ON APPROXIMATION OF FUNCTION $\tilde{f} \in H_w$ CLASS BY *(C, 2)(E, 1)* MEANS OF CONJUGATE SERIES OF FOURIER SERIES.

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ABSTRACT- We studied on degree of approximation of function belonging to Hölder metric by *(C, 2) (E, 1)* mean has been discussed by Rathore, Shrivastava and Mishra. Since (E, 1*)* includes *(E, q)* method, so for obtaining more generalized result we replace *(E, q)* by *(E, 1)* mean. The Euler mean (E, 1) contains the summability method of generalized *Borel, Euler, Taylor etc.* In this chapter we obtain on approximation of function $\tilde{f} \in H_w$ class by *(C, 2)(E, 1)* means of conjugate series of Fourier series has been proved.

Mathematics Subject Classification: 42B05, 42B08
Keywords & Phrases: Degree of approximation, Hölder metric, *(C, 2)* summability, (E, 1*)* summability, *(C, 2)(E, 1)* product summability, Fourier series, Lebesgue integral.

1. INTRODUCTION

In this direction we studied on approximation of *f* belong to many classes also Hölder metric by Cesằro mean, Nörlund mean, Euler mean has been discussed by several investigator like respectively Alexits [2], Khan [6], Chandra [3], Mohapatra and Chandra [11], Das, Ghosh and Ray[4], etc. Further in this field several researchers like Lal and Kushwaha [8], Lal and Singh [9], Rathore and Shrivastava [14], Nigam [12], Albayrak, Koklu and Bayramov [1], Rathore, Shrivastava and Mishra ([15], [16],), Kushwaha [7], Singh and Mahajan [18], Mishra and Khatri [10] etc. Recently Rathore, Shrivastava and Mishra [17] has been determined on approximation of function in the Hölder metric by (C, 2) (E, q) product summability method of Fourier series. We extend the result on approximation of function $\tilde{f} \in H_w$ class by *(C, 2)(E, 1)* mean of conjugate series of Fourier series, has been proved.

2. DEFINITION AND NOTATIONS

Let *f (x)* be periodic with period *-2π* and integrable in the sense of Lebesgue. The Fourier series of *f(x)* is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx) \cong \sum_{n=0}^{\infty} A_n\,(x) \tag{2.1}$$

with n^{th} partial sum $S_n\,(f; x)$.
The conjugate series of Fourier series (2.1) is given by

$$\sum_{n=1}^{\infty}(b_n cosnx - a_n sinnx) \cong \sum_{n=1}^{\infty} B_n\,(x) \tag{2.2}$$

with n^{th} partial sum $\widetilde{S_n}\,(f; x)$
Let $w(t)$ and $w^*(t)$ denote two given moduli of continuity such that
$(w(t))^{\beta/\alpha} = O(w^*(t))$ as $t \to 0^+$ for $0< \beta \le \alpha \le 1$

Let $C_{2\pi}$ denote the Banach Spaces of all $2\,\pi -$ periodic continuous function defined on $[-\pi, \pi]$ under "sup'' norm for $0<\alpha \le 1$ and some positive constant K the function space H_w is defined by

$$H_w = \{f \epsilon\, C_{2\pi}: |f(x)-f(y)| \le K\, w\, |x-y|\}. \tag{2.3}$$

with the norm $\|.\|_{w^*}$ defined by

$$\|f\|_{w^*} = \|f\|_c + \operatorname*{Sup}_{x,y} \Delta^{w^*}\,[f(x,y)], \tag{2.4}$$

where

$$\|f\|_c = \operatorname*{Sup}_{-\pi \le x \le \pi} |f(x)|. \tag{2.5}$$

and

$\Delta^{w^*}\{f(\mathrm{x},\mathrm{y})\}= \frac{|\mathrm{f(x)-f(y)}|}{w^*\,(|\mathrm{x-y}|)}$, (x ≠ y). (2.6)

the convention that Δ^0 f(x, y)=0. If there exit positive constant B and K such that w |x-y| $\leq B\ |\mathrm{x-y}|^{\alpha}$ and w^*|x-y| $\leq K\ |\mathrm{x-y}|^{\beta}$ then

