Graceful labeling on cycle related graphs

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1 Introduction

The theory of Graphs is now branched off in various directions. Decomposition of graphs, Theory of Domination, Chromatic Graph Theory, Algebraic Graph Theory, Labeling of Graphs, Enumeration of Graphs are just to name a few. Further we have an enormous number of conjectures and open problems in graph labelings. For an excellent and up to date dynamic survey on graph labeling we refer to Gallian[4]. All the graphs considered here are finite and undirected. The terms not defined here are used in the sense of Harary [3].

1.1 Graceful labeling on graphs

Initiation of graph labeling were taken in 1960's. Tremendous work of literature has to been developed around graph labeling over the most recent couple of years. It also provides a mathematical structure with a broad range of application.

The utilization of labeled graph models require imposing of additional constraints which characterize the problem being investigated. To label the graphs, we have several variations for labeling such as graceful, harmonious mean, heron mean, sequential, magic, vertex total magic, cordial, k-equitable, radio, and many other have been introduced by several authors. These all techniques are motivated by real life problems.

The name **"Graceful Labeling"** is because of Solomon W. Golomb and this type of labeling was first given by the name "beta labeling" by Alexander Rosa in 1967.

Definition 1.1. *Let G be a graph of order p and size q. A graceful labeling of G is an injection $f : V \to 0,1,...,q$ such that when each edge uv is assigned the label $f^*(uv) = |f(u)-f(v)|$, the resulting edge labels are all distinct. Such a function f^* is called the induced edge function and a graph which admits such a labeling is called a graceful graph.*

The following results are due to Golomb[15]:

1. A necessary condition for a (p,q)-graph $G(V,E)$ to be graceful is that, it be possible to partition its vertex set V (G) into two subsets V_0 and V_e such that there are exactly $\lceil \frac{q}{2} \rceil$ edges each of which joins a vertex of V_0 with one of V_e.

2. A complete graph K_p is graceful if and only if $p \leq 4$.

3. The following results are due to Rosa[16]:

   - A cycle C_n of order n is graceful if and only if $n \equiv 0$ *or* $3(mod\ 4)$.
   - A friendship graph F_k on k triangles is graceful if and only if $n \equiv 0$ *or* $1(mod\ 4)$.
   - If G is a graceful eulerian graph of size q, then $q \equiv 0$ *or* $3(mod\ 4)$.

One of the still unsolved problems on graceful graphs is the now famous Ringel Kotzig Conjecture [19, 38, 48]

4. **Conjecture:** All trees are graceful.

Motivated by the notion of graceful labeling of graphs, a labeling called **pronic graceful labeling is discussed** in this work.

2 Graceful labeling using pronic numbers

Definition 2.1. *Pronic Number:*

A pronic number is a number which is the product of two consecutive integers, that is, a number of the form $n(n + 1)$*. The study of these numbers dates back to Aristotle. They are also called oblong numbers, heteromecic or rectangular numbers. The* n^{th} *pronic number is the sum of the first n even integers. From the definition, it is seen that all pronic numbers are even, and the only prime pronic number is 2. Also the only pronic number in the Fibonacci sequence is 2.*

