EXISTENCE OF SOLUTIONS FOR NONLINEAR IMPLICIT FREDHOLM INTEGRODIFFERENTIAL EQUATION VIA S-ITERATION METHOD

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ABSTRACT. In this chapter, we study the existence and other properties of solutions of nonlinear implicit Fredholm integrodifferential equation. The tool employed in the analysis is based on application of $S-$ iteration method (refer [23] for more details).

Key words: Existence, $S-$iteration, Fredholm integrodifferential equation, Continuous dependence, Parameters.

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1. INTRODUCTION

Consider the nonlinear integrodifferential equation of the type:

$$x(t) = \mathcal{G}(t) + \int_a^b \mathcal{F}\Big(t, s, x(s), x'(s), \cdots, x^{(n-1)}(s), x(a), x'(a), \cdots, x^{(n-1)}(a), x(b), x'(b), \cdots, x^{(n-1)}(b)\Big) ds, \tag{1}$$

for $t \in I = [a, b]$. Let $\mathbb{R}$ denote the set of real numbers, $E = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (n times) be the product space and $\mathbb{R}_+ = [0, \infty)$ be the given subset of $\mathbb{R}$. We assume $\mathcal{F} \in C(I^2 \times E^3, \mathbb{R})$, $\mathcal{G} \in C(I, \mathbb{R})$, and $n \geq 1$.

Several researchers have been introduced many iteration methods for certain classes of operators in the sense of their convergence, equivalence of convergence and rate of convergence etc. (see [1, 3, 5, 8, 9, 15, 16, 17, 18, 19, 20, 23, 24, 25]). The most of iterations devoted for both analytical and numerical approaches. The $S-$ iteration method, due to simplicity and fastness, has attracted the attention and hence, it is used in this chapter.

The sufficient literature exists dealing with the special and even more general version of the equation (1) by using different techniques [2, 6, 10, 11, 12, 13, 14, 21, 22] and some of references cited therein. In recently, Yunus Atalan, Faik Gürsoy and Abdul Rahim Khan [4] has studied the special version of equation (1) for different qualitative properties of solutions. Authors are motivated by the work of D. R. Sahu [23] and influenced by [4].

The main objective of this chapter is to use normal $S-$iteration method to establish the solution of the problem (1). Also, we give a data dependence result for the integral equation (1) with the help of normal $S-$iteration method.

2. EXISTENCE OF SOLUTION VIA $S-$ITERATION

For continuous functions $x^{(j)} : I \to \mathbb{R}$ ($j = 0, 1, \cdots, n-1$), we denote by

$$|x(t)|_E = \sum_{j=0}^{n-1} |x^{(j)}(t)|,$$

for $(x(t), x'(t), \cdots, x^{(n-1)}(t)) \in E,\ t \in I$. We define $B = C^{n-1}(I) = C^{n-1}(I, \mathbb{R})$, is a space of those functions x which are $(n-1)$ times continuously differentiable on I endowed with norm

$$\|x\|_B = \max_{t \in I} \big\{ |x(t)|_E \big\}. \tag{2}$$

It is easy to see that B with norm defined by (2) is a Banach space.

By a solution of equation (1), we mean a continuous function $x(t),\ t \in I$ which is $(n-1)$ times continuously differentiable on I and satisfies the equation (1). It is easy to observe that the solution $x(t)$ of the equation (1) and its derivatives satisfy the integral equations (see [7], p.318)

$$\begin{aligned} x^{(j)}(t) &= \mathcal{G}^{(j)}(t) \\ &\quad + \int_a^b \frac{\partial^j}{\partial t^j} \mathcal{F}\Big(t, s, x(s), x'(s), \cdots, x^{(n-1)}(s), x(a), x'(a), \cdots, x^{(n-1)}(a), x(b), x'(b), \cdots, x^{(n-1)}(b)\Big) ds, \end{aligned} \tag{3}$$

for $t \in I$ and $0 \le j \le n-1$

We need the following pair of known results:

Theorem 1. *(([23], p.194)) Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a contraction operator with contractivity factor $k \in [0, 1)$ and fixed point $x*$. Let α_n and β_n be two real sequences in $[0, 1]$ such that $\alpha \le \alpha_n \le 1$ and $\beta \le \beta_n < 1$ for all $n \in \mathbb{N}$ and for some $\alpha, \beta > 0$. For given $u_1 = v_1 = w_1 \in C$, define sequences u_n, v_n and w_n in C as follows:*

S-iteration process: $\begin{cases} u_{n+1} = (1-\alpha_n)Tu_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)u_n + \beta_n T u_n, n \in \mathbb{N}. \end{cases}$

Picard iteration: $v_{n+1} = Tv_n, n \in \mathbb{N}$.

Mann iteration process: $w_{n+1} = (1-\beta_n)w_n + \beta_n T w_n, n \in \mathbb{N}$.

Then we have the following:

(a) $\|u_{n+1} - x^*\| \le k^n \Big[1 - (1-k)\alpha\beta\Big]^n \|u_1 - x^*\|$, *for all* $n \in \mathbb{N}$.
(b) $\|v_{n+1} - x^*\| \le k^n \|v_1 - x^*\|$, *for all* $n \in \mathbb{N}$.
(c) $\|w_{n+1} - x^*\| \le \Big[1 - (1-k)\beta\Big]^n \|w_1 - x^*\|$, *for all* $n \in \mathbb{N}$.

Moreover, the S-iteration process is faster than the Picard and Mann iteration processes.

In particular, for $\alpha_n = 1,\ n \in \mathbb{N}$, the S-iteration process can be written as:

$$\begin{cases} u_1 \in C, \\ u_{n+1} = T y_n, \\ y_n = (1-\beta_n)u_n + \beta_n T u_n,\ n \in \mathbb{N}. \end{cases} \tag{4}$$

Lemma 1. (([25], p.4)) Let $\{\beta_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which one assumes there exists $n_0 \in \mathbb{N}$, such that for all $n \ge n_0$ one has satisfied the inequality

$$\beta_{n+1} \le (1-\mu_n)\beta_n + \mu_n \gamma_n, \tag{5}$$

where $\mu_n \in (0, 1)$, for all $n \in \mathbb{N}$, $\displaystyle\sum_{n=0}^{\infty} \mu_n = \infty$ and $\gamma_n \ge 0,\ \forall n \in \mathbb{N}$. Then the following inequality holds

$$0 \le \limsup_{n \to \infty} \beta_n \le \limsup_{n \to \infty} \gamma_n. \tag{6}$$

