Existence of Solution for Nonlinear Implicit Fredholm Integrodifferential Equation via S-Iteration Method

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In this research chapter, we investigate the existence and uniqueness of the solution to a nonlinear implicit Fredholm integrodifferential equation. To analyze the problem, we utilize the iteration method. As the study of qualitative properties typically requires differential and integral inequalities, the iteration method proves to be equally important in the analysis of various qualitative properties, such as the continuity dependence and closeness of solutions. We provide an example that supports the established results.

**Key words:** Existence, iteration, Fredholm integrodifferential equation, Continuous dependence, Parameters.

**Mathematics Subject Classification:** 34A12,45B05, 37C25,34K32

# Introduction

Consider the nonlinear integrodifferential equation of the type:

 (1)

for . Let stand for the set of real numbers, be the product space and be the given subset of . We assume , , and .

 Many Iterative methods for certain classes of operators have been introduced by several researchers, including their convergence, equivalence of convergence, and rate of convergence, etc. (see [1, 3, 5, 8, 9, 15, 16, 17, 18, 19, 20, 21, 24, 25, 26]). Most of the iterative methods focus on both analytical and numerical approaches. Due to its simplicity and fastness, the iteration method has attracted attention and hence, it is used in this chapter.

 There is a sufficient amount of literature that deals with the special and even more general version of the equation [(1)](#e1) by using a variety of techniques [2, 6, 10, 11, 12, 13, 14, 22, 23, 27, 28, 29, 30, 31] and some of the references cited therein. Recently, Yunus Atalan, Faik Grsoy and Abdul Rahim Khan [4] have studied the special version of equation [(1)](#e1) for different qualitative properties of solutions. Authors are inspired by the work of D. R. Sahu [24] and influenced by the work in [4] .

 The primary aim of this chapter is to utilize the normal iteration method to establish the existence and uniqueness of the solution for the problem [(1)](#e1). Additionally, we provide a result of the data dependence for the solutions of integrodifferential equation [(1)](#e1) through the normal iteration method.

# Existence of solution via iteration method:

In terms of continuous functions , we denote

for . We define , is the space of all functions which are continuously differentiable on and endowed with the norm

 (2)

It is easy to note that with the norm defined by [(2)](#e2) forms a Banach space.

By a solution of equation [(1)](#e1), it mean a continuous function which is times continuously differentiable on and satisfies the equation [(1)](#e1). It is easy to observe that the solution of the equation [(1)](#e1) and its derivatives satisfy the integral equations (see [7], p.318)

for and

 We require the following pair of known results:

**Theorem 1**. *([24], p.194)* *[thm1] Let be a nonempty closed convex subset of a Banach space and a contraction operator with contract factor and fixed point . Let and be two real sequences in such that and for all and for some . For given , define sequences and in as follows:*

*-iteration process:*

*Picard iteration:*

*Mann iteration process:*

*Then we have the following:*

1. *for all*
2. *for all*
3. *for all*

*Moreover, the -iteration process is faster than the Picard and Mann iteration processes.*

In particular, for , the -iteration process can be written as:

 (4)

**Lemma 1**. ([26], p.4) [lem1] Let be a nonnegative sequence for which one assumes there exists , such that for all one has satisfied the inequality

 (5)

where , for all , and . Then the following inequality holds

 (6)

For our convenience, we list the following hypotheses:

() The function in equation [(1)](#e1) and its derivatives with respect satisfy the condition

for , where and .

() , where denotes a positive constant such that for all

 The following theorem establishes the existence and uniqueness of the solution of equation 1. [(1)](#e1).

**Theorem 2**. *Assume that hypotheses hold. Let be a real sequence in satisfying . Then the equation* [*(1)*](#e1) *has a unique solution and normal iterative method* [*(4)*](#e4) *(with ) converges to with the following estimate:*

 (7)

***Proof.*** For , we define

for .
Differentiating [(8)](#e8) on both sides with respect to (see [7], p. 318), we have

for and .

Let and be iterative sequences generated by normal iteration method [(4)](#e4) for the operators given in [(8)](#e8) and [(9)](#e9) respectively.

We will show that as .

From method [(4)](#e4), equations [(3)](#e3), [(9)](#e9) and hypotheses, we obtain

Similarly,

Taking the supremum in the above inequalities, we get

and

respectively.
Therefore, using [(13)](#e13) in [(12)](#e12), we get

 (14)

Thus, by applying induction on k, we get

 (15)

Since for all , the assumption gives

 (16)

From the classical theory, we have

Hence, by using this fact with [(16)](#e16) in [(15)](#e15), we obtain

 (17)

This is [(7)](#e7). Since ,

 (18)

which implies . This gives as .

**Remark:** It is notable that inequality [(17)](#e17) gives the bounds in terms of known functions, which majorizes the iterations for solution of equation [(1)](#e1) as well as its derivatives.

# 3. Closeness of solution via iteration method:

Now, we discuss the continuous dependency of solutions of [(1)](#e1) on the functions. Consider the problem [(1)](#e1) and the corresponding problem

 (19)

for , where , , and is an arbitrary integer.

A solution to equation [(19)](#e19) refers to a continuous function , where belongs to the interval . This function must be continuously differentiable for times on , and it must also satisfy the equation [(19)](#e19). It is worth noting that the solution , along with its derivatives, meets the integral equations (see [7] , p.318)

for and

Following steps from the proof of Theorem [2](#thm2), for we define the operator for the equation [(19)](#e19)

for .
Differentiating both sides of [(21)](#e21) with respect to (see [7], p. 318), we get

for and .

