Connectivity concepts of an Arithmetic Graph

1.Introduction

Number theory is one of the oldest branches of mathematics which inherited rich contributions from almost all greatest mathematicians ancient and modern. The theory of congruences in Graph theory by Melvyn Bernard Nathanson in [3], paved the way for the emergence of a new class of graphs, namely Arithmetic Graphs. Inspired by the interplay between number theory and graph theory several researches in recent times are carrying out extensive studies on various arithmetic graphs in which adjacency between vertices is defined through various arithmetic functions. Vasumathi and Vangipuram defined the arithmetic graph in such a way that the adjacency between the vertices of same parity is considered in [6]. Suriyanarayana Rao and Sreenivasan.V in [5] defined the arithmetic graph by excluding the condition of adjacency between the vertices of same parity. By the fundamental theorem of arithmetic, every positive integer greater than one can be uniquely represented as a product of primes (i.e) $n = {p_1}^{a_1} \times {p_2}^{a_2} \times \ldots \ldots \times {p_r}^{a_r}$ where $a_i \geq 1$. Here we discuss about the positive integer $n > 1$ other than $n = p^{a_i}$ where p is prime and $a_i \geq 1$.

2.Arithmetic graph

Definition2.1. The **arithmetic graph V_n** is defined as a graph with its vertex set is the set consists of the divisors of n (excluding 1) where n is a positive integer and $n = {p_1}^{a_1} \times {p_2}^{a_2} \times \ldots \ldots \times {p_r}^{a_r}$ where p_i ' s are distinct primes and a_i ' s ≥ 1 and two distinct vertices *a, b* which are not of the same parity are adjacent in this graph if $(a, b) = p_i$, for some i , $1 \leq i \leq r$. The vertices u and v are said to be of same parity if both u and v are the powers of the same prime, for instance $u = p^2$, $v = p^3$. In this graph, vertex 1 becomes an isolated vertex. Hence, we consider arithmetic V_n graph without vertex 1. Therefore, each vertex of V_n is connected to some vertex in V_n . Clearly, V_n is a connected simple graph.

Example 2.2. Consider an arithmetic graph G = V_{210} where 210 = 2 × 3 × 5 × 7. The vertex set V (G) = {2, 3, 5, 7, 2×3, 2×5, 2×7, 3×5, 3×7, 5×7, 2×3×5, 2×3×7, 2× 5×7, 3×5×7, 2×3×5×7}. Since (2, 2×3) = (2, 2×5) = (2, 2×7) = (2, 2×3×5) = (2, 2×3×7) = (2, 2×5×7) = (2, 2×3×5×7) = 2 there exist edges between the vertex 2 and the vertices such as 2×3, 2×5, 2×7, 2×3×5, 2×3×7, 2×5×7, 2×3×5×7. Also, since the (2 × 3, 2 × 3 × 7) = 2 × 3= 6≠ pi , 2 × 3 and 2 × 3 × 7 are non-adjacent vertices. Similarly for the other vertices. This can be shown in Figure 2.1