$H_\alpha = \{f \epsilon\ C_{2\pi}$: |f(x)-f(y)| ≤ K|x-y|$^\alpha$, 0<α≤ 1}. (see Prössdorf's[13]) (2.7)

H_α is Banach space and the metric induced (2.5) by the norm $\|.\|_\alpha$ on the H_α is called the Hölder metric. If can be seen that $\|f\|_\beta \leq (2\pi)^{\alpha-\beta}\|f\|_\alpha$ for 0≤β<α≤1.Thus $\{(H_{\alpha}, \|.\|_\alpha)\}$ is a family of Banach Spaces which decreases as α increase.

The series $\sum_{n=0}^{\infty} u_n$ is said to be (C, 2) summable to S. If the *(C, 2)* transform of S_n is defined as(see Hardy [5])

$$t_n^{\widetilde{(C,2)}}(f:x) = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)\,\widetilde{S_k} \to S \quad \text{as } n\to\infty \tag{2.8}$$

The $t_n^{\widetilde{(E,1)}}(f:x)$ denotes the transform of $\widetilde{(E,1)}$ is defined as

$$t_n^{\widetilde{(E,1)}}(f:x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\widetilde{S_k} \to S, \text{ as } n\to\infty$$

Thus if

$$t_n^{\widetilde{(C,2)(E,1)}}(f:x) = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)\, t_k^{\widetilde{(E,1)}} \to S \quad \text{as } n\to\infty \tag{2.9}$$

where $t_n^{\widetilde{(C,2)(E,1)}}$ denotes the sequence of $\widetilde{(\mathrm{C},2)(\mathrm{E},1)}$ product summability of the sequence S_n, the series $\sum_{n=0}^{\infty}\mathrm{u_n}$ is said to be summable $\widetilde{(\mathrm{C},2)(\mathrm{E},1)}$ to the definite number *S.* If

$$t_n^{\widetilde{(C,2)(E,1)}}(f:x) = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)\sum_{v=0}^{k}\binom{k}{v}\widetilde{S_v} \to S \quad \text{as } n\to\infty \tag{2.10}$$

The conjugate function $\widetilde{f(x)}$ is defined for almost every x by

$$\widetilde{f(x)} = -\frac{1}{2\pi}\int_0^{\pi}\varphi(t)\ \cot\frac{t}{2}\,dt$$
$$= \lim_{h\to 0}\left(-\frac{1}{2\pi}\int_h^{\pi}\varphi(t)\ \cot\frac{t}{2}\,dt\right) \tag{2.11}$$

"The degree of approximation E_n(f) be given by

$$E_n(f) = min\,\|T_n - f\|_p\,, \tag{2.12}$$

where $T_n(x)$ is a trigonometric polynomial of degree n" by (see Zygmund[20]).

We shall use following notation

$$\Phi_x(t) = f(x+t)+f(x-t)-2f(x) \tag{2.13}$$

and

$$\varphi(t) = \Phi_x(t) - \Phi_y(t). \tag{2.14}$$

3. Known Theorem.

Theorem 1 (see [18]). Let w(t) defined in ( ) br such that

$$\int_t^{\pi}\frac{w(u)}{u^2}\,du = O\,(H(t),\ H(t) \geq 0, \tag{3.1}$$

$$\int_0^t H(u)du = O(t\,H(t), \text{ as } t\to 0^+ \tag{3.2}$$

then, for 0< β≤ $\alpha \leq 1$ and f ϵ H_α , we have

$$\|t_n^{C^1.E^1}(f)\text{-}f(x)\|_{w^*} = O\left(\left((n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right)^{1-\beta/\alpha}\right) \tag{3.3}$$

4. MAIN THEOREM

We prove the following theorem

"On approximation of function $\tilde{f} \in H_w$ class by $(C, 2)(E, 1)$ mean of conjugate of Fourier series" has been established.