We list the following hypotheses for our convenience:

(H_1) The function $\mathcal{F}$ in equation (1) and its derivatives with respect t satisfy the condition

$$\begin{aligned}&|\frac{\partial^j}{\partial t^j}\mathcal{F}\Big(t,s,x(t),x'(t),\cdots,x^{(n-1)}(t),x(a),x'(a),\cdots,x^{(n-1)}(a),x(b),x'(b),\cdots,x^{(n-1)}(b)\Big)\\&\quad-\frac{\partial^j}{\partial t^j}\mathcal{F}\Big(t,s,y(s),y'(s),\cdots,y^{(n-1)}(s),y(a),y'(a),\cdots,y^{(n-1)}(a),y(b),y'(b),\cdots,y^{(n-1)}(b)\Big)|\\&\leq p_j(t,s)\Big[\alpha\sum_{i=0}^{n-1}|x^{(i)}(s)-y^{(i)}(s)|+\beta\sum_{i=0}^{n-1}|x^{(i)}(a)-y^{(i)}(a)|+\gamma\sum_{i=0}^{n-1}|x^{(i)}(b)-y^{(i)}(b)|\Big],\end{aligned}$$

for $j=0,1,\cdots,n-1$, where $p_j(t,s)\in C(I^2,\mathbb{R}_+)$ and $\alpha,\beta,\gamma>0$.

(H_2) $M_{\mathcal{F}}\Big(\alpha+\beta+\gamma\Big)(b-a)<1$,

where $M_{\mathcal{F}}$ denotes a positive constant such that for all $t,s\in I$

$$M_{\mathcal{F}}=\max\Big\{\sum_{j=0}^{n-1}p_j(t,s):(t,s)\in I^2\Big\}.$$

Now, we are able to state and prove the following theorem which deals with the existence of solutions of the equation (1).

Theorem 2. *Assume that hypotheses $(H_1)-(H_2)$ hold. Let $\{\xi_k\}_{k=0}^{\infty}$ be a real sequence in $[0,1]$ satisfying $\sum_{k=0}^{\infty}\xi_k=\infty$. Then the equation (1) has a unique solution $x\in B$ and normal $S-$iterative method (4) (with $u_1=x_0$) converges to $x\in B$ with the following estimate:*

$$\|x_{k+1}-x\|_B\leq\frac{\Big[M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\Big]^{k+1}}{e^{\Big[1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\Big]\sum_{i=0}^{k}\xi_i}}\|x_0-x\|_B. \tag{7}$$

Proof. Let $x(t)\in B$ and define the operator

$$\begin{aligned}(Tx)(t)&=\mathcal{G}(t)\\&\quad+\int_a^b\mathcal{F}\Big(t,s,x(s),x'(s),\cdots,x^{(n-1)}(s),x(a),x'(a),\cdots,x^{(n-1)}(a),x(b),x'(b),\cdots,x^{(n-1)}(b)\Big)ds,\end{aligned}\tag{8}$$

for $t\in I=[a,b]$.

Differentiating both sides of (8) with respect to t (see [7], p. 318), we have

$$\begin{aligned}(Tx)^{(j)}(t)&=\mathcal{G}^{(j)}(t)\\&\quad+\int_a^b\frac{\partial^j}{\partial t^j}\mathcal{F}\Big(t,s,x(s),x'(s),\cdots,x^{(n-1)}(s),x(a),x'(a),\cdots,x^{(n-1)}(a),x(b),x'(b),\cdots,x^{(n-1)}(b)\Big)ds,\end{aligned}\tag{9}$$

for $t\in I$ and $0\leq j\leq n-1$.

Let $\{x_k\}_{k=0}^{\infty}$ and $\{x_k^{(j)}\}_{k=0}^{\infty}$, $(j=1,\cdots,n-1)$ be iterative sequences generated by normal $S-$iteration method (4) for the operators given in (8) and (9) respectively.

We will show that $x_k\to x$ as $k\to\infty$.

From iteration (4) and equations (3), (9) and hypotheses, we obtain

$$|x_{k+1}(t)-x(t)|_E$$

$$
\begin{aligned}
&= \sum_{j=0}^{n-1} |{x_{k+1}}^{(j)}(t) - x^{(j)}(t)| \\
&= \sum_{j=0}^{n-1} |(Ty_k)^{(j)}(t) - (Tx)^{(j)}(t)| \\
&= \sum_{j=0}^{n-1} \Big| \mathcal{G}^{(j)}(t) \\
&\quad + \int_a^b \frac{\partial^j}{\partial t^j} \mathcal{F}\Big(t, s, y_k(s), y_k'(s), \cdots, {y_k}^{(n-1)}(s), y_k(a), {y_k}'(a), \cdots, {y_k}^{(n-1)}(a), y_k(b), {y_k}'(b), \cdots, {y_k}^{(n-1)}(b)\Big) ds \\
&\quad - \mathcal{G}^{(j)}(t) \\
&\quad - \int_a^b \frac{\partial^j}{\partial t^j} \mathcal{F}\Big(t, s, x(s), x'(s), \cdots, x^{(n-1)}(s), x(a), x'(a), \cdots, x^{(n-1)}(a), x(b), x'(b), \cdots, x^{(n-1)}(b)\Big) ds \Big| \\
&= \sum_{j=0}^{n-1} \int_a^b \Big| \frac{\partial^j}{\partial t^j} \mathcal{F}\Big(t, s, y_k(s), y_k'(s), \cdots, {y_k}^{(n-1)}(s), y_k(a), {y_k}'(a), \cdots, {y_k}^{(n-1)}(a), y_k(b), {y_k}'(b), \cdots, {y_k}^{(n-1)}(b)\Big) \\
&\quad - \frac{\partial^j}{\partial t^j} \mathcal{F}\Big(t, s, x(s), x'(s), \cdots, x^{(n-1)}(s), x(a), x'(a), \cdots, x^{(n-1)}(a), x(b), x'(b), \cdots, x^{(n-1)}(b)\Big) \Big| ds \\
&\le \sum_{j=0}^{n-1} \int_a^b p_j(t,s) \Big[\alpha \sum_{i=0}^{n-1} |(y_k)^{(i)}(s) - x^{(i)}(s)| + \beta \sum_{i=0}^{n-1} |(y_k)^{(i)}(a) - x^{(i)}(a)| + \gamma \sum_{i=0}^{n-1} |(y_k)^{(i)}(b) - x^{(i)}(b)| \Big] ds \\
&\le M_{\mathcal{F}} \int_a^b \Big[\alpha \sum_{i=0}^{n-1} |(y_k)^{(i)}(s) - x^{(i)}(s)| + \beta \sum_{i=0}^{n-1} |(y_k)^{(i)}(a) - x^{(i)}(a)| + \gamma \sum_{i=0}^{n-1} |(y_k)^{(i)}(b) - x^{(i)}(b)| \Big] ds. \qquad (10)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&|y_k(t) - x(t)|_E \\
&= \sum_{j=0}^{n-1} |{y_k}^{(j)}(t) - x^{(j)}(t)| \\
&= \sum_{j=0}^{n-1} \Big[(1-\xi_k) |{x_k}^{(j)}(t) - x^{(j)}(t)| + \xi_k |(Tx_k)^{(j)}(t) - (Tx)^{(j)}(t)| \Big] \\
&= \Big[(1-\xi_k) \sum_{j=0}^{n-1} |{x_k}^{(j)}(t) - x^{(j)}(t)| + \xi_k \sum_{j=0}^{n-1} |(Tx_k)^{(j)}(t) - (Tx)^{(j)}(t)| \Big] \\
&\le \Big[(1-\xi_k) \sum_{j=0}^{n-1} |{x_k}^{(j)}(t) - x^{(j)}(t)| \\
&\quad + \xi_k M_{\mathcal{F}} \int_a^b \Big[\alpha \sum_{i=0}^{n-1} |(x_k)^{(i)}(s) - x^{(i)}(s)| + \beta \sum_{i=0}^{n-1} |(x_k)^{(i)}(a) - x^{(i)}(a)| + \gamma \sum_{i=0}^{n-1} |(x_k)^{(i)}(b) - x^{(i)}(b)| \Big] ds. \qquad (11)
\end{aligned}
$$

