A Survey Study of Some Graph Labeling Techniques.

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Abstract

Graph theory is a well-known branch of Mathematical sciences (Arithmetic). Mathematics gives the same name to different things. But graph theory creates a graph labeling techniques in which different labeling are given to same graphical structures. Graph is a mathematical structure describing dots, curves, bars, or traces.In Graph labeling non-negative integers are used to label the vertices or edges, or both, satisfying certain mathematical conditions. Graphs are used to describe various mathematical models in Operations Research, control, and Engineering, in studies domains of laptop technological know-how which include statistics mining, photograph segmentation, clustering, photograph capturing, networking that's used in structural fashions, and so forth. Graph labeling has many applications in the social community, verbal exchange (communication) community, circuit design, Database management, coding principle, radar, astronomy, and X-ray crystallography. Depending on the trouble scenario a type of graph is used for representing the hassle (problem) and by way of applying appropriate graph labeling techniques the hassle may be solved. Graph labeling is a flourishing as well as application-oriented area of research in Mathematics.
In this chapter, we discuss the different graph labeling techniques related to different graphs.

Keywords- Graph, Graph labeling, Cordial, Magic, Mean, Radio, Power mean, Permutation, combination.

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1 Introduction

The field graph theory has been invented in 1735 with the Koinsberg Seven Bridge problem. The phrase graph is derived from the Greek word 'graphein'. A "graph" in this paper consists of "vertices" or"nodes" and lines called edges that connect with vertices. Graph labeling have been introduced within the mid 1960's by Alexander Rosa. Over the past six decades,the research in graph labeling developed very fast. Graph labeling is a flourishing as well as an application-oriented area of research in the theory of Graphs. To date, more than 200 types of graph labeling techniques had been introduced. In this chapter we try to collect some important graph labeling techniques with suitable graphs. Here we consider only simple, finite, connected and non-directed graphs having countable number of edges and vertices. Here we consider the terminology and symbols of graph from Harary's 'Graph Theory'.

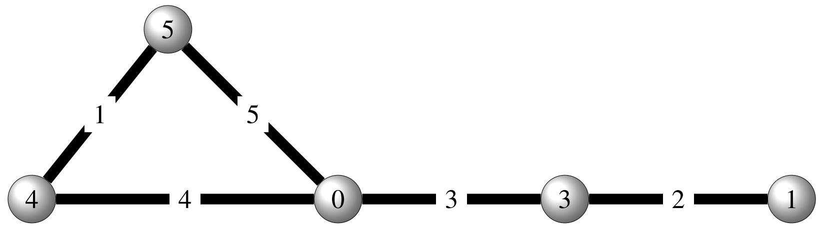
1.1 Graph :

Definition: 1. *A linear graph(simple graph) $G=(V(G),E(G),F(G))$,consisting of a non-empty set $V(G)=\{v_1,v_2,---------\}$ of vertices,and a set $E(G)=\{e_1,e_2,-----\}$ whose elements are called edges and an incidence function $F(G)$ associated with each element of $E(G)$ an unordered pair of elements of $V(G)$.*
If e is an edge and x and y are vertices such that $F(e)=xy$ then the vertices x and y are called the end vertices of e.

Order of a graph: The number of elements in a vertex set $V(G)$ is called order of a graph.
Size of a graph : The number of elements in an edge set $E(G)$ is called size of a graph.
Degree of a vertex: The total number of edges incident on a vertex is called a degree of a vertex .
Degree of a graph: The sum of all degrees of vertices in a graph G is called degree of a graph G.