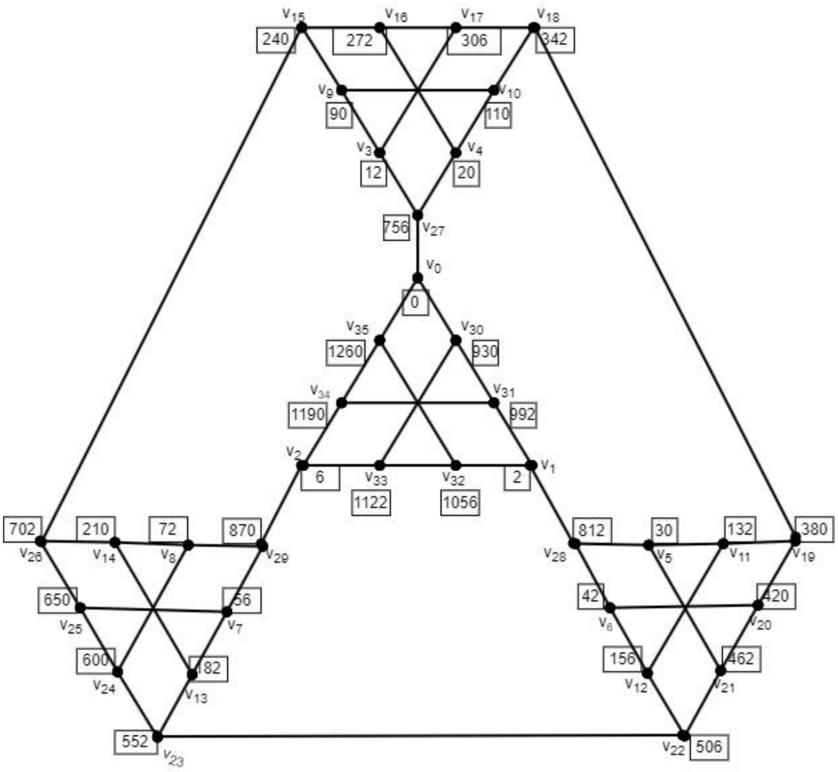

Figure 1: Zamfirescue Graph-Pronic Graceful

Note 2.2. *A pronic number is squarefree if and only if n and n + 1 are also squarefree. The number of distinct prime factors of a pronic number is the sum of the number of distinct prime factors of n and* $n+1$*. 0, 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240, 272, 306, 342, 380, 420, 462 are few among them.*

Definition 2.3. *Pronic Graceful Labeling:*

Let $G(p,q)$ *be graph with* $p \geq 2$*. A pronic graceful labeling of G is a bijection* $f : V(G) \rightarrow \{0,2,6,12,...,p(p + 1)\}$ *such that the resulting edge labels obtained by* $|f(u) - f(v)|$ *on every edge uv are pairwise disjoint. A graph G is called pronic graceful if it admits pronic graceful labeling.*

Example 2.4. *An example for a graph which admits pronic graceful labeling is given in 1*

In this chapter, the pronic graceful labeling on graphs with some graph operations have been discussed.

2.1 Main theorems

Theorem 2.5. *Cycle graph* C_n*,* $n \geq 3$ *is a pronic graceful graph*

Theorem 2.6. *Star graph* $K_{1,n}$*,* $n \geq 3$ *is a pronic graceful graph.*

Theorem 2.7. *Path graph P_n, $n \geq 3$ is a pronic graceful graph.*

Theorem 2.8. *Path graph P_n, $n \geq 3$ is a pronic graceful graph.*

Theorem 2.9. *Complete graph K_n, $n \geq 4$ does not admit pronic graceful labeling.*

2.2 Wheel related graphs

Theorem 2.10. *The wheel graph $K_1 + C_n$, $n \geq 4$ admits pronic graceful labeling.*

Theorem 2.11. *Gear graph G_n admits pronic graceful labeling*

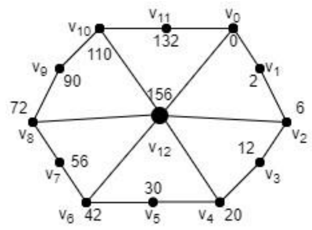

Figure 2: Gear Graph

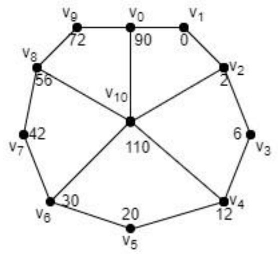

Figure 3: G_5 Graph

Proof : Let v_n be the apex vertex and $\{v_0, v_1, v_2 \ldots, v_{n-1}\}$ be the rim vertices of G_n, $n \geq 3$ and $\{v_i v_{i+1}, i = 0,1,\ldots n-2, v_{n-1}v_0, v_n v_i, i = 0,2,4,\ldots,n-2$ be the edges of G_n.
Define a bijection $f : V(G) \to \{p_0, p_1, \ldots, p_{n-1}\}$ by