Now, by taking supremum in the above inequalities, we obtain

$$
\begin{aligned}
\|x_{k+1} - x\|_B &\le M_{\mathcal{F}} \int_a^b \Big[\alpha + \beta + \gamma \Big] \|y_k - x\|_B ds \\
&= M_{\mathcal{F}} \Big[\alpha + \beta + \gamma \Big] (b-a) \|y_k - x\|_B, \qquad (12)
\end{aligned}
$$

and

$$
\begin{aligned}
\|y_k - x\|_B &\le \Big[(1-\xi_k)\|x_k - x\|_B + \xi_k M_{\mathcal{F}} \int_a^b \Big(\alpha + \beta + \gamma \Big) \|x_k - x\|_B ds \Big] \\
&= \Big[(1-\xi_k)\|x_k - x\|_B + \xi_k M_{\mathcal{F}} \Big(\alpha + \beta + \gamma \Big) (b-a) \|x_k - x\|_B \Big]
\end{aligned}
$$

$$= \Big[(1-\xi_k) + \xi_k M_{\mathcal{F}}\Big(\alpha+\beta+\gamma\Big)(b-a)\Big]\|x_k - x\|_B$$

$$= \Big[1-\xi_k\Big(1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)\Big]\|x_k - x\|_B, \tag{13}$$

respectively.

Therefore, using (13) in (12), we have

$$\|x_{k+1} - x\|_B \le \Big(M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)\Big[1-\xi_k\Big(1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)\Big]\|x_k - x\|_B. \tag{14}$$

Thus, by induction, we get

$$\|x_{k+1} - x\|_B \le \Big(M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)^{k+1}\prod_{j=0}^{k}\Big[1-\xi_j\Big(1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)\Big]\|x_0 - x\|_B. \tag{15}$$

Since $\xi_k \in [0,1]$ for all $k \in \mathbb{N}$, the assumption (H_2) yields

$$\xi_k \le 1 \quad \text{and} \quad M_{\mathcal{F}}\Big(\alpha+\beta+\gamma\Big)(b-a) < 1$$

$$\Rightarrow \xi_k M_{\mathcal{F}}\Big(\alpha+\beta+\gamma\Big)(b-a) < \xi_k$$

$$\Rightarrow \xi_k\Big[1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big] < 1, \ \forall\, k \in \mathbb{N}. \tag{16}$$

From the classical analysis, we know that

$$1 - x \le e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots, \ x \in [0,1].$$

Hence by utilizing this fact with (16) in (15), we obtain

$$\|x_{k+1} - x\|_B \le \Big(M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)^{k+1} e^{-\Big(1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)\sum_{j=0}^{k}\xi_j}\|x_0 - x\|_B. \tag{17}$$

This is (7). Since $\sum_{k=0}^{\infty}\xi_k = \infty$,

$$e^{-\Big(1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)\sum_{j=0}^{k}\xi_j} \to 0 \quad \text{as} \quad k \to \infty, \tag{18}$$

which implies $\lim_{k\to\infty}\|x_{k+1} - x\|_B = 0$. This gives $x_k \to x$ as $k \to \infty$. □

Remark: It is an interesting to note that the inequality (17) gives the bounds in terms of known functions, which majorizes the iterations for solutions of the equation (1) as well as its derivatives $x^{(j)}(t)$, $(j = 1, 2, \cdots, n-1)$ for $t \in I$.

3. Closeness of solution via $S-$iteration

We study the continuous dependence of solutions of (1) on the function involved therein.

Now, we consider the problem (1) and the corresponding problem

$$\overline{x}(t) = \mathcal{H}(t) + \int_a^b \mathcal{L}\Big(t, s, \overline{x}(s), \overline{x}'(s), \cdots, \overline{x}^{(n-1)}(s), \overline{x}(a), \overline{x}'(a), \cdots, \overline{x}^{(n-1)}(a), \overline{x}(b), \overline{x}'(b), \cdots, \overline{x}^{(n-1)}(b)\Big)ds, \tag{19}$$

for $t \in I = [a,b]$, where $\mathcal{L} \in C(I^2 \times E^3, \mathbb{R})$, $\mathcal{H} \in C(I, \mathbb{R})$, and $n \ge 1$ is an arbitrary integer.

