**Product Signed Dominating Function**

**T. M. Velammal, Research Scholar (Reg. No. 21212232092010)**

PG & Research Department of Mathematics

V.O. Chidambaram College, Thoothukudi-628008, Tamil Nadu, India

Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India

[avk.0912@gmail.com](mailto:avk.0912@gmail.com)

**A. Nagarajan, Head & Associate Professor (Retd.)**

PG & Research Department of Mathematics

V.O. Chidambaram College, Thoothukudi-628008, Tamil Nadu, India

Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India

[nagarajan.voc@gmail.com](mailto:nagarajan.voc@gmail.com)

**K. Palani, Head & Associate Professor**

PG & Research Department of Mathematics

A.P.C. Mahalaxmi College For Women, Thoothukudi-628002, Tamil Nadu, India

Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.

[palani@apcmcollege.ac.in](mailto:palani@apcmcollege.ac.in)

**ABSTRACT:**

Let be a simple graph. A function is called a product signed dominating function, if where and denotes the closed neighborhood of . The weight of a function is defined as . The minimum positive weight of a product signed dominating function is called product signed domination number of a graph and is denoted by In this paper, we discuss product signed dominating functions for some special graphs.

**Keywords:** Fan graph, wheel graph, helm graph, flower graph, product signed dominating function, product signed domination number.

**AMS Subject Classification:** 05C69.

**I. INTRODUCTION**

The domination problem was studied from 1950s onwards. Richard Karp proved the set cover problem to be NP-complete which had implications for the dominating set problem. Dunbar et al. introduced signed domination number [2],[3],[4],[5]. The concept of product signed domination was introduced in [11]. Hereafter, we denote the weight of a graph with respect to the function as Definitions of fan graph, wheel graph and helm graph are from [1]. Seoud and Youssef defined flower graph in [1]. In this paper, we find product signed domination number for fan graph, wheel graph, helm graph and flower graph.

**II. Main Results**

**2.1 Theorem**

For

**Proof:**

Let be a fan graph on vertices.

Let and

**Case 1:**

**Subcase 1.1:**

If , to get , set

Again to get , set

Proceeding like this, we define as

For

This may be a product signed dominating function. If it is, the weight will be negative since . **[11]**

**Subcase 1.2:**

If , to get , set

Again to get , set

Proceeding like this, we have

In this case the weight is , the total number of vertices,

**Case 2:**

For , it is observed that

If then cases arise

(i) if , then

(ii) if , then

And if then cases arise

(i) if , then

(ii) if , then

**Subcase 2.1:**

If , to get , set

Again to get , set

Again to get , set

Proceeding like this, we define as

For

**Subcase 2.2:**

If , to get , set

Again to get , set

Again to get , set

Again to get , set

Proceeding like this, we define as

For

**When**

By subcase 1.2, .By subcase 2.1, , a negative integer.By subcase 2.2, , a negative integer.Therefore,

**When**

By subcase 1.2, . By subcase 2.1, . By subcase 2.2, . Therefore,

**When**

By subcase 1.2, . By subcase 2.1, . By subcase 2.2, . Therefore,

**When**

By subcase 1.2, . By subcases 2.1 and 2.2, . Therefore,

**When**

By subcase 1.2, . By subcase 2.1, . By subcase 2.2, . Therefore,

**When**

By subcase 1.2, . By subcase 2.1, . By subcase 2.2, . Therefore,

**Consider**

**For**

By subcase 1.2, . By subcases 2.1 and 2.2, . Therefore,

**For**

By subcase 1.2, . By subcase 2.1, . By subcase 2.2, . Therefore,

**For**

By subcase 1.2, . By subcase 2.1, . By subcase 2.2, . Therefore,

**For**

By subcase 1.2, . By subcase 2.1, . By subcase 2.2, . Therefore,

**For**

By subcase 1.2, . By subcase 2.1, . By subcase 2.2, . Therefore,

**For**

By subcase 1.2, . By subcase 2.1, . By subcase 2.2, . Therefore,

Also from the above discussion, it is clear that, by subcase 2.1, is not a product signed dominating function when and by subcase 2.2, is not a product signed dominating function when

Therefore,

**2.2 Illustration**

**Figure 1**

**Product signed dominating function for fan graph on vertices.**

.

**2.3 Illustration**

**Figure 2**

**Product signed dominating function for fan graph on vertices by subcase 2.1 of 2.1**

**Figure 3**

**Product signed dominating function for fan graph on vertices by subcase 2.2 of 2.1**

By subcase 2.1 of 2.1, . By subcase 2.2 of 2.1, . Therefore,

**2.4 Illustration**

**Figure 4**

**Product signed dominating function for fan graph on vertices.**

.

**2.5 Theorem**

For

**Proof:**

Let represent a wheel graph on vertices.