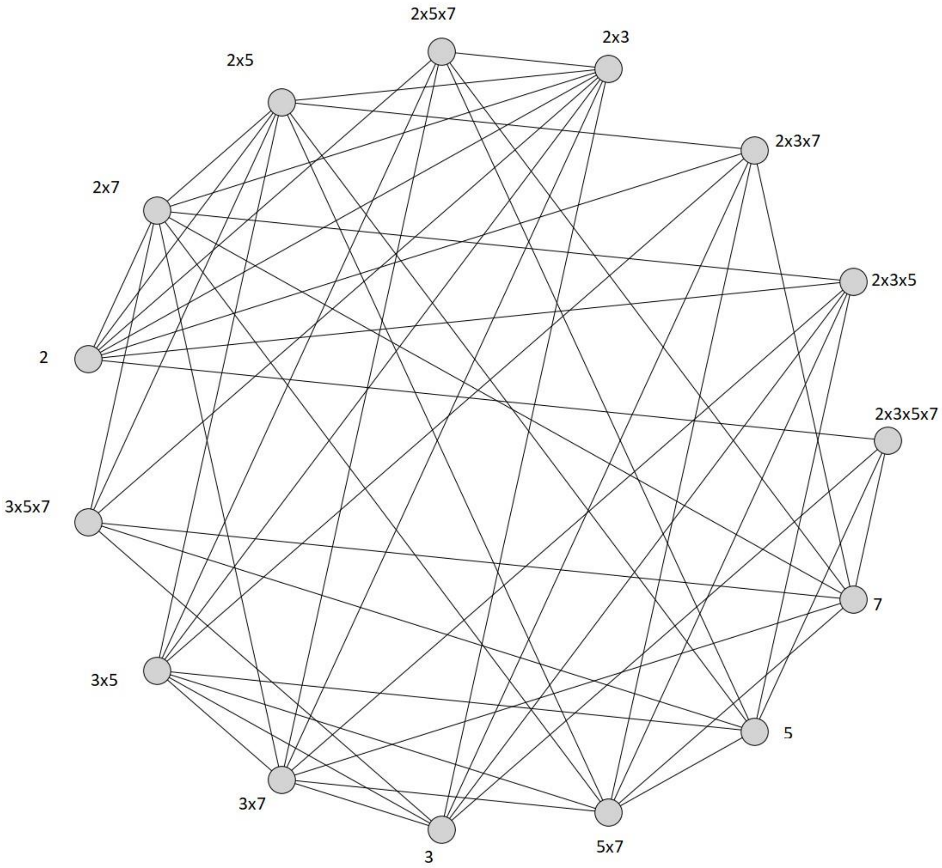

Figure 2.1: **Arithmetic Graph** $G=V_{210}$

Theorem 2.3.[4] If $G = V_n$ is an arithmetic graph, where $n = {p_1}^{a_1} \times {p_2}^{a_2} \times \dots \dots \times {p_r}^{a_r}$, $a_i \geq 1$ for $i \in \{1, 2, \dots, r\}$, then the number of vertices of G can be calculated using the formula $|V| = \prod_{i=1}^{r}(a_i + 1) - 1$.

Example 2.4. Consider the graph $G = V_{210}$ where $210 = 2 \times 3 \times 5 \times 7$ given in Figure 2.1. By Theorem 2.3 the number of vertices is $|V_{210}| = (1 + 1)(1 + 1)(1 + 1)(1 + 1) - 1 = 15$.

Theorem 2.5. Let V_n be an arithmetic graph with $n = \prod_{i=1}^{r} {p_i}^{a_i}$, for any vertex $u = \prod_{i \in B} {p_i}^{\propto_i}$ where $B \subseteq \{1, 2, 3, \dots r\}$, $1 \leq \alpha_i \leq a_i$, and $B' \subseteq B$, one has $\deg(u) = (|B - B'| + \sum_{i \in B'} a_i) \prod_{i \notin B}(a_i + 1) - \delta_{1|B|} - (\sum_{i \in B'} a_i - 1)\, \delta_{1B'}$, where B is the number of distinct prime factors in a chosen vertex u, B' is the number of prime factors having power 1 in chosen vertex u, empty summation equals zero and δ is the Kronecker's delta function defined by $\delta_{ij} = \begin{cases} 0 \; for \; i \neq j \\ 1 \; for \; i = j \end{cases}$

Note 2.6. For an arithmetic graph the vertices are divisors of n, hence the degree of all types of vertices say primes, power of primes, product of primes, product of powers of primes can be calculated using the formulae given in Theorem 2.5.

Theorem 2.7. Let $G = V_n$ be an arithmetic graph where $n = {p_1}^{a_1} \times {p_2}^{a_2}$, then

$$\Delta(G) = \begin{cases} [a_j(a_i + 1) - 1] - |a_j - 1| & for \; a_j \geq a_i \geq 2 \\ a_i + a_j & for \; a_i \; or \; a_j = 1 \end{cases}$$

$$\delta(G) = \begin{cases} 2 & for\ a_i, a_j > 1 \\ 1 & for\ a_i\ or\ a_j = 1 \end{cases}.$$

Theorem 2.8.[4] Let $G = V_n$ be an arithmetic graph where $n = p_1 \times p_2 \times p_3 \times \cdots \times p_r$, then (i) $\Delta(G) = 2r-1$

$$\text{(ii)}\delta(G) = \begin{cases} r & if\ r \geq 3 \\ 1 & \text{if}\ r = 2 \end{cases}$$

Theorem 2.9. Let G be a V_n arithmetic graph, where $n = p_1^{a_1} \times p_2^{a_2} \times \dots \dots \times p_r^{a_r}$, such that at least one of $i \in \{1, 2, \dots, r\}$ does not equal one. Then,

(i) $\Delta(G) = a_j \prod_{i=1, i\neq j}^{r} (a_i + 1) - 1$ where a_j is the maximum exponent of p_i, $i \in \{1, 2, \dots, r\}$

(ii)$\delta(G) = r$.