1.2 Graph Labeling :

Definition: 2. *An allotment of non-negative integers to the vertices or edges, or both, satisfying certain mathematical conditions is called a Graph labeling .*

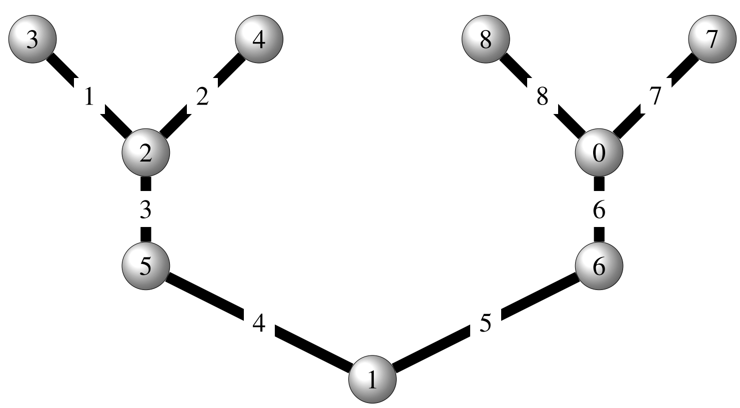


1.3 Different Graph Labeling Techniques :

1.3.1 Graceful Labeling:

Definition: 3. *Let $G = (V(G), E(G))$ be a connected graph with p vertices and q edges. A graceful labeling is a particular graph labeling of a graph in which the vertices are labeled with a subset of distinct non-negative integers from 0 to q and the graph edges are labeled with the absolute differences between vertex label values. The edge labels are numbered from 1 to q both must be included, the labeling is a graceful labeling and the graph is said to be a graceful graph.*

Exa. Banana tree is graceful. ,trees with vertices ≤ 35,Caterpillars,bananas etc. are always graceful.

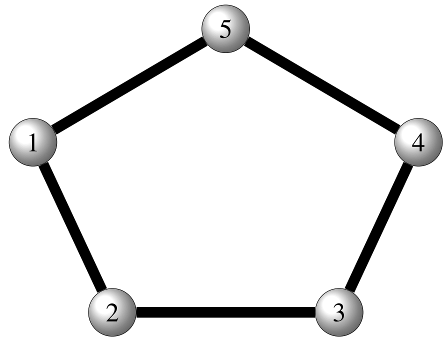


$B_{2,4}$

Examples of graph having Graceful Labeling.

- All symmetrical trees
- Trees with vertices ≤ 35.
- Caterpillars, banana trees etc.

Non-graceful graph: A simple graph G in which we can not make graceful labeling is said to be non-graceful.



$5-cycle$

Result

In graceful graph with q (q is positive integer) edges and $q+1$ vertices the following holds:

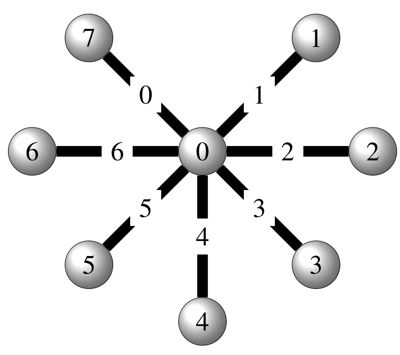
- We can draw distinct graceful labeling for the same graph i.e. it is not unique.
- The vertices labeled with 0 and q are always in neighbouring.
- If the graph has q edges then each graceful labeling must contains vertex label as q.
- Graceful labeling graph may contains a triangle graph.
- The complementarity property for graceful labeling is satisfied. That is for a given graph with graceful labeling if we swap every vertex label q with $q-k$, the resulting labeling is also graceful since the edge labels will not have changed the extreme vertices of an edge. The labels a and b become $q-a$ and $q-b$.

1.3.2 Harmonious labeling:

Graham and Sloane [33]

Definition: 4. *In a graph $G=(V(G),E(G))$ with q edges, a function f is called as harmonious labeling of graph G if there exists an injective function f from vertex set $V(G)$ to $E(G)$ and the induced function f^* from $V(G)$ to $E(G)$ defined by $f^*(e=xy)=(f(x)+f(y))(modq)$ is both one-one and onto.*
A graph having harmonious labeling is called a harmonious graph.