$$f(v_i) = p_i, i = 0,1,2,\ldots,n-1 \qquad\qquad f(v_n) = p_n.$$

For the vertices labeled above, an induced function $f^* : E(G) \to \{2,4,6\ldots,p_{n-1}\}$ is defined by

$$f^*(v_i v_{i+1}) = 2(i+1), i = 0,1,2,\ldots,n-2;$$

$$f^*(v_n v_i) = n(n+1) - i(i+1), i = 0,2,4,\ldots.n-2;$$

$$f^*(v_0 v_{n-1}) = (n-1)n.$$

Let A_1 and A_2 denote the set of edge labels of $\{v_i v_{i+1} (0 \leq i \leq n-2), v_{n-1}v_0\}$, $\{v_i v_{i+1}, i = 0,1,2,\ldots,n-2\}$. Then:

$$A_1 = \{2,4,6,\ldots,2(n-1), n(n-1)\};$$

$$A_2 = \{n(n+1), n(n+1)-6, n(n+1)-20, \ldots 4n-2\}.$$

Hence $A_1 \cap A_2 = \varphi$ which results that the gear graph admits pronic graceful labeling.

□

Theorem 2.12. *Helm Graph HG_n, admits pronic graceful labeling*

Proof : Let v_n be the apex vertex and $\{v_0,v_1,v_2...,v_{n-1}\}$ be the rim vertices of HG_n, $n \geq 3$. Let $\{v_iv_{i+1}, i = n,n+1,...2n-2, v_nv_{2n-1}, v_{2n}v_i, i = n,n+1,...2n-1$ be the edges of HG_n.
Define a bijection $f : V(G) \to \{p_0,p_1,...,p_{2n}\}$ by

$$f(v_i) = p_i, i = 0,1,2,...,n-1 \qquad\qquad f(v_{2n}) = p_{2n}.$$

For the vertices labeled above, an induced function $f^* : E(G) \to \{2,4,6...,p_{n-1}\}$ is defined by

$$f^*(v_iv_{i+1}) = 2(i+1), i = n-1,n,n+1,n+2,...,2n-2;$$

$$f^*(v_{2n}v_i) = 3n^2 - (n+1)i - 1, i = 0,1,2,...n-1;$$

$$f^*(v_iv_{i+(n+1)}) = p_{n+1} + 2i(n+1), i = 0,1,2,...,n-2;$$

$$f^*(v_nv_{2n-1}) = 3n^2 - 3n.$$

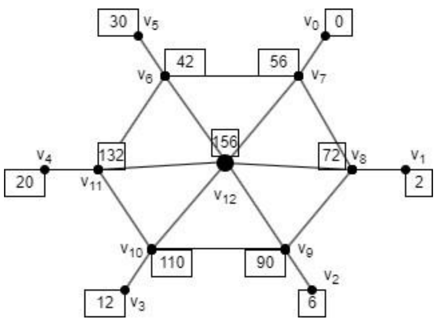

Figure 4: Helm Graph

Let A_1 and A_2 denote the set of edge labels of $\{v_iv_{i+1}(0 \leq i \leq n-2), v_{n-1}v_0\}$, $\{v_iv_{i+1}, i = 0,1,2,...,n-2\}$. Then:

$$A_1 = \{2,4,6,...,2(n-1), n(n-1)\};$$

$$A_2 = \{n(n+1), n(n+1) - 6, n(n+1) - 20,...4n-2\}.$$

Hence $A_1 \cap A_2 = \varphi$ which results that the helm graph admits pronic graceful labeling.

□

2.3 Ladder Graph and Mobius Ladder Graph

Definition 2.13. *Ladder Graph $L_{n,1}$*

The ladder graph, denoted by $L_{n,1}$ is a planar undirected graph which is defined as the cartesian product of two path graphs, one of which has only one edge: $Ln,1 = P_n \times P_2$ with $2n$ vertices and $3n-2$ edges.