Results 2.10.

1) It is identified that given arithmetic graph $G = V_n$, $n = p_1^{a_1} \times p_2^{a_2}$; $a_i \geq 1$ are bipartite.

2) The size of the arithmetic graph $G = V_n$, $n = p_1^{a_1} \times p_2^{a_2}$ where $a_1, a_2 \geq 1$ is determined using the formula $\epsilon = 4a_1a_2 - a_1 - a_2$.

3)The diameter of an arithmetic graph is $diam(G) \leq 3$ and its radius is $radius(G) \leq 2$.

4)Arithmetic graph $G = V_n$, is Hamiltonian, if $n = p_1^{a_1} \times p_2^{a_2} \times \dots \dots \times p_r^{a_r}$, $3 \leq r \leq 6$.

5)All arithmetic graphs $G = V_n$ is not a Eulerian graph.

3.Connectivity Number of an Arithmetic graph $G = V_n$

Connectedness plays an essential role in graph theory, the graph representing a communication network needs to be connected for communications to be possible between all nodes(vertices). Numerous networks such as transport networks, road networks, electrical networks, distributed computing, block chain network, telecommunication systems or networks of servers can be modelled by a graph. Many researchers are made to determine how well a network is connected or can be splitted for sake of effectiveness. Two classical measures that indicate how the graph G is reliable are the edge-connectivity $\kappa'(G)$ and the vertex-connectivity or simply the connectivity $\kappa(G)$ of G. The connectivity of G, written $\kappa(G)$, is the minimum order of a vertex set $S \subset V(G)$ such that $G - S$ is disconnected or has only one vertex.

Definition 3.1. The **connectivity or vertex connectivity $\kappa(G)$** is the number of vertices of a minimal vertex cut. A graph is called $k -$ connected or $k -$ vertex connected if its vertex connectivity is k or greater. Any graph G is said to be k-connected if it contains at least k vertices, but does not contain a set of $k-1$ vertices whose removal disconnects the graph and $\kappa(G)$ is defined as the largest k such that G is $k-$ connected. Thus $\kappa(G) = 0$ if G is either trivial or disconnected. All non-trivial connected graphs are $1 -$ connected.

Definition 3.2. An edge cut of G is a set of edges whose removal renders the graph G disconnected. The **edge-connectivity $\kappa'(G)$** is the size of a smallest edge cut. A graph is called $k-$ edge $-$ connected if its edge connectivity is k or greater. All non-trivial graphs are one edge connected.

Theorem 3.3. For an arithmetic graph $G = V_n$, $n = p_1^{a_1} \times p_2^{a_2}$ where p_1 and p_2 are distinct primes, then $\kappa(V_n) = \kappa'(V_n) = \begin{cases} 1 \; for \; a_i = 1 \& a_j > 1 \\ 2 \; for \; a_i > 1; i = 1,2 \end{cases}$

Theorem 3.4. For an arithmetic graph $G = V_n$, $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \ldots \times p_r^{a_r}$ where p_i, i = 1, 2, . . . , r, $r > 2$ are distinct primes and $a_i = 1$ for all i = 1, 2, . . . , r, $\kappa(V_n) = \kappa'(V_n) = r$

Theorem 3.5. For an arithmetic graph $G = V_n$, $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \ldots \times p_r^{a_r}$ where $p_1, p_2, \ldots p_r$ are distinct primes and a_i' s ≥ 1 for all i = 1, 2, 3, . . . , r and $r > 2$ then $\kappa(V_n) = \kappa'(V_n) = r$.

Proof. We prove the theorem by considering the following four cases

Case (i) All the a_i's, i = 1, 2, 3, . . . r is equal to one. By Theorem3.2., the result follows.

