**Radial Radio Number and some other labeling parameters**

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**ABSTRACT**

Let be a simple, connected and undirected graph. A radial radio labeling of G is an assignment of positive integers to the vertices satisfying the condition that for any two distinct vertices, where and denote the distance between the vertices u and v and the radius of the graph G, respectively. The span of a radial radio labeling is the largest integer in the range of *f* and is denoted by *span f*. The radial radio number of *G*, , is the minimum span taken over all radial radio labelings of *G*. In this paper, we determine the relationship between the radial radio number, the radio number, and the labeling number. In addition, we construct graphs for which the radio number is the algebraic sum of radial radio number and given nonnegative integer. Similarly, we prove the existence of graphs for which the L(2,1) – labeling number is the algebraic sum of radial radio number and given nonnegative integer.

**Keywords –** radial radio number, radio number, labeling number.

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**I. INTRODUCTION**

In this paper, we only consider a simple, connected, finite and undirected graph. For a vertex *v* in a connected graph *G*, the **eccentricity**, *e*(*v*), of *v* is the distance between *v* and a vertex farthest from *v* in *G*. The minimum eccentricity among the vertices of *G* is called the **radius** of *G* and is denoted by *rad*(*G*) or *r*(*G*) or *r*. The maximum eccentricity among the vertices of *G* is called the **diameter** of *G* and is denoted by *diam*(*G*) or *d*(*G*) or *d*. The **centre** is the subgraph of G, induced by the set of vertices of minimum eccentricity. Any graph which is isomorphic to its centre is called a **self-centered graph**. Note that, in a self-centered graph G, *rad*(*G*)*=diam*(*G*)*.* For further details, one can refer [3].

Graph labeling is an assignment of nonnegative integers, sometimes called colors, to the vertices or edges or both. Motivated by the Frequency Assignment Problem, many mathematicians introduced various graph labeling concepts. Some of them are discussed in this paper, namely, L(2,1) – labeling[6], radio labeling[4] and radial radio labeling[8].

The **L(2,1) – labeling** was introduced by Griggs and Yeh[6], which is defined as a function satisfying the conditions that, (i), if (ii), if , for any two distinct vertices u and v in G. A **k – L(2,1) labeling**is an L(2,1) – labeling such that no label is greater than *k*. The **L(2,1) – labeling number**of *G* is the smallest number *k* such that G has a *k* – L(2,1) labeling and is denoted by .

The concept radio labeling was first introduced by Chartrand et al., A function is said to be a **radio labeling**, if it satisfies the following condition:

 (1)

for any two distinct vertices u and v in G. The condition (1) is called radio condition. The **span** of a radio labeling is the largest integer in the range of *f* and is denoted by *span*(*f*). The **radio number, rn(G),** is defined as the minimum span taken overall radio labelings of G.

Inspired by frequency assignment problem and radio labeling, KM. Kathiresan and S. Vimalajenifer introduced a new graph labeling called, radial radio labeling which is defined as an assignment *f*, of positive integers to the set of all vertices satisfying the condition that:

 (2)

for any two distinct vertices u and v in G. The condition (2) is called as radial radio condition. The **span** of a radial radio labeling, *f*,is the largest integer in the range of *f* and is denoted by *span f*. The **radial radio number, rr(G),** is defined as the minimum span taken overall radial radio labelings of G. That is, . A radial radio labeling , for which is called **radial radio labeling**.

Some basic results, which are helpful in further development of this paper are listed below:

**Theorem A:** , .[4]

**Theorem B:** , .[6]

**Theorem C:** For any self centered graph G, .[1]

**Theorem D:** If , then .[1]

**Theorem E:** For any simple connected graph G, , where is the clique number.[1]

**Theorem F:** For any simple connected graph , and where is the maximum degree in G.[1]

 In this paper, we determine the relationships between the radial radio number, radio number and the L(2,1) – labeling number. In addition, we construct graphs for which the radio number is the algebraic sum of its radial radio number and any non negative integer. Also, we construct graphs for which the L(2,1) – labeling number is the algebraic sum of its radial radio number and any non negative integer.

**II. RELATIONS CONNECTING RADIAL RADIO NUMBER, RADIO NUMBER AND L(2,1) – LABELING NUMBER**

 Throughout this section, we only consider simple connected graphs on n vertices. The first two theorems provide the relationship between and .

**Theorem 2.1**

 **If rad(G)=1, then .**

**Proof**

 Assume that and is an *rr(G)* – radial radio labeling of G. Then by definition, for any two distinct vertices u and v of G, *f* satisfies:

 (3)

Inequality (3) implies that,

1. if , then
2. if , then

From i) and ii), we observe that *f* does not satisfy the L(2,1) – labeling condition so that the set of positive integers is not sufficient to label the vertices of G under – labeling condition. This forces that, .

**Theorem 2.2**

 **If , then .**

**Proof**

 Let be an rr(G) – radial radio labeling of G and let . By the radial radio condition satisfies

. If u and v are adjacent, then and if u and v are non adjacent, then . This implies that, .

**Theorem 2.3**

**If diam(G)=1, then .**

**Proof**

If diam(G)=1, then G must be isomorphic to the complete graph . By Theorems A and B, it is clear that .

**Corollary 2.4**

 **If diam(G)=1, then .**

**Proof**

 If diam(G)=1, then G is self centered with radius 1 and so [D]. By Theorem 2.1, .