Theorem: "If $0\leq\beta<\alpha\leq1$ and $\tilde{f}\in H_w$ then

$$\| t_n^{(C,2)\widetilde{(E,1)}}(f;x)-\tilde{f}(x)\|_{w^*} = O\left\{\frac{w(|x-y|)^{\beta/\alpha}}{w^*(|x-y|)}(\log(n+1))^{\beta/\alpha}\left[(n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}\right\} \quad (4.1)$$

where $t_n^{(C,2)\widetilde{(E,1)}}$ is the product summability $(C,2)\widetilde{(E,1)}$ mean of $S_n(f;x)$".

5. **Lemmas**: We shall use the following lemmas-

Lemma 1. Let $\widetilde{M_n}(t) = \frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin t/2}\right\}\right]$

then $\widetilde{M_n}(t) = O\left(\frac{1}{t}\right)$, for $0 \leq t \leq \frac{\pi}{(n+1)}$

Proof Using $|\sin\frac{t}{2}| \geq \frac{t}{\pi}$ and $|\cos\left(v+\frac{1}{2}\right)t| \leq 1$, for $0 \leq t \leq \frac{\pi}{(n+1)}$

$$\begin{aligned}
|\widetilde{M_n(t)}| &= \frac{1}{\pi(n+1)(n+2)}\left|\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin t/2}\right\}\right]\right| \\
&= \frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\frac{|\cos\left(v+\frac{1}{2}\right)t|}{|\sin t/2|}\right\}\right] \\
&= \frac{1}{t(n+1)(n+2)}\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\right\}\right] \\
&= \frac{1}{t(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1) \qquad \left(\because \sum_{v=0}^{k}\binom{k}{v}=2^k\right) \\
&= \frac{(n+1)}{t(n+1)(n+2)} - \frac{1}{t(n+1)(n+2)}\sum_{k=0}^{n}k \\
&= \frac{1}{t(n+2)} - \frac{n(n+1)}{2t(n+1)(n+2)} \\
&= \frac{1}{t(n+2)} - \frac{n}{2t(n+2)} \\
&= O\left(\frac{1}{t}\right)
\end{aligned} \quad (5.1)$$

Lemma2. Let $\widetilde{M_n}(t) = \frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin t/2}\right\}\right]$

then $\widetilde{M_n}(t) = O\left(\frac{1}{t^2(n+2)}\right)$, for $\frac{\pi}{(n+1)} \leq t \leq \pi$

Proof- Using $|\sin\frac{t}{2}| \geq \frac{t}{\pi}$ and $|\sin t| \leq 1$ for $\frac{\pi}{(n+1)} \leq t \leq \pi$

$$\begin{aligned}
|\widetilde{M_n(t)}| &= \frac{1}{\pi(n+1)(n+2)}\left|\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin t/2}\right\}\right]\right| \\
&= \frac{1}{t(n+1)(n+2)}\left|\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\cos\left(v+\frac{1}{2}\right)t\right\}\right]\right| \\
&= \frac{1}{t^2(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1) \qquad \text{(see [])} \\
&= \frac{(n+1)}{t^2(n+1)(n+2)} - \frac{n(n+1)}{2t^2(n+1)(n+2)} \\
&= \frac{1}{t^2(n+2)}
\end{aligned} \quad (5.2)$$

Lemma 3. (see [18]). If w(t) satisfies condition (3.1) and (3.2) then

$$\int_0^u t^{-1}w(t)dt = O(u\,H(u)), \quad \text{as } u\to 0^+. \quad (5.3)$$

Lemma 4 Let $\Phi_x(t)$ defines (2.13) for $\tilde{f}\in H_w$

$$|\Phi_x(t) - \Phi_y(t)| \leq 2M\,w|x-y| \quad (5.4)$$

also

$$|\Phi_x(t) - \Phi_y(t)| \leq 2M\,w|t| \quad (5.5)$$

It is easy to verify.