Case (ii) Some of the a_i's are equal to one and the others are greater than 1. Consider the vertex set of V_n as $V(V_n) = \{p_1, p_2, \ldots, p_r, p_1 \times p_2, \ldots, p_1^{a_1} \times p_2^{a_2} \times \ldots \ldots \times p_r^{a_r}\}$. Let the last vertex be $p_1^{a_1} \times p_2^{a_2} \times \ldots \ldots \times p_r^{a_r}$ say v_1, where a_i 's are the maximum powers of the given distinct primes. By the definition of an arithmetic graph, we see that the only vertices which are adjacent to v_1 are $p_1, p_2, \ldots, p_r$. Hence $d(v_1) = r$. Also, the minimum degree of V_n occurs at the vertex v_1. That is, $\delta(V_n) = r = d(v_1)$. Hence, $\kappa(V_n) = \kappa'(V_n) \leq \delta(V_n) = r$. But the removal of r vertices adjacent to v_1 makes the graph disconnected. Hence, we obtain the result $\kappa(V_n) = r$. The edge connectivity $\kappa'(V_n) = r$ is same as Theorem 3.2.

Case(iii) All the a_i 's are equal and greater than 1. Here also consider the last vertex of $V(V_n)$, say $p_1^{a_1} \times p_2^{a_2} \times \ldots \ldots \times p_r^{a_r}$ where the a_i's are the maximum power of given distinct primes. By the definition of an arithmetic graph, it is clear that $p_1, p_2, \ldots, p_r$ are the only vertices which are adjacent to the vertex $p_1^{a_1} \times p_2^{a_2} \times \ldots \ldots \times p_r^{a_r}$. The remaining proof is similar to case (ii).

Case (iv) All the a_i's are distinct and greater than one. Consider the last vertex in the vertex set of V_n, say $p_1^{a_1} \times p_2^{a_2} \times \ldots \ldots \times p_r^{a_r}$ where the a_i's are the maximum power of the given distinct primes. By the definition of an arithmetic graph, this vertex is adjacent to exactly r vertices namely $p_1, p_2, \ldots, p_r$. Suppose it is adjacent to any other vertex except p_i then, it contradicts the definition of an arithmetic graph. The remaining proof is similar to case (ii).

Example 3.6. Consider an arithmetic graph $G = V_{2310}$ where $2310 = 2 \times 3 \times 5 \times 7 \times 11$, given in Figure 3.1. The set $S = \{2, 3, 5, 7, 11\}$ is a minimum vertex cut so the cardinality of the set S is a connectivity number $\kappa(G)$ which is equal to 5. Also, the removal of the set of edges say $F = \{2 \times 3 \times 5 \times 7 \times 11\ 2, 2 \times 3 \times 5 \times 7 \times 11\ 3, 2 \times 3 \times 5 \times 7 \times 11\ 5, 2 \times 3 \times 5 \times 7 \times 11\ 7, 2 \times 3 \times 5 \times 7 \times 11\ 11\}$ makes the graph disconnected, since F is a minimum edge cut, cardinality of F is the edge connectivity number. Thus, we have $\kappa'(G) = 5$