Exa. $K_{1,7}$



$K_{1,7}$

Examples of graph having Harmonious Labeling.

- Cycle $C_n(n \geq 3)$ has harmonious labeling if n is an odd number and vice versa.
- Ladder graph $L_n, n \neq= 2$.
- F_n, Friendship graph excluding n is congruent to 2(mod4).
- The fan graph f_n .

- The graph g_n for $(n \geq 2)$.

Result

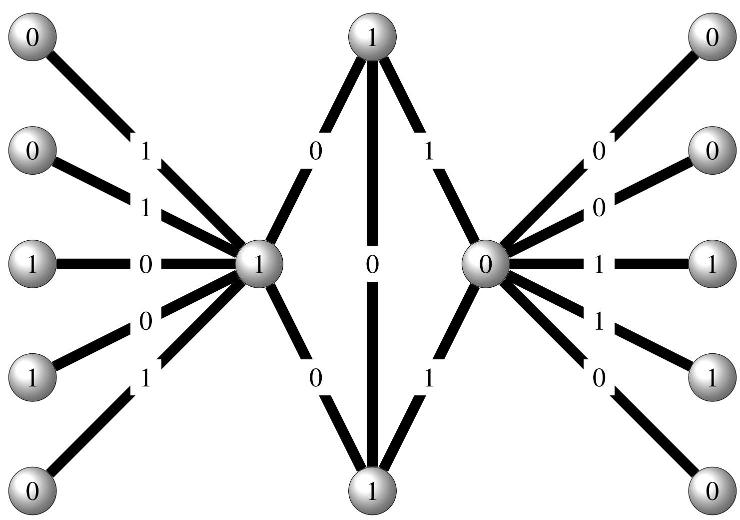
- Harmonious labeling is non distinctive.
- $af(x)+b$ is harmonious labeling of a graph G if f is, where a is invertible element of set q and b is any random element of q. (i.e. Z_q).
- We can label 0 to any vertex in a graph.
- All trees are harmonious.
- Complete graph C_n is harmonious iff $n \leq 4$.
- The Peterson graph is Harmonious.
- Wheel graph $W_n = C_n + K_1$ is harmonious.
- If $n = 4$, $K_n^{(2)}$ is harmonious, if f n is odd or $n = 6$ it is not harmonious..

1.3.3 Cordial Labeling :

Cahit [9]

Definition: 5. *Let $G = (V(G), E(G))$ be a connected graph with p vertices and q edges.A labeling in a graph is called cordial if we label its vertices with $0's$ and $1's$ so that when the edges are labeled with the difference of the labels at their end vertices, the number of vertices (edges) labeled with ones and zeros differ at most by one.*

Exa. Cordial labeling graph of Jelly Fish $J_{5,5}$



$J_{5,5}$

Examples of standard graph having Cordial Labeling.

- All trees.
- Complete graph K_n iff n is less than or equal to 3.
- Complete bipartite graph $K_{m,n}$ for all positive integers m and n
- The friendship graph $C_3^{(t)}$ iff t is not congruent to 2 modulo 4.
- All fan graph f_n.
- The wheel graph W_n iff n is not congruent to 3 modulo 4.

- mK_n for particular values of m and n.
- Every Skolem-graceful graph is cordial.

Types of Cordial Labeling.

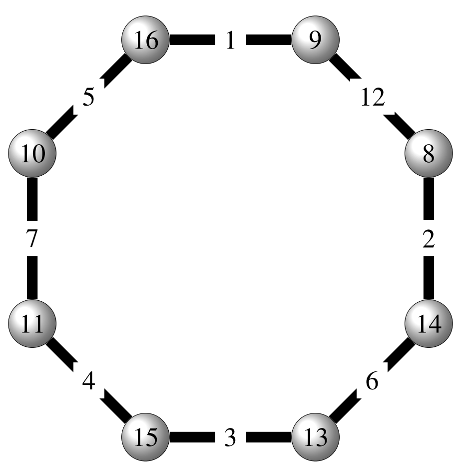
- Difference cordial graph.
- Edge product cordial graph.
- Prime cordial graph.
- Planar grid cordial graph.
- Context of Duplication cordial graphs.
- Second order cordial labeling graph.
- SET cordial graph.
- Integer cordial graph.
- Signed product cordial graph.
- Mean cordial graph.
- Geometric Mean cordial graph.
- Harmonic Mean cordial graph.