6. PROOF OF THE MAIN THEOREM

Using (Titchmarsh [19]) and Riemann – Lebesgue theorem, the partial sum $S_n(f; x)$ of the series (2.1) is given by

$$\widetilde{S_n}\,(f;x) - \tilde{f}\,(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\phi_x(t)}{sin\frac{t}{2}}cos\left(n+\tfrac{1}{2}\right)t\ dt \tag{6.1}$$

If $t_n^{\widetilde{(E,1)}}$ denotes $\widetilde{(E,1)}$ transform of $\widetilde{S_n}\,(f;x)$ then

$$t_n^{\widetilde{(E,1)}}(f;x) - \tilde{f}(x) = \frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin^{t}/_{2}}\sum_{k=0}^{n}\binom{n}{k}\cos\left(k+\frac{1}{2}\right)t\ dt\ , \tag{6.2}$$

If $t_n^{(C,2)\widetilde{(E,1)}}$ denotes $(C,2)\widetilde{(E,1)}$ transform of $\widetilde{S_n}\,(f;x)$,

We write

$$t_n^{(C,2)\widetilde{(E,1)}}(f;x) - \tilde{f}(x) = \frac{1}{\pi(n+1)(n+2)}\ \sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\int_0^{\pi}\frac{\phi_x(t)}{\sin^{t}/_{2}}\left\{\sum_{v=0}^{k}\binom{k}{v}\cos\left(v+\frac{1}{2}\right)t\right\}\right] \tag{6.3}$$

Writing $I_n(x) = t_n^{(C,2)\widetilde{(E,1)}}(f;x) - \tilde{f}(x)$ we have

$$\left|I_n(x)\right| = \left|\ t_n^{(C,2)\widetilde{(E,1)}}(f;x) - \tilde{f}(x)\right|$$

$$\leq \left|\frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\int_0^{\pi}\frac{\phi_x(t)}{\sin^{t}/_{2}}\left\{\sum_{v=0}^{k}\binom{k}{v}\cos\left(v+\frac{1}{2}\right)t\right\}\right]\right|\ dt \tag{6.4}$$

$$\left|I_n(x) - I_n(y)\right|$$

$$= \left|\frac{1}{\pi(n+1)(n+2)}\ \sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\int_0^{\pi}\frac{\phi_x(t)-\phi_y(t)}{\sin^{t}/_{2}}\left\{\sum_{v=0}^{k}\binom{k}{v}\cos\left(v+\frac{1}{2}\right)t\right\}\right]\right| dt \tag{6.5}$$

$$= \frac{1}{\pi(n+1)(n+2)}\ \sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\int_0^{\pi}\frac{|\,\phi_x(t)-\phi_y(t)\,|}{\sin^{t}/_{2}}\left\{\sum_{v=0}^{k}\binom{k}{v}\cos\left(v+\frac{1}{2}\right)t\right\}\right] dt$$

$$= \frac{1}{\pi(n+1)(n+2)}\ \sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\int_0^{\pi}\frac{|\,\phi(t)\,|}{\sin^{t}/_{2}}\left\{\sum_{v=0}^{k}\binom{k}{v}\cos\left(v+\frac{1}{2}\right)t\right\}\right] dt$$

$$= \int_0^{\pi}|\,\phi(t)\,|\ |\,M_n(t)\,|\,dt \qquad \text{using Lemma 1}$$

$$= \left[\int_0^{\pi/_{n+1}} + \int_{\pi/_{n+1}}^{\pi}\right]|\,\phi(t)\,|\ |\,M_n(t)\,|\,dt$$

$$= I_1 + I_2 \tag{6.6}$$

Now using (5.5) and Lemma3

$$|I_1| = \int_0^{\pi/_{n+1}}|\phi(t)||M_n\,(t)|\,dt$$

$$= O\,(1)\int_0^{\pi/_{(n+1)}} t^{-1}\ w(t)dt$$

$$= O\left((n+1)^{-1}\,H\left(\frac{\pi}{n+1}\right)\right). \tag{6.7}$$

Now

$$|I_2| = \int_{\pi/_{n+1}}^{\pi}|\phi(t)||M_n\,(t)|dt \qquad \text{using (5.5) and Lemma 2}$$

$$= O(1)\int_{\pi/_{(n+1)}}^{\pi} t^{-2}\,w(t)\ dt$$

$$=O\left((n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right). \quad (6.8)$$