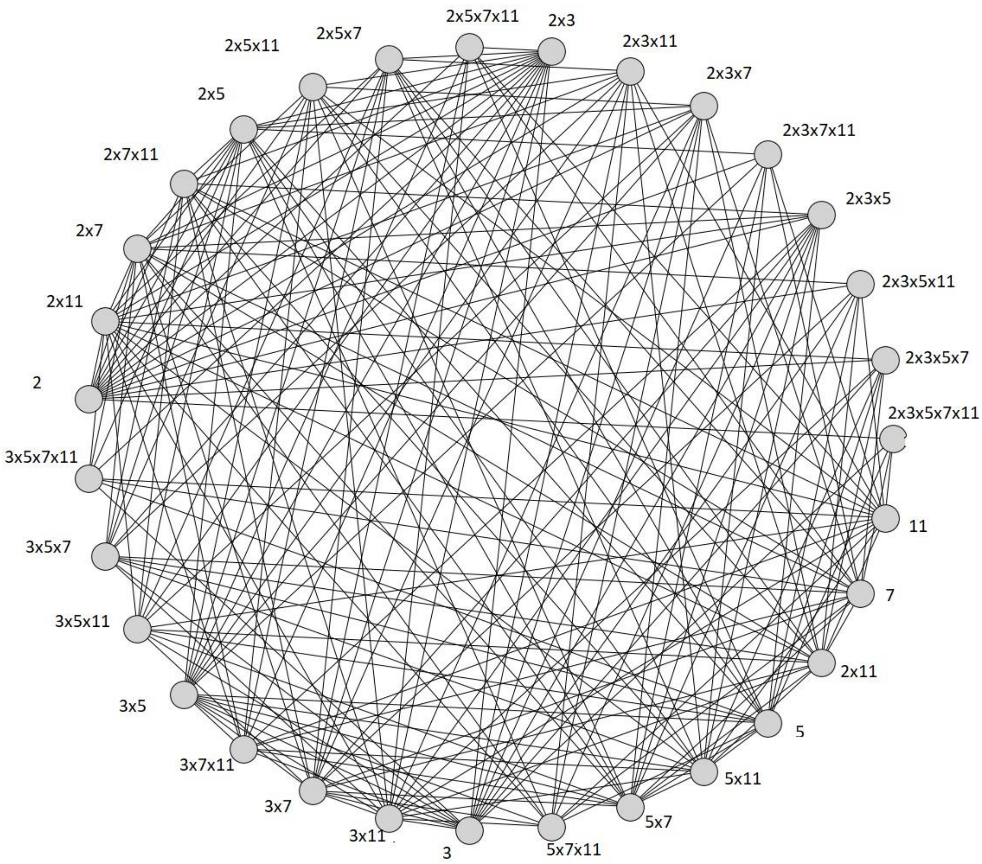

Figure 3.1: Arithmetic graph $G = V_{2310}$

Remark 3.7. The arithmetic graph V_n is a maximally connected graph

4. The connectivity number of complement of an arithmetic graph

In this section, we identified the connectivity number for complement of an arithmetic graph $G = V_n$, where n is a product of two primes, product of powers of two primes, product of r primes, product of powers of r primes.

Definition 4.1 Let G be a graph, The complement $\bar{G}$ of a graph G is the graph with vertex set $V(\bar{G})$ such that two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G

Theorem 4.2 For an arithmetic graph $G = V_n$, $n = {p_1}^{a_1} \times {p_2}^{a_2}$ where $a_j > 1$; $i \neq j$, $\kappa(\bar{G}) = a_j - 1$

Theorem 4.3 For an arithmetic graph $G = V_n$, $n = {p_i}^{a_i} \times {p_j}^{a_j}$where $a_j \geq a_i \geq 2$, $\kappa(\bar{G}) = a_i + a_j - 1$

Example 4.5 Consider the graph $G = V_{36}$, $36 = 2^2 \times 3^2$ given in Figure 4.1. The set $S_1 = \{2, 3\}$ is a minimum vertex cut, and the Figure 4.2 shows the complement of an arithmetic graph V_{36} and the set $S_2 = \{2^2 , 3, 3^2\}$ is the minimum vertex cut of G. Thus $\kappa(\bar{G}) = 3$ which satisfies the value of $a_i + a_j - 1 = 2 + 2 - 1 = 3$

