ 0 i.e. , line in z- plane does not pass through origin then equation (4.24) becomes a circle in w – plane .

Case (ii) If c = 0 i.e., line in z – plane passes through origin then equation (4.24) becomes a straight line .

5 . Conclusions :

In our book chapter, we basically highlight about the topic “ Möbius transformation ” and see how it transforms different curves and regions from one complex plane to the other complex plane . We have also discussed some of its remarkable properties like conformality (angle preserving property), circle preserving property and what a Möbius transformation is composed of .

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