Now using (5.4), Lemma 1, we get

$$I_1 = O\left(\frac{1}{n+2}\right)\int_0^{\pi/(n+1)} t^{-1}\ w(|x-y|)dt$$

$$= O\ (w(|x-y|))\int_0^{\pi/(n+1)} t^{-1}\ dt$$

$$= O\ (\log\ (n+1)\ w(|x-y|)) \quad (6.9)$$

Now using (5.4) and Lemma2

$$I_2 = O\left(\frac{1}{n+2}\right)\int_{\pi/(n+1)}^{\pi} t^{-2}\ w(|x-y|)\ dt$$

$$= O\ (w(|x-y|)). \quad (6.10)$$

We observe that

$$|I_k| = |I_k|^{1-\beta/\alpha}\ |I_k|^{\beta/\alpha}. \quad \text{when} \quad k = 1, 2 \quad (6.11)$$

By using (6.7) and (6.9) respectively in the first and the second factor on the right of the above identify (6.11) for k = 1 we obtain that

$$|I_1| = O\left(\left[(n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}.\left[\log(n+1)\ w(|x-y|)\right]^{\beta/\alpha}\right) \quad (6.12)$$

Again using (6.8) and (6.10) in the first and second factor on the right of the identify (6.11) for k = 2 we have

$$|I_2| = O\left(\left[(n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}.\left[w(|x-y|)\right]^{\beta/\alpha}\right) \quad (6.13)$$

Thus from (2.6), (6.12) and (6.13) we get

$$\sup_{x\neq y}\left|\ \Delta^{w^*} I_n\ (x, y)\ \right| = \sup_{\substack{(x,y)\\ x\neq y}}\frac{|I_n(x)-I_n(y)|}{w^*(|x-y|)}$$

$$=O\left\{\frac{w(|x-y|)^{\beta/\alpha}}{w^*(|x-y|)}(\log(n+1))^{\beta/\alpha}\left[(n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}\right\} \quad (6.14)$$

Using the fact that $\tilde{f}\epsilon\ H_w \Rightarrow \phi_x(t) = O\ (w(t))$

Proceeding as above we obtain

$$\|\ I_n\|_c = \sup_{-\pi\leq x\leq\pi}\ \|\ t_n^{(C,2)\widetilde{(E,1)}}(f;x)-\tilde{f}(x)\|$$

$$=O\ \left\{(n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right\}. \quad (6.15)$$

Combining the result of (6.14) and (6.15), we get

$$\|\ t_n^{(C,2)\widetilde{(E,1)}}(f;x)-\tilde{f}(x)\|_{w^*} = O\left\{\frac{w(|x-y|)^{\beta/\alpha}}{w^*(|x-y|)}(\log(n+1))^{\beta/\alpha}\left[(n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}\right\} \quad (6.16)$$

Proof of the main theorem is completes.

7. Corollaries:
From our main theorem can be derived following corollaries.

Corollary7. 1: "If β = 0 and $\tilde{f} \in Lip(\alpha,\ p)$, 0< $\alpha \leq 1$ then

$$\| t_n^{\widetilde{(C,2)(E,1)}}(f;x) - \tilde{f}(x)\|_c = O\left\{\frac{1}{(n+1)^{\alpha}}\right\} \quad \text{for } 0< \alpha < 1.$$

$$= O\left(\frac{\log(n+1)}{(n+1)}\right), \text{ for } \alpha = 1$$

Acknowledgement- We are highly thankful to **Dr. U. K. Shrivastava,** Head, Department of Mathematics, Govt. E. R. Rao science PG College Bilaspur, Chhattisgarh, India for his encouragement and support to this work.

Conclusion

The summability method F(a, q) includes method of summability like Borel, *(E, 1), (E, q), (e, c)* and *[F, d_n]* then by using the result of main theorem we can derive more generalizing result and also the result of J. K. Kushwaha [6] can be derived directly.

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