Let and

**Case 1:**

**Subcase 1.1:**

If , to get , set

Again to get , set

Proceeding like this, we define as

For

This may be a product signed dominating function. If it is, the weight will be negative since . **[11]**

**Subcase 1.2:**

If , to get , set

Again to get , set

Proceeding like this, we have

In this case the weight is , the total number of vertices,

**Case 2:**

For , it is observed that

If then cases arise

(i) if , then

(ii) if , then

And if then cases arise

(i) if , then

(ii) if , then

**Subcase 2.1:**

If , to get , set

Again to get , set

Again to get , set

Proceeding like this, we define as

For

**Subcase 2.2:**

If , to get , set

Again to get , set

Again to get , set

Again to get , set

Proceeding like this, we define as

For

**When**

By subcase 1.2, . By subcases 2.1 and 2.2, . Therefore,

**When**

By subcase 1.2, . By subcase 2.1, . By subcase 2.2, . Therefore,

**Consider**

**For**

By subcase 1.2, . By subcase 2.1, . By subcase 2.2, . Therefore,

**For**

By subcase 1.2, . By subcases 2.1 and 2.2, . Therefore,

**For**

By subcase 1.2, . By subcase 2.1, . By subcase 2.2, . Therefore,

**For**

By subcase 1.2, . By subcase 2.1, . By subcase 2.2, . Therefore,

**For**

By subcase 1.2, . By subcases 2.1 and 2.2, . Therefore,

**For**

By subcase 1.2, . By subcase 2.1, . By subcase 2.2, . Therefore,

Also from the above discussion, it is clear that the functions defined in subcases 2.1 and 2.2 are not product signed dominating functions when

Therefore,

**2.6 Illustration**

**Figure 5**

**Product signed dominating function for fan graph on vertices by subcase 2.1 of 2.5**

**Figure 6**

**Product signed dominating function for fan graph on vertices by subcase 2.2 of 2.5**

By subcase 2.1 of 2.5, .

By subcase 2.2 of 2.5, .

Therefore,

**2.7 Theorem:**

Let be any integer and , a helm graph on vertices. Then

**Proof:**

Let with as the pendant vertices and

Here and where must be assigned the same functional value **[11]**.

**Let .**

To get as , odd number of s where must be assigned .

**Suppose is even,**

Assign to and take . Correspondingly, for and .

Now obviously.

Hence is not a product signed dominating function.

Assign to and to

Correspondingly,

Here also obviously.

Hence is not a valid product signed dominating function.

Assign to and to

Correspondingly,

Clearly, here also .

Hence is not a valid product signed dominating function.

Continuing like this,

Assign to and to where

Correspondingly,

Clearly, .

Hence is not a valid product signed dominating function.

**Suppose is odd,**

Assign to . Correspondingly, for .

Now obviously.

Hence is not a product signed dominating function.

Assign to and to

Correspondingly,

Here also obviously.

Hence is not a valid product signed dominating function.

Assign to and to

Correspondingly,

Clearly, here also .

Hence is not a valid product signed dominating function.

Continuing like this,

Assign to and to where

Correspondingly,

Clearly, .

Hence is not a valid product signed dominating function.

Therefore, assigning or to continuous s fails to give a product signed dominating function.

Redefine as and

Correspondingly,

Now only when is odd such that is odd.

But here,

Therefore this also does not lead to any product signed dominating function.

Assign . Then

Correspondingly,

if and only if and are of opposite sign.

Without loss of generality, assume and

Then and

Correspondingly,

if and only if

Let . Then .

Correspondingly,

if and only if .

Let . Then .

Repeating the above procedure, and so on. (i.e) where follows the pattern for every four vertices starting from Therefore, if then the function is defined by

and for all to Correspondingly, and for all to

Now by construction,

1)

Also by construction,

Hence is not a product signed dominating function.

Suppose for any odd if the above pattern of assignment of functional values is followed, then

but in this case,

Hence fails to be a product signed dominating function.

**Therefore, assigning to under fails to give a product signed dominating function.**

**Let .**

Assign . Then .

Correspondingly,

if and only if and are of same sign.

Suppose . This procedure leads assigning to all the vertices of which gives a maximum weight.

So let us assign . Then .

Now,

if and only if

Let . Then

Now,

if and only if

Repeating the above procedure, and so on. (i.e) where follows the pattern for every two vertices starting from Therefore, if then the function is defined by and for all to Correspondingly, and for all to

Now by construction,

if and only if is even

if and only if is a multiple of

Therefore, is a product signed dominating function when

Now,

Therefore,

**2.8 Illustration:**

**Figure 7**

**Product signed dominating function for graph on vertices.**

.

**2.9 Theorem:**

**Proof:**

Let represent a flower graph on vertices.

Let and

**Case 1:**

Here to get any as , one of must be equal to . But in this case, to get , should assign values to and for such that . Finally, which is negative.

Further to get as positive among the remaining vertices atleast vertices must get under .

But in this case, if one of for gets , then and so that .

**Subcase 1.1: is even**

Here is odd.

In this case is a valid product signed dominating function with negative.

**Subcase 1.2: is odd**

Then is even.

Here in which fails to be a product signed dominating function.

**Case 2:**

Here for every , both and must have the same functional value. That is, or

Suppose for some . Then the neighbor vertices of in the inner cycle must get to get minimum weight. --- (I)

At the same time the neighbors of and must get (in the inner cycle) so that .

Repeating this procedure, the vertices of the inner and outer cycle get and alternately.

**Subcase 2.1: is even**

Therefore is odd.

Here the above procedure fails to give a valid product signed dominating function.

**Subcase 2.2: is odd**

Here is even.

In this case, the procedure yields a valid product signed dominating function and the corresponding

Hence this is a product signed dominating function with a positive weight.

As the weight is , this is minimum and the corresponding .

Further by statement (I) and subcase 2.1, the only product signed dominating function giving positive weight is when is odd.

Hence when is even and the corresponding .

**2.10 Illustration:**

**Figure 8**

**Product signed dominating function for flower graph on vertices.**

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