1.3.4 Magic labeling of a graph:

Using the number theory concept of magic squares, magic labeling had been discovered by Sedlacek in 1963 [41]. **a)Vertex magic labeling of a graph(VML):**

Definition: 6. *Let $G = (V(G), E(G))$ be a connected graph with p vertices and q edges is said to be vertex magic labeled graph if there exists a one-one and onto function $f : V(G) \bigcup E(G) \to \{1, 2, ..., p+q\}$, such that for all vertices of G, the sum of a vertex labels and its incident edges labels is constant. This bijective mapping is known as vertex magic labeling of G.*

Exa.VML cycle graph C_8 with magic constant K=22.

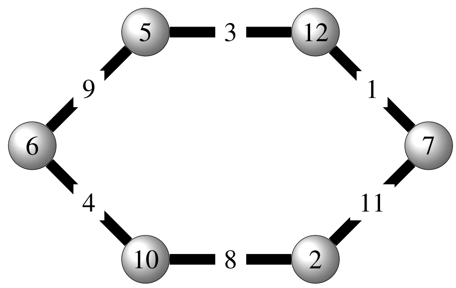


Cycle C_8

b) Edge magic labeling of a graph

Definition: 7. *Let $G = (V(G), E(G))$ be a connected graph with p vertices and q edges is said to be an Edge magic labeled graph if there exists a one-one and onto function $f: V(G) \bigcup E(G) \to \{1, 2, ..., p+q\}$, such that for all edges of G, the sum of an edge label and it's end vertices labels are constant.This bijective function is known as Edge magic labeling of G. A graph having edge magic labeling is called an edge magic graph.*
s a connected graph semi-magic if there is a labeling of edges with integers such that for each vertex v the sum of all edges incident with v is the same for all v.

Edge magic labeling of cycle graph C_6 with magic constant K=20.



Cycle C_6

Examples of a graph having Magic Labeling.

- Complete graph K_n for $n = 2$ and n greater than or equal to 5.
- Complete bipartite graph $K_{n,n}$ for all $n \geq 3$
- The fan graph f_n iff n=odd and n greater than or equal to 3.
- The wheel graph W_n for $n \geq 4$.
- A connected (p, q) graph iff $5p/4 < q \leq p(p-1)/2$.

Types of Magic Labeling.

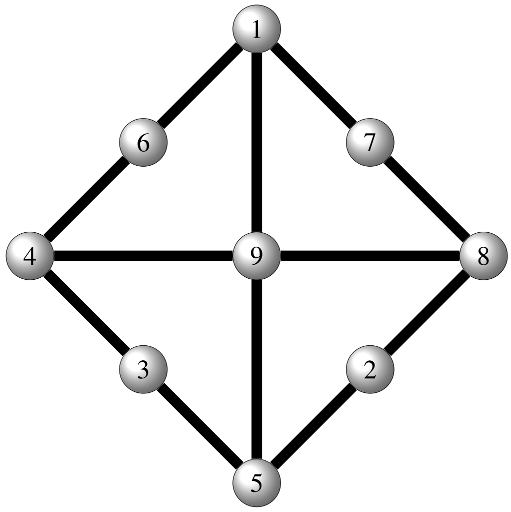
- Semi-Magic
- Super-Magic
- Anti-magic
- Prime-magic
- H-magic
- Magic labeling of type (a,b,c)
- Sigma labeling/ Distance-magic labeling

1.3.5 Radio labeling:

Chartrand, Erwin, Zhang, and Harary [15]
Radio labeling of graph is used for regulations for channel assignments of FM radio stations.