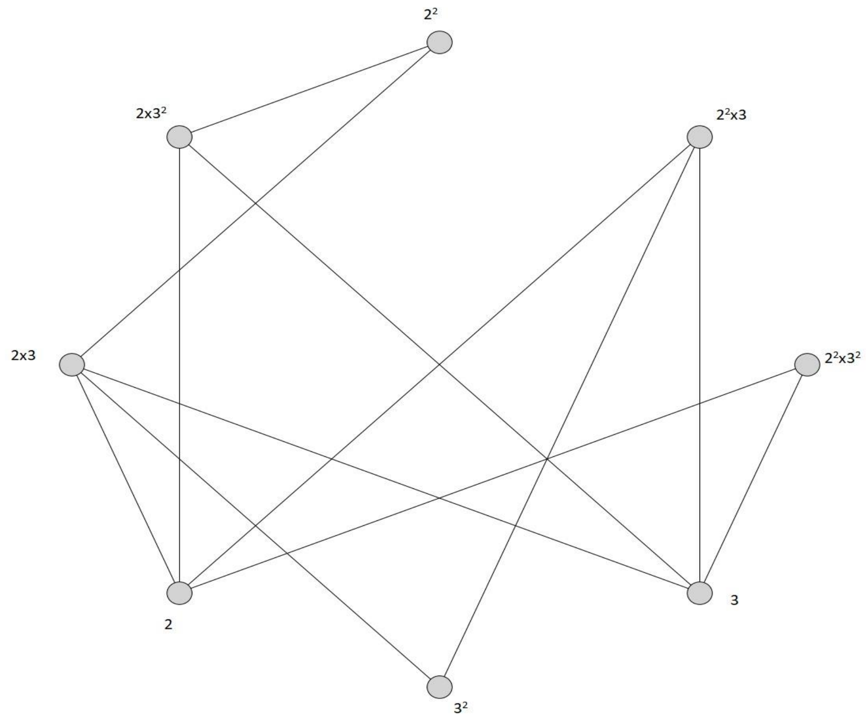

Figure 4.1: Arithmetic Graph $G = V_{36}$

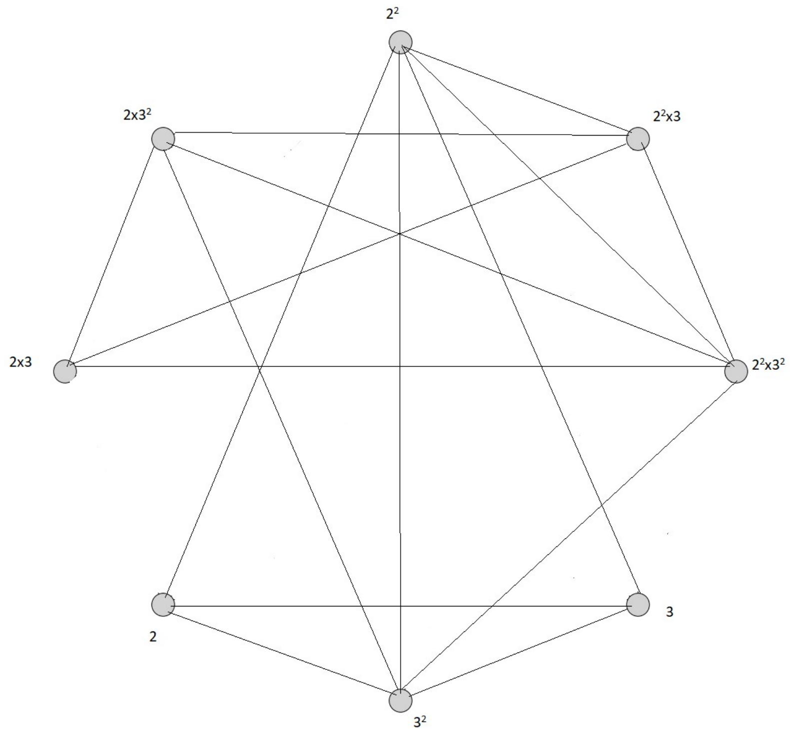

Figure 4.2: Arithmetic graph $\bar{G} = \overline{V_{36}}$

Theorem 4.6 For an arithmetic graph $G = V_n$, $n = p_1 \times p_2 \times p_3 \times \cdots \times p_r$, $r > 2$, $\kappa(\bar{G}) = (2r - 4)/2$.

Proof. Let $G = V_n$ be an arithmetic graph. The vertex set $V(G) = V(\bar{G}) = \{p_1, p_2, \ldots, p_r, p_1 \times p_2, \ldots, p_1 \times p_r, \ldots, p_{r-1} \times p_r, p_1 \times p_2 \times p_3, \ldots, p_1 \times p_2 \times p_3 \times \cdots \times p_r\}$. By Theorem 2.2.4, the maximum degree of the graph G is $\Delta(G) = 2r-1$. The vertices which are of having maximum degree is $p_i \times p_j$;i, j $\in \{1, 2, \ldots, r\}$, $i \neq j$. Hence the number of vertices having maximum degree is rC_2. These rC_2 vertices will have minimum degree in $\bar{G}$. Thus, there are rC_2 set of minimum vertex cuts. Also, the minimum degree $\delta(\bar{G}) = |V(G) - 1| - \Delta(G) = (2r - 4)/2$. Thus $d(p_i \times p_j) = (2r - 4)/2$. So we have $\kappa(\bar{G}) = (2r - 4)/2$.

Theorem 4.7 For an arithmetic graph $G = V_n$, $n = {p_1}^{a_1} \times {p_2}^{a_2} \times \ldots \ldots \times {p_r}^{a_r}$, $r > 2$ and at least one of a_i , $i \in \{1, 2, \ldots, r\}$ does not equal to one, $\kappa(\bar{G}) = \prod_{i=1, i\neq j}^{r}(a_i + 1) + a_j - 2$ where a_j is the maximum exponent of p_i , $i \in \{1, 2, \ldots, r\}$.

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