By a solution of equation (19), we mean a continuous function $\overline{x}(t),\ t \in I$ which is $(n-1)$ times continuously differentiable on I and satisfies the equation (19). It is easy to observe that the solution $\overline{x}(t)$ of the equation (19) and its derivatives satisfy the integral equations (see [7], p.318)

$$\overline{x}^{(j)}(t) = \mathcal{H}^{(j)}(t) + \int_a^b \frac{\partial^j}{\partial t^j}\mathcal{L}\Big(t, s, \overline{x}(s), \overline{x}'(s), \cdots, \overline{x}^{(n-1)}(s), \overline{x}(a), \overline{x}'(a), \cdots, \overline{x}^{(n-1)}(a), \overline{x}(b), \overline{x}'(b), \cdots, \overline{x}^{(n-1)}(b)\Big)ds, \tag{20}$$

for $t \in I$ and $0 \le j \le n-1$

Let $\overline{x}(t) \in B$ and following steps from the proof of Theorem 2, define the operator for the equation (19)

$$(\overline{T}\overline{x})(t) = \mathcal{H}(t) + \int_a^b \mathcal{L}\Big(t, s, \overline{x}(s), \overline{x}'(s), \cdots, \overline{x}^{(n-1)}(s), \overline{x}(a), \overline{x}'(a), \cdots, \overline{x}^{(n-1)}(a), \overline{x}(b), \overline{x}'(b), \cdots, \overline{x}^{(n-1)}(b)\Big)ds, \tag{21}$$

for $t \in I = [a, b]$.

Differentiating both sides of (21) with respect to t (see [7], p. 318), we have

$$(\overline{T}\overline{x})^{(j)}(t) = \mathcal{H}^{(j)}(t) + \int_a^b \frac{\partial^j}{\partial t^j}\mathcal{L}\Big(t, s, \overline{x}(s), \overline{x}'(s), \cdots, \overline{x}^{(n-1)}(s), \overline{x}(a), \overline{x}'(a), \cdots, \overline{x}^{(n-1)}(a), \overline{x}(b), \overline{x}'(b), \cdots, \overline{x}^{(n-1)}(b)\Big)ds, \tag{22}$$

for $t \in I$ and $0 \le j \le n-1$.

The next theorem deals with the closeness of solutions of the problems (1) and (19).

Theorem 3. *Consider the sequences* $\{x_k\}_{k=0}^{\infty}$ *and* $\{\overline{x}_k\}_{k=0}^{\infty}$ *generated normal* $S-$ *iterative method associated with operators* T *in* (9) *and* $\overline{T}$ *in* (22), *respectively with the real sequence* $\{\xi_k\}_{k=0}^{\infty}$ *in* $[0, 1]$ *satisfying* $\frac{1}{2} \le \xi_k$ *for all* $k \in \mathbb{N}$. *Assume that*

(i) *all conditions of Theorem 2 hold, and* $x(t)$ *and* $\overline{x}(t)$ *are solutions of* (1) *and* (19) *respectively.*

(ii) *there exist non negative constants* ϵ_j *and* $\overline{\epsilon}_j$ *such that*

$$|\mathcal{G}^{(j)}(t) - \mathcal{H}^{(j)}(t)| \le \epsilon_j,\ \forall\ t \in I,\ (j = 0, 1, \cdots, n-1), \tag{23}$$

and

$$\begin{aligned}&\Big|\frac{\partial^j}{\partial t^j}\mathcal{F}\Big(t, s, x(t), x'(t), \cdots, x^{(n-1)}(t), x(a), x'(a), \cdots, x^{(n-1)}(a), x(b), x'(b), \cdots, x^{(n-1)}(b)\Big)\\ &\quad - \frac{\partial^j}{\partial t^j}\mathcal{L}\Big(t, s, x(t), x'(t), \cdots, x^{(n-1)}(t), x(a), x'(a), \cdots, x^{(n-1)}(a), x(b), x'(b), \cdots, x^{(n-1)}(b)\Big)\Big|\\ &\le p_j(t, s)\overline{\epsilon}_j,\ \forall\ t \in I,\ (j = 0, 1, \cdots, n-1).\end{aligned} \tag{24}$$

If the sequence $\{\overline{x}_k\}_{k=0}^{\infty}$ *converges to* $\overline{x}$, *then we have*

$$\|x - \overline{x}\|_B \le \frac{3\Big[M_{\mathcal{F}}\big((b-a)\big)\overline{\epsilon} + \epsilon\Big]}{1 - M_{\mathcal{F}}\big(\alpha + \beta + \gamma\big)(b-a)}, \tag{25}$$

where $\epsilon = \sum_{j=0}^{n-1} \epsilon_j$ *and* $\overline{\epsilon} = \sum_{j=0}^{n-1} \overline{\epsilon}_j$.

Proof. Suppose the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{\overline{x}_k\}_{k=0}^{\infty}$ generated normal $S-$ iterative method associated with operators T in (9)and $\overline{T}$ in (22), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N}$. From iteration (4) and equations (3) with (9); (20) with (22) and hypotheses, we obtain