Theorem 2.14. *Ladder graph $L_{n,1}$ is pronic graceful.*

Proof : Let $L_{n,1}$ be the ladder graph with vertex set $V(L_{n,1}) = \{u_i,v_i, 0 \leq i \leq n-1\}$ and edge set $E(L_{n,1}) = \{u_iu_{i+1}, v_iv_{i+1}, 0 \leq i \leq n-2\} \cup \{u_iv_i, 0 \leq i \leq n-1\}$.

Define a bijection $f : V(G) \to \{p_0, p_1, \ldots, p_{2n-1}\}$ by

$$f(u_i) = p_i, i = 0,1,2,\ldots,n-1; f(v_i) = p_{i+n}, i = 0,1,2,\ldots,n-1.$$

For the vertices labeled above, an induced function $f^* : E(G) \to \{2,4,6\ldots,p_{n-1}\}$ is defined by

$$f^*(u_iu_{i+1}) = 2(i+1), i = 0,1,2,\ldots,n-2;$$

$$f^*(v_iv_{i+1}) = 2(n+i+1), i = 0,1,2,\ldots,n-2;$$

$$f^*(u_iv_i) = n^2 + n(1+2i), i = 0,1,2\ldots,n-1.$$

Let A_1, A_2 and A_3 denote the set of edge labels of $\{u_iu_{i+1}(0 \le i \le n-2)\}$, $\{v_iv_{i+1}, i = 0,1,\ldots n-2\}$ and $\{u_iv_i, i = 0,1,\ldots n-1\}$. Then:

$$A_1 = \{2,4,6,\ldots,2(n-1)\};$$
$$A_2 = \{2(n+1), 2(n+2), \ldots, 2(2n-1)\}; A_3 = \{n^2+n, n^2+3n, \ldots, n(3n-1)\}.$$

Hence $A_1 \cap A_2 = \varphi$ which results that the ladder graph admits pronic graceful labeling.

□

Definition 2.15. *Mobius Ladder Graph M_n*

A Mobius ladder graph M_n is a simple cubic graph on $2n$ vertices and $3n$ edges. A Mobius ladder graph M_n is a graph obtained from the ladder P_nP_2 by joining the opposite end points of the two copies of P_n.

Theorem 2.16. *Mobius Ladder Graph M_n is pronic graceful.*

Proof : Let M_n be the Mobius Ladder graph with vertex set $V(M_n) = \{u_i, v_i, 0 \le i \le n-1\}$ and edge set $E(M_n) = \{u_iu_{i+1}, v_iv_{i+1}, 0 \le i \le n-2\} \cup \{u_0v_{n-1}, v_0u_{n-1}\}$.

Define a bijection $f : V(G) \to \{p_0, p_1, \ldots, p_{2n-1}\}$ by

$$f(u_i) = p_i, i = 0,1,2,\ldots,n-1; f(v_i) = p_{i+n}, i = 0,1,2,\ldots,n-1.$$

For the vertices labeled above, an induced function $f^* : E(G) \to \{2,4,6\ldots,p_{n-1}\}$ is defined by

$$f^*(u_iu_{i+1}) = 2(i+1), i = 0,1,2,\ldots,n-2; \qquad f^*(u_0v_{n-1}) = 2n(2n-1);$$

$$f^*(v_iv_{i+1}) = 2(n+i+1), i = 0,1,2,\ldots,n-2;$$

$$f^*(v_0u_{n-1}) = 2n; f^*(u_iv_i) = n^2 + n(1+2i), i = 0,1,2\ldots,n-1.$$

Let A_1, A_2, A_3 and A_4 denote the set of edge labels of $\{u_iu_{i+1}(0 \le i \le n-2)\}$, $\{v_iv_{i+1}, i = 0,1,\ldots n-2\}$, $\{u_iv_i, i = 0,1,\ldots n-1\}$ and $\{u_0v_{n-1}, v_0u_{n-1}\}$ Then:

$$A_1 = \{2,4,6,\ldots,2(n-1)\};$$
$$A_2 = \{2(n+1), 2(n+2), \ldots, 2(2n-1)\};$$
$$A_3 = \{n^2+n, n^2+3n, \ldots, n(3n-1)\}; A_4 = \{2n(2n-1), 2n\}.$$

Hence $A_i \cap A_j = \varphi$ for all $i \neq j$ which results that the mobious ladder graph admits pronic graceful labeling. □

2.4 Shell related graphs

From the excellent survey of Gallion, one can find many families of cycle related graphs on which important is the Shell graph family.