Definition: 8. *A connected graph $G(V, E)$ is said to have a radio labeling if there exists a one-one mapping $c: V(G) \to N$, set of natural numbers satisfying the condition $d(x, y) + |c(x) - c(y)| \geq 1 + diam(G)$ for every pair of distinct vertices x and y of G.*

The maximum number alloted to any vertex of G by c is called a radio number and is denoted by $rn(c)$.
The radio number of G, denoted by $rn(G)$ is the minimum value of radio numbers $rn(c)$ taken from radio labeling c of G.
Exa. Radio labeling of a graph



Cycle C_8

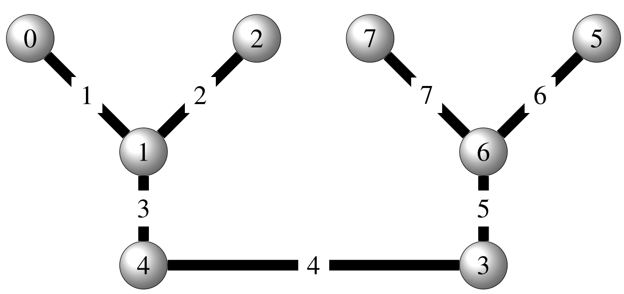
Examples of Radio Labeling graph.

- The $rn(G)$ of a sunflower graph SF_n is equal to its order.
- The $rn(G)$ of a Helm graph H_n is an odd no. i.e.$2n+1$.
- The $rn(G)$ of a gear graph G_n is also an odd no. i.e.$2n+1$.

1.3.6 Mean labeling of graphs:

Somasundaram and Ponraj [40].

Definition: 9. *Let $G=(V(G),E(G))$ be a connected graph with p vertices and q edges is said to be a mean labeled graph if there exists a one-one function $f: V(G) \to \{0,1,2,...,q\}$, such that each edge (x,y) is labeled with $(f(x)+f(y)+1)/2$ if $f(x)+f(y)$ is odd and $(f(x)+f(y))/2$ if $f(x)+f(y)$ is even.*
Here the resulting edge labels are distinct.



$F_{2,4}$

Examples of Mean Labeling graph.

- The graph P_n, C_n, the bipartite graph $K_{2,n}$, triangular snakes, quadrilateral snakes etc. all are mean graphs..
- K_n if and only if n is less than 3.
- $K_{1,n}$ iff n is less than3.
- The friendship graph $C_3^{(t)}$ iff t less than 2.
- Bistars $B_{m,n}(m>n)$ iff $m<n+2$

Types of Mean Labeling graph.

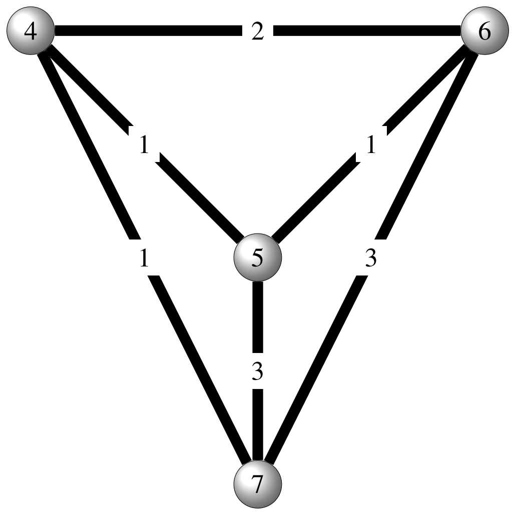
- Vertex even and odd mean graph.
- Super Mean labeling graph.