$$
\begin{aligned}
&|x_{k+1}(t) - \overline{x}_{k+1}(t)|_E \\
&= \sum_{j=0}^{n-1} |{x_{k+1}}^{(j)}(t) - {\overline{x}_{k+1}}^{(j)}(t)| \\
&= \sum_{j=0}^{n-1} |(Ty_k)^{(j)}(t) - (\overline{T}\overline{y}_k)^{(j)}(t)| \\
&= \sum_{j=0}^{n-1} \Big| \mathcal{G}^{(j)}(t) - \mathcal{H}^{(j)}(t) \\
&\quad + \int_a^b \frac{\partial^j}{\partial t^j} \mathcal{F}\Big(t, s, y_k(s), y_k'(s), \cdots, {y_k}^{(n-1)}(s), y_k(a), {y_k}'(a), \cdots, {y_k}^{(n-1)}(a), y_k(b), {y_k}'(b), \cdots, {y_k}^{(n-1)}(b)\Big) ds \\
&\quad - \int_a^b \frac{\partial^j}{\partial t^j} \mathcal{L}\Big(t, s, \overline{y}_k(s), {\overline{y}_k}'(s), \cdots, {\overline{y}_k}^{(n-1)}(s), \overline{y}_k(a), {\overline{y}_k}'(a), \cdots, {\overline{y}_k}^{(n-1)}(a), \overline{y}_k(b), {\overline{y}_k}'(b), \cdots, {\overline{y}_k}^{(n-1)}(b)\Big) ds \Big| \\
&\leq \sum_{j=0}^{n-1} \epsilon_j \\
&\quad + \sum_{j=0}^{n-1} \int_a^b \Big| \frac{\partial^j}{\partial t^j} \mathcal{F}\Big(t, s, y_k(s), y_k'(s), \cdots, {y_k}^{(n-1)}(s), y_k(a), {y_k}'(a), \cdots, {y_k}^{(n-1)}(a), y_k(b), {y_k}'(b), \cdots, {y_k}^{(n-1)}(b)\Big) \\
&\quad - \frac{\partial^j}{\partial t^j} \mathcal{L}\Big(t, s, \overline{y}_k(s), {\overline{y}_k}'(s), \cdots, {\overline{y}_k}^{(n-1)}(s), \overline{y}_k(a), {\overline{y}_k}'(a), \cdots, {\overline{y}_k}^{(n-1)}(a), \overline{y}_k(b), {\overline{y}_k}'(b), \cdots, {\overline{y}_k}^{(n-1)}(b)\Big) \Big| ds \\
&\leq \sum_{j=0}^{n-1} \epsilon_j \\
&\quad + \sum_{j=0}^{n-1} \int_a^b \Big| \frac{\partial^j}{\partial t^j} \mathcal{F}\Big(t, s, y_k(s), y_k'(s), \cdots, {y_k}^{(n-1)}(s), y_k(a), {y_k}'(a), \cdots, {y_k}^{(n-1)}(a), y_k(b), {y_k}'(b), \cdots, {y_k}^{(n-1)}(b)\Big) \\
&\quad - \frac{\partial^j}{\partial t^j} \mathcal{F}\Big(t, s, \overline{y}_k(s), {\overline{y}_k}'(s), \cdots, {\overline{y}_k}^{(n-1)}(s), \overline{y}_k(a), {\overline{y}_k}'(a), \cdots, {\overline{y}_k}^{(n-1)}(a), \overline{y}_k(b), {\overline{y}_k}'(b), \cdots, {\overline{y}_k}^{(n-1)}(b)\Big) \Big| ds \\
&\quad + \sum_{j=0}^{n-1} \int_a^b \Big| \frac{\partial^j}{\partial t^j} \mathcal{F}\Big(t, s, \overline{y}_k(s), {\overline{y}_k}'(s), \cdots, {\overline{y}_k}^{(n-1)}(s), \overline{y}_k(a), {\overline{y}_k}'(a), \cdots, {\overline{y}_k}^{(n-1)}(a), \overline{y}_k(b), {\overline{y}_k}'(b), \cdots, {\overline{y}_k}^{(n-1)}(b)\Big) \\
&\quad - \frac{\partial^j}{\partial t^j} \mathcal{L}\Big(t, s, \overline{y}_k(s), {\overline{y}_k}'(s), \cdots, {\overline{y}_k}^{(n-1)}(s), \overline{y}_k(a), {\overline{y}_k}'(a), \cdots, {\overline{y}_k}^{(n-1)}(a), \overline{y}_k(b), {\overline{y}_k}'(b), \cdots, {\overline{y}_k}^{(n-1)}(b)\Big) \Big| ds \\
&\leq \sum_{j=0}^{n-1} \epsilon_j + M_{\mathcal{F}} \sum_{j=0}^{n-1} \int_a^b \overline{\epsilon}_j ds \\
&\quad + \sum_{j=0}^{n-1} \int_a^b p_j(t,s) \Big[\alpha \sum_{i=0}^{n-1} |(y_k)^{(i)}(s) - {\overline{y}_k}^{(i)}(s)| + \beta \sum_{i=0}^{n-1} |(y_k)^{(i)}(a) - {\overline{y}_k}^{(i)}(a)| + \gamma \sum_{i=0}^{n-1} |(y_k)^{(i)}(b) - {\overline{y}_k}^{(i)}(b)| \Big] ds \\
&\leq \epsilon + M_{\mathcal{F}} \overline{\epsilon}(b-a) \\
&\quad + M_{\mathcal{F}} \int_a^b \Big[\alpha \sum_{i=0}^{n-1} |(y_k)^{(i)}(s) - {\overline{y}_k}^{(i)}(s)| + \beta \sum_{i=0}^{n-1} |(y_k)^{(i)}(a) - {\overline{y}_k}^{(i)}(a)| + \gamma \sum_{i=0}^{n-1} |(y_k)^{(i)}(b) - {\overline{y}_k}^{(i)}(b)| \Big] ds. \qquad (26)
\end{aligned}
$$

Similarly, we have

$$|y_k(t) - \overline{y}_k(t)|_E$$

$$
\begin{aligned}
&= \sum_{j=0}^{n-1} |{y_k}^{(j)}(t) - {\overline{y}_k}^{(j)}(t)| \\
&= \sum_{j=0}^{n-1} \Big[(1-\xi_k)|{x_k}^{(j)}(t) - {\overline{x}_k}^{(j)}(t)| + \xi_k |(Tx_k)^{(j)}(t) - (\overline{T}\overline{x}_k)^{(j)}(t)|\Big] \\
&= \Big[(1-\xi_k)\sum_{j=0}^{n-1} |{x_k}^{(j)}(t) - {\overline{x}_k}^{(j)}(t)| + \xi_k \sum_{j=0}^{n-1} |(Tx_k)^{(j)}(t) - (\overline{T}\overline{x}_k)^{(j)}(t)|\Big] \\
&\leq (1-\xi_k)\sum_{j=0}^{n-1} |{x_k}^{(j)}(t) - {\overline{x}_k}^{(j)}(t)| + \xi_k\Big[\epsilon + M_{\mathcal{F}}\overline{\epsilon}(b-a)\Big] \\
&\quad + \xi_k M_{\mathcal{F}} \int_a^b \Big[\alpha \sum_{i=0}^{n-1} |(x_k)^{(i)}(s) - {\overline{x}_k}^{(i)}(s)| + \beta \sum_{i=0}^{n-1} |(x_k)^{(i)}(a) - {\overline{x}_k}^{(i)}(a)| + \gamma \sum_{i=0}^{n-1} |(x_k)^{(i)}(b) - {\overline{x}_k}^{(i)}(b)|\Big] ds. \qquad (27)
\end{aligned}
$$