The following theorem addresses the closeness of the solutions for problems [(1)](#e1) and [(19)](#e19).

**Theorem 3**. *Consider the sequences and generated by normal iterative method associated with operators in* [*(9)*](#e9) *and in* [*(22)*](#e22)*, respectively with the real sequence in satisfying for all . Assume that*

1. *all the conditions of Theorem 2 hold, and and are solutions of* [*(1)*](#e1) *and* [*(19)*](#e19) *respectively.*
2. *there exist nonnegative constants and such that*

(23)

*and*

*If the sequence converges to , then we have*

(25)

*where and .*

***Proof.*** Suppose the sequences and generated by normal iterative method associated with operators in [(9)](#e9)and in [(22)](#e22), respectively with the real control sequence in satisfying for all . From iterative method [(4)](#e4) and equations [(3)](#e3) with [(9)](#e9); [(20)](#e20) with [(22)](#e22) and hypotheses, we obtain

Similarly,

Taking supremum in the above inequalities, we get

and

respectively.
Therefore, using [(29)](#e29) in [(28)](#e28) and using hypothesis , and for all , the resulting inequality become

We denote

It is to be observed that inequality [(30)](#e30) satisfies all the conditions of Lemma 1, therefore, we get

By (i), we have . Using this fact and the assumption , we get from [(31)](#e31) that

 (32)

**Remark:** The inequality [(32)](#e32) shows how the solutions of the problems [(1)](#e1) and [(19)](#e19) are related. If the functions and are close to and , respectively, then not only are the solutions of the problems [(1)](#e1) and [(19)](#e19) closer to each other (i.e. ), but they also depend continuously on the functions involved. Additionally, this inequality estimates the derivatives of the solutions.

Now, we focus on analyzing how solutions depend continuously on certain parameters.
Consider the problems

 (33)

and

 (34)

for . The functions are defined as in [(1)](#e1) and are real parameters.

A solution to equation [(33)](#e33) is a continuous function defined on the interval , which is differentiable times and satisfies the equation [(33)](#e33). We can observe that both and its derivatives satisfy integral equations. (see [7], p.318)

for and

Now, following the steps from the proof of Theorem [2](#thm2), for , we define the operator for the equation [(33)](#e33)

for .
Taking derivatives on both sides of [(36)](#e36) with respect to (see [7], p. 318), we get

for and .
Similarly, for the equation [(34)](#e34), we define

for and

Again, following the steps from the proof of Theorem 2, for , we define the operator for the equation [(34)](#e34)

for .
Taking derivatives on both sides of [(39)](#e39) with respect to (see [7], p. 318), we get

for and .

The next theorem asserts that the solutions depend continuously on the parameters.

**Theorem 4**. *Consider the sequences and generated by normal iterative method associated with operators in* [*(37)*](#e37)*and in* [*(40)*](#e40)*, respectively with the real sequence in satisfying for all . Assume that*

1. *the hypothesis holds.*
2. *the function satisfy the conditions:*

 *for , where and .*

*Suppose and are solutions of* [*(33)*](#e33) *and* [*(34)*](#e34) *respectively and if the sequence converges to , then we have*

(43)

***Proof.*** Suppose the sequences and generated by normal iterative method associated with operators in [(37)](#e37)and in [(40)](#e40), respectively with the real sequence in satisfying for all . From iterative method [(4)](#e4) and equations [(35)](#e35) with [(37)](#e37); [(38)](#e38) with [(40)](#e40) and hypotheses, we obtain

Similarly, we have

Taking supremum in the above inequalities, we get

and

respectively.
Therefore, using [(47)](#e47) in [(46)](#e46) and using hypothesis , and for all , the resulting inequality become

We denote

Now, it is to be observed that, the inequality [(48)](#e48) satisfies all the conditions of Lemma 1, therefore, we get

By (i), we have . Using this fact and the assumption , we get from [(49)](#e49) that

 (50)

**Remark:** The concept of "dependence of solutions on parameters" refers to how the properties of a solution change when certain scalar parameters are varied. It is important to note that the initial conditions do not involve any parameters. However, the dependence on parameters plays a crucial role in many physical problems.

# 4. Example

We consider the following integral equation:

 (51)

Comparing this equation with proposed equation [(1)](#e1) for , we get

Now, we have

Taking sup norm, we obtain

where .
Therefore, we the estimate

 (54)

We define the operator by

 (55)

Since, all the conditions of Theorem 2 are satisfied, we can conclude that the sequence associated with the normal iterative method [(4)](#e4) for the operator in [(55)](#e55) converges to a unique solution .

Further, we also have for any

 (56)

for . The estimate obtained from [(56)](#e56) is a bound for the truncation error at the -th iteration of computation.

Next, we consider the perturbed integral equation:

 (57)

Similarly, comparing it with the equation [(19)](#e19) for , we have

Now, we define the mapping by

 (58)

According to Theorem 2, all conditions of the perturbed integral equation are satisfied. Consequently, the sequence related to the normal iterative method [(4)](#e4) for the operator in [(58)](#e58) converges to a unique solution .

The estimates below are what we have now:

 (59)

Let us consider two sequences and generated by the normal iterative method associated with operators in [(55)](#e55) and in [(58)](#e58), respectively. Here, approaches a limit as and approaches a limit as . Let be a real sequence in the interval such that for all . From Theorem 3, we can conclude that:

The statement highlights the extent to which solutions are influenced by the functions involved and how closely related these solutions are.

Now, we demonstrate how the solutions are dependent on real parameters. Let’s examine the integral equations that involve real parameters.:

 (62)

and

 (63)

Therefore, by using similar arguments and referencing Theorem 4, one can have

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