Shell Graph

Theorem 2.17. *A Shell Graph $C(n,n-3)$, for $n \geq 3$ is a pronic graceful graph.*

Proof : Let $\{v_0,v_1,v_2...,v_{n-1}\}$ be the vertices of $C(n,n-3)$.

Define a bijection $f : V(G) \to \{p_0,p_1,...,p_{n-1}\}$ by

$$f(v_i) = p_i, i = 0,1,2,...,n-1.$$

For the vertices labeled above, an induced function $f^* : E(G) \to \{2,4,6...,p_{n-1}\}$ is defined by

$$f^*(v_iv_{i+1}) = 2(i+1), i = 0,1,2,...,n-3;\ f^*(v_nv_i) = n(n+1) - i(i+1), i = 0,1,2...,n-2.$$

The edge labels are thus $\{2,4,8...,2(n-2),p_{n-1},p_{n-1}-2,p_{n-1}-6,...p_{n-1}-p_{n-2}\}$ and hence shell graph $C(n,n-3)$, for $n \geq 3$ admits pronic graceful labeling. □

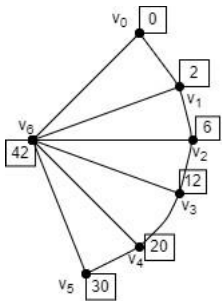

Figure 5: Shell Graph

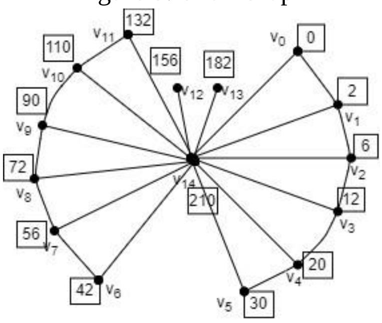

Figure 6: Shell Butterfly Graph

2.5 Shell Butterfly Graph

J.J. Jesintha, K.E. Hilda[17] defined a Shell -butterfly graph as a double shell in which each shell has any order with exactly two pendant edges at the apex and proved that all shell- butterfly graphs with shells of order l and m(shell order excludes the apex) are graceful. Note that G has $n = 2m + 3$ vertices and $q = 4m$ edges. Here in the following theorem, we consider the shell butterfly graph of same order.

Theorem 2.18. *A shell butterfly graph G is a pronic graceful graph.*

Proof : Let G be a shell-butterfly graph with n vertices and q edges and have the shell orders as m(odd or even) and l where $l = 2t + 1$. Note that shell orders exclude the apex. Let the shell that is present to the left of the apex be called as the left wing of the G. Let the shell that is present to the right of the apex is called the right wing of G.

Denote the apex of G be v_{2m+2} and the vertices of right wing of the graph from top to bottom as $v_0,v_1,...,v_{m-1}$. Similarly the left wing vertices by $\{v_m,v_{m+1},...,v_{2m-1}\}$. Let $\{v_{2m},v_{2m+1}\}$ be the two pendant vertices of G.