Now, by taking supremum in the above inequalities, we obtain

$$
\begin{aligned}
\|x_{k+1} - \overline{x}_{k+1}\|_B &\leq \epsilon + M_{\mathcal{F}}\overline{\epsilon}(b-a) + M_{\mathcal{F}} \int_a^b \Big[\alpha+\beta+\gamma\Big] \|y_k - \overline{y}_k\|_B ds \\
&= \epsilon + M_{\mathcal{F}}\overline{\epsilon}(b-a) + M_{\mathcal{F}}\Big[\alpha+\beta+\gamma\Big](b-a)\|y_k - \overline{y}_k\|_B, \qquad (28)
\end{aligned}
$$

and

$$
\begin{aligned}
\|y_k - \overline{y}_k\|_B &\leq (1-\xi_k)\|x_k - \overline{x}_k\|_B + \xi_k\Big[\epsilon + M_{\mathcal{F}}\overline{\epsilon}(b-a) + M_{\mathcal{F}}\int_a^b \Big(\alpha+\beta+\gamma\Big)\|x_k - \overline{x}_k\|_B ds\Big] \\
&= (1-\xi_k)\|x_k - \overline{x}_k\|_B + \xi_k\Big[\epsilon + M_{\mathcal{F}}\overline{\epsilon}(b-a) + M_{\mathcal{F}}\Big(\alpha+\beta+\gamma\Big)(b-a)\|x_k - \overline{x}_k\|_B\Big] \\
&= \xi_k\Big(\epsilon + M_{\mathcal{F}}\overline{\epsilon}(b-a)\Big) + \Big[(1-\xi_k) + \xi_k M_{\mathcal{F}}\Big(\alpha+\beta+\gamma\Big)(b-a)\Big)\Big]\|x_k - \overline{x}_k\|_B \\
&= \xi_k\Big(\epsilon + M_{\mathcal{F}}\overline{\epsilon}(b-a)\Big) + \Big[1 - \xi_k\Big(1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)\Big]\|x_k - \overline{x}_k\|_B, \qquad (29)
\end{aligned}
$$

respectively.

Therefore, using (29) in (28) and using hypothesis (H_2), and $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N}$, the resulting inequality become

$$
\begin{aligned}
\|x_{k+1} - \overline{x}_{k+1}\|_B &\leq \Big[1 - \xi_k\Big(1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)\Big]\|x_k - \overline{x}_k\|_B \\
&\quad + \xi_k\Big(\epsilon + M_{\mathcal{F}}\overline{\epsilon}(b-a)\Big) + 2\xi_k\Big(\epsilon + M_{\mathcal{F}}\overline{\epsilon}(b-a)\Big) \\
&\leq \Big[1 - \xi_k\Big(1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)\Big]\|x_k - \overline{x}_k\|_B \\
&\quad + \xi_k\Big(1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)\frac{3\Big(\epsilon + M_{\mathcal{F}}\overline{\epsilon}(b-a)\Big)}{\Big(1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)}. \qquad (30)
\end{aligned}
$$

We denote

$$
\begin{aligned}
\beta_k &= \|x_k - \overline{x}_k\|_B, \\
\mu_k &= \xi_k\Big(1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big) \in (0,1), \\
\gamma_k &= \frac{3\Big(\epsilon + M_{\mathcal{F}}\overline{\epsilon}(b-a)\Big)}{\Big(1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)} \geq 0.
\end{aligned}
$$

The assumption $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Now, it can be easily see that (30) satisfies all the conditions of Lemma 1 and hence we have

$$\begin{aligned} &0 \leq \limsup_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \gamma_k \\ &\Rightarrow 0 \leq \limsup_{k\to\infty} \|x_k - \overline{x}_k\|_B \leq \limsup_{k\to\infty} \frac{3\Big(\epsilon + M_{\mathcal{F}}\overline{\epsilon}(b-a)\Big)}{\Big(1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)} \\ &\Rightarrow 0 \leq \limsup_{k\to\infty} \|x_k - \overline{x}_k\|_B \leq \frac{3\Big(\epsilon + M_{\mathcal{F}}\overline{\epsilon}(b-a)\Big)}{\Big(1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)}. \end{aligned} \tag{31}$$

By (i), we have $\lim_{k\to\infty} x_k = x$. Using this fact and the assumption $\lim_{k\to\infty} \overline{x}_k = \overline{x}$, we get from (31) that

$$\|x - \overline{x}\|_B \leq \frac{3\Big[M_{\mathcal{F}}\big((b-a)\big)\overline{\epsilon} + \epsilon\Big]}{1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)}. \tag{32}$$

□

Remark: The inequality (32) relates the solutions of the problems (1) and (19) in the sense that if $\mathcal{F}$ and $\mathcal{G}$ are close to $\mathcal{L}$ and $\mathcal{H}$ respectively, then not only the solutions of the problems (1) and (19) are close to each other (i.e. $\|x - \overline{x}\|_B \to 0$), but also depend continuously on the functions involved therein. Moreover, this inequality majores derivatives of the solutions.

In the last, we study the continuous dependence of solutions on certain parameters.

$$x(t) = \mathcal{G}(t) + \int_a^b \mathcal{F}\Big(t, s, x(s), x'(s), \cdots, x^{(n-1)}(s), x(a), x'(a), \cdots, x^{(n-1)}(a), x(b), x'(b), \cdots, x^{(n-1)}(b), \mu_1\Big) ds, \tag{33}$$

$$x(t) = \mathcal{G}(t) + \int_a^b \mathcal{F}\Big(t, s, x(s), x'(s), \cdots, x^{(n-1)}(s), x(a), x'(a), \cdots, x^{(n-1)}(a), x(b), x'(b), \cdots, x^{(n-1)}(b), \mu_2\Big) ds, \tag{34}$$

for $t \in I = [a, b]$. The functions $\mathcal{F}$, $\mathcal{G}$ are defined as in (1) and μ_1, μ_2 are real parameters.

By a solution of equation (33), we mean a continuous function $x(t),\ t \in I$ which is $(n-1)$ times continuously differentiable on I and satisfies the equation (33). It is easy to observe that the solution $x(t)$ of the equation (33) and its derivatives satisfy the integral equations (see [7], p.318)

$$\begin{aligned} x^{(j)}(t) &= \mathcal{G}^{(j)}(t) \\ &+ \int_a^b \frac{\partial^j}{\partial t^j} \mathcal{F}\Big(t, s, x(s), x'(s), \cdots, x^{(n-1)}(s), x(a), x'(a), \cdots, x^{(n-1)}(a), x(b), x'(b), \cdots, x^{(n-1)}(b), \mu_1\Big) ds, \end{aligned} \tag{35}$$

for $t \in I$ and $0 \leq j \leq n-1$.

Let $x(t) \in B$ and following steps from the proof of Theorem 2, define the operator for the equation (33)

$$\begin{aligned} (Tx)(t) &= \mathcal{G}(t) \\ &+ \int_a^b \mathcal{F}\Big(t, s, x(s), x'(s), \cdots, x^{(n-1)}(s), x(a), x'(a), \cdots, x^{(n-1)}(a), x(b), x'(b), \cdots, x^{(n-1)}(b), \mu_1\Big) ds, \end{aligned} \tag{36}$$

for $t \in I = [a, b]$.