1.3.7 Irregular labeling of graph:

Chartrand, Jacobson, Lehel, Oellermann, Ruiz, and Saba [14]

Definition: 10. *Let $G = (V(G), E(G))$ be a connected weighted graph with p vertices and q edges is said to have an irregular labeling if a graph G has no isolated vertices and there is an assignment of positive integer weights to the edges of G such that the sums of all the positive integer weights of the edges at each vertex are unequal.*

If we make different irregular labeling to a graph, then the minimum value of the largest weight of an edge over all irregular labeling is called the irregularity strength which is denoted by $s(G)$ of G.
If there does not exists such weight for a graph then it's irregularity strength is ∞.

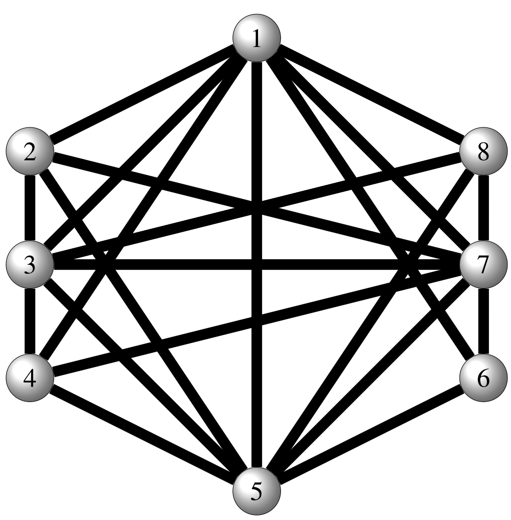


An irregular 3-labeling of the wheel W_3

1.3.8 Prime labeling of graph:

Tout, Dabboucy, and Howalla.[20]

Definition: 11. *Let $G = (V(G), E(G))$ be a connected graph with p vertices and q edges is said to have a prime labeling if there exists a bijective mapping $f\colon V(G) \to \{1, 2, ..., p\}$ such that the vertices x and y are join to form an edge if $(f(x)$ and $f(y)$ are relatively prime i.e. gcd $(f(x), f(y)) = 1$.*



R_8

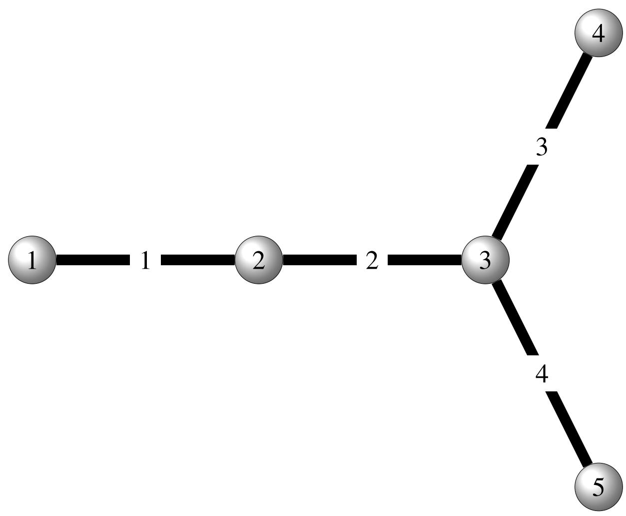
Examples of Prime Labeling of graph.

- Caterpillar graph with maximum degree 5.
- Path graph, star graph, complete binary trees, spider graph.

1.3.9 Power Mean Labeling of graph:

Mercy P.[37]

Definition: 12. *Let $G = (V(G), E(G))$ be a connected graph with p vertices and q edges is said to have a power mean labeling if it is possible to assign a label to each vertex $v \in V(G)$ with distinct labels $f(v)$ from $1, 2, 3, ..., q+1$ such that when each edge $e = xy$ is labeled with*
$f(e = xy) = \lfloor (f(x)^{f(y)} * f(y)^{f(x)})^{\frac{1}{f(x)+f(y)}} \rfloor$
$f(e = xy) = \lceil (f(x)^{f(y)} * f(y)^{f(x)})^{\frac{1}{f(x)+f(y)}} \rceil$
Here the edge labels are from $\{1, 2, 3, ..., q\}$ and all are distinct.