Define a bijection $f : V(G) \to \{p_0,p_1,...,p_{n-1}\}$ by

$$f(v_i) = p_i,\ i = 0,1,2,...,2n+2.$$

For the vertices labeled above, an induced function $f^* : E(G) \to \{2,4,6...,p_{n-1}\}$ is defined by

$$f^*(v_iv_{i+1}) = 2(i+1), i = 0,1,2,...,m-1,m,m+1,...,2m-2;$$

$$f^*(v_{2m+2}v_i) = (2m+2)(2m+3) - i(i+1), i = 0,1,2...,m-1,m,m+1,...,2m+2.$$

The edge labels are thus $\{2,4,8...,2(m-1),2(m+1),2(m+2),...,2(2m-1)\}$. The labels of the edges $v_{2m+2}v_i$ are of the form $(2m+2)(2m+3)-i(i+1), i = 0,1,2...,m-1,m,m+1,...,2m-1$ and begins with p_{2m+2} and the difference of each label is of the form $2i$, $i = 1,2,...m-1,m+1,m+2,...,2m+1$. and hence shell butterfly graph admits pronic graceful labeling. □

2.5.1 PGL on corona product and joint sum of graphs

Definition 2.19. *Corona Product of C_n and mK_1*

The corona product of C_n and mK_1, denoted by $C_n \circ mK_1$ is the graph with the vertex set

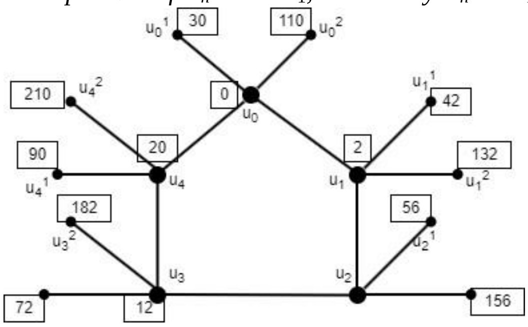

Figure 7: Corona graph $C_5 \circ 2K_1$

$V(C_n \circ mK_1) = \{x_i,y_i^j : 1 \le i \le n, 1 \le j \le m\}$ *and the edge set*

$E(C_n \circ mK_1) = \{x_i,x_{i+1} : 1 \le i \le n-1\} \cup \{x_i,y_i^j : 1 \le i \le n, 1 \le j \le m\} \cup \{x_n,x_1\}$.

Theorem 2.20. *Corona product $C_n \circ mK_1$ is a pronic graceful graph.*

Proof : Let $\{u_0,u_1,u_2,...,u_{n-1}\}$ be the vertices of the cycle C_n and $\{u_0^{(j)}, u_1^{(j)}, ...u_{n-1}^{(j)}, j = 1,2,...,m\}$ be the corresponding pendant vertices attached to the $u_0,u_1,u_2,...,u_{n-1}$.
Define a bijection $f : V(G) \to \{0,2,6...(nm+n)(nm+n-1)\}$ by

$$f(u_i) = p_i, i = 0,1,...,n-1;$$

$$f(u^j_i) = p_{nj+i}, i = 0,1,2,...,n-1, j = 1,2,...,m.$$

And the induced edge labeling $f^* : E(G) \to N$ is defined by

$$f^*(u_iu_{i+1}) = 2(i+1), i = 0,1,2...,n-2;\ f^*(u_0u_{n-1}) = n(n-1);$$

$$f^*(u_iu_i^{(j)}) = n[2ij + j(nj+1)], i = 0,1,2...,n-1, j = 1,2,...,m.$$

Let A_1, A_2, A_3 denote the set of edge labels of $\{u_iu_{i+1}, i = 0,1,...n-2\}$, $\{u_{n-1}u_0\}$ and $\{u_iu_i^{(j)}, i = 0,1,2...,n-1, j = 1,2,...,m\}$ respectively.
Clearly the labels of the edges for the above sets are of the form as follows:

A_1 contains the edges of the form $2k$, $k = 1,2,...(n-1)$ and each label differs by 2 and hence they are distinct.
A_2 contains the edge of the form $n(n-1)$ and is differed from the above labeling by p_{n-1}.

Consider the labels of A_3

For $j = 1$, the set contains edges of the form $\{p_n, p_n + 10i, ..., n(2i + (n + 1)\}$ For $j = 2$, the set contains edges of the form $\{p_{2n}, p_{2n} + 20i, ..., n(4i + 2(2n + 1)\}$

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.....