Differentiating both sides of (36) with respect to t (see [7], p. 318), we have

$$(Tx)^{(j)}(t) = \mathcal{G}^{(j)}(t) + \int_a^b \frac{\partial^j}{\partial t^j} \mathcal{F}\Big(t, s, x(s), x'(s), \cdots, x^{(n-1)}(s), x(a), x'(a), \cdots, x^{(n-1)}(a), x(b), x'(b), \cdots, x^{(n-1)}(b), \mu_1\Big) ds, \tag{37}$$

for $t \in I$ and $0 \le j \le n-1$.

Similarly we define for the equation (34)

$$\overline{x}^{(j)}(t) = \mathcal{G}^{(j)}(t) + \int_a^b \frac{\partial^j}{\partial t^j} \mathcal{L}\Big(t, s, \overline{x}(s), \overline{x}'(s), \cdots, \overline{x}^{(n-1)}(s), \overline{x}(a), \overline{x}'(a), \cdots, \overline{x}^{(n-1)}(a), \overline{x}(b), \overline{x}'(b), \cdots, \overline{x}^{(n-1)}(b), \mu_2\Big) ds, \tag{38}$$

for $t \in I$ and $0 \le j \le n-1$

Let $\overline{x}(t) \in B$ and following steps from the proof of Theorem 2, define the operator for the equation (34)

$$(\overline{T}\overline{x})(t) = \mathcal{G}(t) + \int_a^b \mathcal{F}\Big(t, s, \overline{x}(s), \overline{x}'(s), \cdots, \overline{x}^{(n-1)}(s), \overline{x}(a), \overline{x}'(a), \cdots, \overline{x}^{(n-1)}(a), \overline{x}(b), \overline{x}'(b), \cdots, \overline{x}^{(n-1)}(b), \mu_2\Big) ds, \tag{39}$$

for $t \in I = [a, b]$.

Differentiating both sides of (39) with respect to t (see [7], p. 318), we have

$$(\overline{T}\overline{x})^{(j)}(t) = \mathcal{G}^{(j)}(t) + \int_a^b \frac{\partial^j}{\partial t^j} \mathcal{F}\Big(t, s, \overline{x}(s), \overline{x}'(s), \cdots, \overline{x}^{(n-1)}(s), \overline{x}(a), \overline{x}'(a), \cdots, \overline{x}^{(n-1)}(a), \overline{x}(b), \overline{x}'(b), \cdots, \overline{x}^{(n-1)}(b), \mu_2\Big) ds, \tag{40}$$

for $t \in I$ and $0 \le j \le n-1$.

The following theorem states the continuous dependency of solutions on parameters.

Theorem 4. *Consider the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{\overline{x}_k\}_{k=0}^{\infty}$ generated normal $S-$ iterative method associated with operators T in (37)and $\overline{T}$ in (40), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0, 1]$ satisfying $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N}$. Assume that*

(i) *the hypothesis (H_2) holds.*

(ii) *the function $\mathcal{F}$ satisfy the conditions:*

$$\begin{aligned}&\Big| \frac{\partial^j}{\partial t^j} \mathcal{F}\Big(t, s, x(t), x'(t), \cdots, x^{(n-1)}(t), x(a), x'(a), \cdots, x^{(n-1)}(a), x(b), x'(b), \cdots, x^{(n-1)}(b), \mu_1\Big) \\ &\quad - \frac{\partial^j}{\partial t^j} \mathcal{F}\Big(t, s, y(s), y'(s), \cdots, y^{(n-1)}(s), y(a), y'(a), \cdots, y^{(n-1)}(a), y(b), y'(b), \cdots, y^{(n-1)}(b), \mu_1\Big) \Big| \\ &\le p_j(t, s) \Big[\alpha \sum_{i=0}^{n-1} |x^{(i)}(s) - y^{(i)}(s)| + \beta \sum_{i=0}^{n-1} |x^{(i)}(a) - y^{(i)}(a)| + \gamma \sum_{i=0}^{n-1} |x^{(i)}(b) - y^{(i)}(b)| \Big],\end{aligned} \tag{41}$$

$$\Big| \frac{\partial^j}{\partial t^j} \mathcal{F}\Big(t, s, x(t), x'(t), \cdots, x^{(n-1)}(t), x(a), x'(a), \cdots, x^{(n-1)}(a), x(b), x'(b), \cdots, x^{(n-1)}(b), \mu_1\Big)$$

$$-\frac{\partial^j}{\partial t^j}\mathcal{F}\Big(t,s,x(t),x'(t),\cdots,x^{(n-1)}(t),x(a),x'(a),\cdots,x^{(n-1)}(a),x(b),x'(b),\cdots,x^{(n-1)}(b),\mu_2\Big)\Big|$$
$$\leq p_j(t,s)|\mu_1-\mu_2|, \tag{42}$$

for $j=0,1,\cdots,n-1$, where $p_j(t,s)\in C(I^2,\mathbb{R}_+)$ and $\alpha,\beta,\gamma>0$.

Suppose $x(t)$ and $\overline{x}(t)$ are solutions of (33) and (34) respectively and if the sequence $\{\overline{x}_k\}_{k=0}^{\infty}$ converges to $\overline{x}$, then we have

$$\|x-\overline{x}\|_B\leq\frac{3\Big[M_{\mathcal{F}}|\mu_1-\mu_2|(b-a)\Big]}{1-M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)}. \tag{43}$$

Proof. Suppose the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{\overline{x}_k\}_{k=0}^{\infty}$ generated normal $S-$ iterative method associated with operators T in (37)and $\overline{T}$ in (40), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2}\leq\xi_k$ for all $k\in\mathbb{N}$. From iteration (4) and equations (35) with (37); (38) with (40) and hypotheses, we obtain