Examples of Power Mean Labeling of a graph.

- Tadpoles $T(n, k)$.

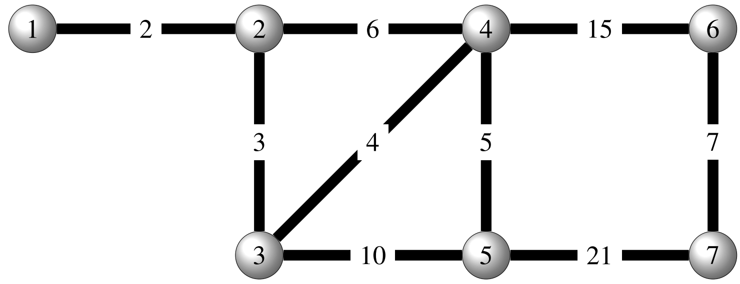
Result:

- The graph obtained by joining any two cycles C_m and C_n by a pathP_n is a Power mean graph
- For $m \geq 2$, $T(P_m)$ is a Power mean graph.
- For $n \geq 2$ Subdivision of any path is a Power mean graph.
- For $n \geq> 3$, a subdivision of any cycle C_n is a Power mean graph.

1.3.10 Combination and Permutation labeling of graph:

Hegde and Shetty [20]
a)Combination labeling of graph:

Definition: 13. *Let $G = (V(G), E(G))$ be a connected graph with p vertices and q edges. A bijective mapping $f : V(G) \to \{1, 2, ..., p\}$ is called as combination labeling of graph G if each edge (x, y) is assigned with the label $(f(x))!/[f(x) - f(y)]!(f(y))!$ where $f(x) > f(y)$.Here all edge labels are all distinct.*

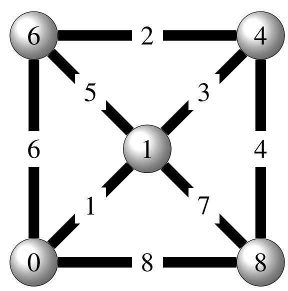


Examples of Combination Labeling of a graph.

- K_n if and only if n is less than or equal to 5.
- C_n, n greater than 3.
- $K_{n,n}$ iff n is less equals to 2. .

b) Permutation Labeling of graph:

Definition: 14. *Let $G = (V(G), E(G))$ be a connected graph with p vertices and q edges. An one-one and onto function $f : V(G) \to \{1, 2, ..., p\}$ is called as permutation labeling of graph G if each edge (x, y) is assigned with the label $(f(x))!/[f(x) - f(y)]!$ where $f(x) > f(y)$.Here all edge labels are distinct.*



grap **Examples of Permutation Labeling of a graph.**

- K_n iff n is less equals 5.

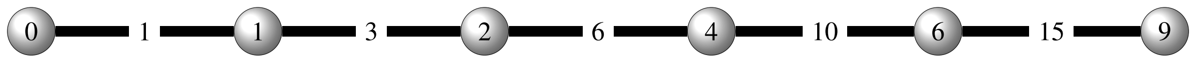
Result:

- In a permutation graph G if we remove any edge from this graph then the resultant graph is also a permutation graph.
- The graph k-wheel admits a permutation labeling for all integer $n \geq 2$.
- The graph k-fan $F_{n,k}$.
- The gear graph G_{2n} is also a permutation grph.

1.3.11 Triangular sum labeling of graph:

Hegade and Shankaran[36]

Definition: 15. *Let $G = (V(G), E(G))$ be a connected graph with p vertices and q edges.A graph G is said to have a triangular sum labeling if the vertices are assigned with non-negative integers, not all same in such a way that, when an edge whose end vertices are labeled with i and j then the edges are labeled with the value $i + j$.Here the vertices and edges are labeled with distinct labels.*



P_6

Examples of Triangular sum Labeling of graph.

- Paths, stars, complete n-ary trees.
- K_n iff $n = 1 or n = 2$

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