For $j = m$, the set contains edges of the form $\{p_{mn}, p_{mn} + 10mi, ..., n[2im + j(mn + 1)]\}$

It is observed that the labels in the above sets are distinct, that is $A_1 \cap A_2 \cap A_3 = \varphi j$ and hence $C_n \circ mK_1$ is a pronic graceful graph. □

2.6 Barycentric Subdivision of a graph

Definition 2.21. *Creating a barycentric subdivision is a recursive process. In this section we consider the concept of barycentric subdivision of cycles introduced by Vaidya et al. An edge $e = uv$ of a graph G is said to be subdivided when it is deleted and replaced by path of length 2 . Let $C_n = u_1...u_n$ be a cycle on n vertices. Subdivide each edge u_iu_{i+1} of C_n and let the new vertex be $u_i, 1 \leq i \leq n$. Join u_i with $u_{i+1}, 1 \leq i \leq n$. All suffixes are taken modulo n. The resulting graph is denoted as $(C_n)^2$. This graph is called the barycentric subdivision of C_n and it is denoted by $C_n(C_n)$ as it look like C_n inscribed in C_n. The barycentric subdivision subdivides each edge of the graph.*

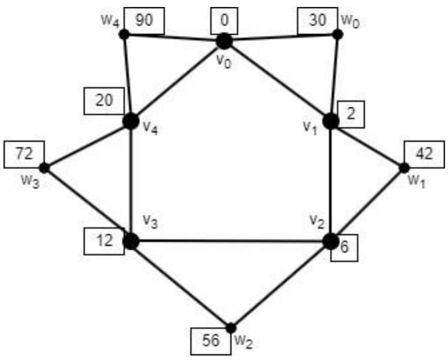

Figure 8: $C_5(C_5)$

Theorem 2.22. *Barycentric subdivision of cycle $C_n(C_n)$ is a pronic graceful graph.*

Proof : Let $\{v_0, v_1, ..., v_{n-1}\}$ be the vertices of $n-$ cycle and $\{w_0, w_1, w_2 ..., w_{n-1}\}$ such that w_i connected to v_i and v_{i+1} for $0 \leq i \leq n-2$ and w_{n-1} is connected to v_{n-1} and v_1.

Define a bijection $f : V(G) \longrightarrow \{p_0, p_1, p_2, ..., p_{2n-1}\}$ by

$$f(v_i) = p_i, i = 0,1,2,...,n-1; f(w_i) = p_{n+i}, i = 0,1,2,...,n-1.$$

Clearly f is a bijection. For the above vertices labeled above, the edge labeling $f^* : E(G) \longrightarrow N$ is defined by

$f^*(v_iv_{i+1}) = 2(i+1), i = 0,1,2...,n-2;$

$f^*(v_0v_{n-1}) = n(n-1); f^*(v_iw_i) = p_n + 2ni, i = 0,1,2,...,n-1.$

$f^*(w_iv_{i+1}) = p_n - 2 + i(2n-2), i = 0,1,2,...,n-2;$

$f^*(v_0w_{n-1}) = 2n(2n-1);$

Let A_1, A_2, A_3 and A_4 denote the set of edge labels of $\{v_iv_{i+1}, i = 0,1,...n-2\}$, $\{v_iw_i, i = 0,1,2,...,n-1\}$, $\{v_iw_{i-1}, i = 0,1,2,...,n-1\}$ and $\{v_0v_{n-1}, v_0w_{n-1}$ respectively. Then:

$A_1 = \{2,4,6,...,2(n-1)\}$; $A_2 = \{n(n+1),n(n+3),n(n+5)...,n(3n-1)\}$;

$A_3 = \{(n-1)(n+3),(n-1)(n+5),(n-1)(n+5),...,(n-1)(3n-2)\}$; $A_4 = \{2n(2n-1),2n\}$.

Hence $A_1 \cap A_2 = \varphi$ which results that the barycentric subdivision of cycle $C_n(C_n)$ admits pronic graceful labeling.

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