**Theorem 2.5**

 **If , then .**

**Proof**

Assume that, and are and radio labeling of G, respectively, such that and . Then satisfies the radio condition

 (4)

for any two distinct vertices u and v of G.

**Case 1:** whendiam(G)=2

1. if , then (4) becomes and
2. if , then (4) becomes

Here the statements in i) and ii) are as same as the – labeling conditions and hence . Thus, in this case, .

**Case 2:** when diam(G)>2

1. If , then (4) becomes and
2. if , then (4) becomes

In this case also, satisfies the – labeling conditions. But the strict inequalities in iii) and iv) forces that, . Thus .

This completes the proof.

**Corollary 2.6**

If G is self centered with , then .

Overall, from this section, we can assert that:

**Theorem 2.7**

For any simple connected graph G,

1. if and , then .
2. if and , then .
3. if G is self centered with , then .
4. if G is not self centered and , then

**III. RADIAL RADIO NUMBER AND RADIO NUMBER**

In this section, we prove the existence of graphs for which the radio number is the algebraic sum of the radial radio number and given nonnegative integer.

**Theorem 3.1**

 **For , there exists a graph such that .**

**Proof**

 Let us take . Since , .

**Theorem 3.2**

 **There is no graph exists, for which .**

**Proof**

Since for each self centered graph , .[1] Assume that, is not self centered. That is, , which implies that, . By Theorem F, we have and so

 (5)

Also, by Theorem F, we have

 (6)

 implies that, , and so , since G is connected. Thus, there exists no graph such that, .

**Theorem 3.3**

 **For any , there exists a graph G such that .**

**Proof**

Assume that G is constructed by using two copies of , . Let and let . Then and .

We now find the radial radio number for G.

Define such that , where . We have to prove that, for every pair of vertices u and v of G, satisfies,

 . (7)

**Case 1a:** Consider the pair

Here, for all . Since, the pair satisfies (7). In a similar manner, we can show that the pair also satisfies (7), for all .

**Case 2a:** Consider the pair

Since and , the pair satisfies (7) for all , . Similarly, we can prove that the pair satisfies (3) for all , .

**Case 3a:** Consider the pair .

In this case, . We have, . Thus pair also satisfies (7).

 From the three cases, we can say that, is a radial radio labeling of G and so , which implies that Also, , by Theorem E, . Thus .

Now, we determine the radio number for G. Define g such that , ; , . We have to prove that, for every pair of vertices u and v of G, satisfies,

 . (8)

**Case 1b:** Consider the pair , .

Since , for all , , and hence the pair , satisfies (8), for all . Proceeding like this, we can show that the pair , satisfies (8), for all ,

**Case 2b:** Consider the pair

Here Also, , the pair satisfies (4) for all , . Similarly, we can prove that the pair satisfies (8) for all , .

**Case 3b:** Consider the pair .

We have, and . Thus pair also satisfies (8).

 From cases 1b, 2b, 3b, we conclude that, is a radio labeling for . Also, we have , which forces that, . Since , by Theorem E, . Thus . Finally, for this graph and , which implies that, . For m=5 the graph G is illustrated in Figure 1. The corresponding radio labeling and radial radio labeling are illustrated in Figure 2 and Figure 3, respectively.

**Figure 1**

**Figure 2 Figure 3**

From Theorems 3.1, 3.2 and 3.3, we note that:

**Theorem 3.4**

**For and any , there exists a graph , for which .**

**IV. RADIAL RADIO NUMBER AND L(2,1) – LABELING NUMBER**

 In this section, we show the existence of graphs for which the L(2,1) – labeling number is the algebraic sum of the radial radio number and any given non negative integer.

**Theorem 4.1**

 **For , there exists graph G such that .**

**Proof**

**Case 1:** m=0

Take , where . We have . In this case, is the required graph.

**Case 2:** m=1

Consider G as . We know that, and .

**Case 3:** m=2

In this case, is the desired graph, since and .

**Theorem 4.2**

**For , there exists graph G such that .**

**Proof**

Let and . Then rad(G)=1 and diam(G)=2.

Define such that , ; ; ; ; , . We now show that, *f* is an L(2,1) – labeling of G.

1. for the pair , ,
2. for the pair , ,
3. for the pair , ,
4. for the pair , , ,

Form this discussion, if two vertices u and v are adjacent, then the label difference between them is atleast 2 and if two vertices u and v are non adjacent, then the label difference between them is atleast 1 and so *f* is an L(2,1) – labeling. Also, . We know that, . Since , . Thus we conclude that, .

Now, we determine the radial radio number of G. Define such that , and , . Here, we note the following:

 , ;

 , and ;

 , ;

 , .

This implies that, every pair of vertices of G satisfies the radial radio condition that and hence is a radial radio labeling of G and and hence . Since , by Theorem E, . This gives that, . Hence G is the required graph. For m=7, the constructed graph is drawn in Figure 4. The corresponding L(2,1) – labeling and radial radio labeling are shown in Figure 5 and Figure 6, respectively.

**Figure 4**

**Figure 5 Figure 6**

**V. CONCLUSION**

In this paper, we compared three labeling parameters, which are based on the distance between two vertices of a graph G. Also, we prove the existence of graphs whose radio number and L(2, 1) – labeling number as the algebraic sum of radial radio number and any given non negative integer. In a similar manner, we may compare other graph labeling parameters.

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