$$
\begin{aligned}
&|x_{k+1}(t)-\overline{x}_{k+1}(t)|_E\\
&=\sum_{j=0}^{n-1}|{x_{k+1}}^{(j)}(t)-{\overline{x}_{k+1}}^{(j)}(t)|\\
&=\sum_{j=0}^{n-1}|(Ty_k)^{(j)}(t)-(\overline{T}\overline{y}_k)^{(j)}(t)|\\
&=\sum_{j=0}^{n-1}\Big|\mathcal{G}^{(j)}(t)-\mathcal{G}^{(j)}(t)\\
&\quad+\int_a^b\frac{\partial^j}{\partial t^j}\mathcal{F}\Big(t,s,y_k(s),y_k'(s),\cdots,{y_k}^{(n-1)}(s),y_k(a),{y_k}'(a),\cdots,{y_k}^{(n-1)}(a),y_k(b),{y_k}'(b),\cdots,{y_k}^{(n-1)}(b),\mu_1\Big)ds\\
&\quad-\int_a^b\frac{\partial^j}{\partial t^j}\mathcal{F}\Big(t,s,\overline{y}_k(s),{\overline{y}_k}'(s),\cdots,{\overline{y}_k}^{(n-1)}(s),\overline{y}_k(a),{\overline{y}_k}'(a),\cdots,{\overline{y}_k}^{(n-1)}(a),\overline{y}_k(b),{\overline{y}_k}'(b),\cdots,{\overline{y}_k}^{(n-1)}(b),\mu_2\Big)ds\Big|\\
&\leq\sum_{j=0}^{n-1}\int_a^b\Big|\frac{\partial^j}{\partial t^j}\mathcal{F}\Big(t,s,y_k(s),y_k'(s),\cdots,{y_k}^{(n-1)}(s),y_k(a),{y_k}'(a),\cdots,{y_k}^{(n-1)}(a),y_k(b),{y_k}'(b),\cdots,{y_k}^{(n-1)}(b),\mu_1\Big)\\
&\quad-\frac{\partial^j}{\partial t^j}\mathcal{F}\Big(t,s,\overline{y}_k(s),{\overline{y}_k}'(s),\cdots,{\overline{y}_k}^{(n-1)}(s),\overline{y}_k(a),{\overline{y}_k}'(a),\cdots,{\overline{y}_k}^{(n-1)}(a),\overline{y}_k(b),{\overline{y}_k}'(b),\cdots,{\overline{y}_k}^{(n-1)}(b),\mu_2\Big)\Big|ds\\
&\leq\sum_{j=0}^{n-1}\int_a^b\Big|\frac{\partial^j}{\partial t^j}\mathcal{F}\Big(t,s,y_k(s),y_k'(s),\cdots,{y_k}^{(n-1)}(s),y_k(a),{y_k}'(a),\cdots,{y_k}^{(n-1)}(a),y_k(b),{y_k}'(b),\cdots,{y_k}^{(n-1)}(b),\mu_1\Big)\\
&\quad-\frac{\partial^j}{\partial t^j}\mathcal{F}\Big(t,s,\overline{y}_k(s),{\overline{y}_k}'(s),\cdots,{\overline{y}_k}^{(n-1)}(s),\overline{y}_k(a),{\overline{y}_k}'(a),\cdots,{\overline{y}_k}^{(n-1)}(a),\overline{y}_k(b),{\overline{y}_k}'(b),\cdots,{\overline{y}_k}^{(n-1)}(b),\mu_1\Big)\Big|ds\\
&\quad+\sum_{j=0}^{n-1}\int_a^b\Big|\frac{\partial^j}{\partial t^j}\mathcal{F}\Big(t,s,\overline{y}_k(s),{\overline{y}_k}'(s),\cdots,{\overline{y}_k}^{(n-1)}(s),\overline{y}_k(a),{\overline{y}_k}'(a),\cdots,{\overline{y}_k}^{(n-1)}(a),\overline{y}_k(b),{\overline{y}_k}'(b),\cdots,{\overline{y}_k}^{(n-1)}(b),\mu_1\Big)\\
&\quad-\frac{\partial^j}{\partial t^j}\mathcal{F}\Big(t,s,\overline{y}_k(s),{\overline{y}_k}'(s),\cdots,{\overline{y}_k}^{(n-1)}(s),\overline{y}_k(a),{\overline{y}_k}'(a),\cdots,{\overline{y}_k}^{(n-1)}(a),\overline{y}_k(b),{\overline{y}_k}'(b),\cdots,{\overline{y}_k}^{(n-1)}(b),\mu_2\Big)\Big|ds\\
&\leq M_{\mathcal{F}}\int_a^b|\mu_1-\mu_2|ds\\
&\quad+\sum_{j=0}^{n-1}\int_a^b p_j(t,s)\Big[\alpha\sum_{i=0}^{n-1}|(y_k)^{(i)}(s)-{\overline{y}_k}^{(i)}(s)|+\beta\sum_{i=0}^{n-1}|(y_k)^{(i)}(a)-{\overline{y}_k}^{(i)}(a)|+\gamma\sum_{i=0}^{n-1}|(y_k)^{(i)}(b)-{\overline{y}_k}^{(i)}(b)|\Big]ds\\
&\leq M_{\mathcal{F}}|\mu_1-\mu_2|(b-a)
\end{aligned}
$$

$$+ M_{\mathcal{F}} \int_a^b \Big[\alpha \sum_{i=0}^{n-1} |(y_k)^{(i)}(s) - \overline{y}_k{}^{(i)}(s)| + \beta \sum_{i=0}^{n-1} |(y_k)^{(i)}(a) - \overline{y}_k{}^{(i)}(a)| + \gamma \sum_{i=0}^{n-1} |(y_k)^{(i)}(b) - \overline{y}_k{}^{(i)}(b)|\Big] ds. \tag{44}$$

Similarly, we have

$$\begin{aligned}
&|y_k(t) - \overline{y}_k(t)|_E \\
&= \sum_{j=0}^{n-1} |y_k{}^{(j)}(t) - \overline{y}_k{}^{(j)}(t)| \\
&= \sum_{j=0}^{n-1} \Big[(1-\xi_k)|x_k{}^{(j)}(t) - \overline{x}_k{}^{(j)}(t)| + \xi_k |(Tx_k)^{(j)}(t) - (\overline{T}\overline{x}_k)^{(j)}(t)|\Big] \\
&= \Big[(1-\xi_k) \sum_{j=0}^{n-1} |x_k{}^{(j)}(t) - \overline{x}_k{}^{(j)}(t)| + \xi_k \sum_{j=0}^{n-1} |(Tx_k)^{(j)}(t) - (\overline{T}\overline{x}_k)^{(j)}(t)|\Big] \\
&\le (1-\xi_k) \sum_{j=0}^{n-1} |x_k{}^{(j)}(t) - \overline{x}_k{}^{(j)}(t)| + \xi_k \Big[M_{\mathcal{F}}|\mu_1 - \mu_2|(b-a)\Big] \\
&\quad + \xi_k M_{\mathcal{F}} \int_a^b \Big[\alpha \sum_{i=0}^{n-1} |(x_k)^{(i)}(s) - \overline{x}_k{}^{(i)}(s)| + \beta \sum_{i=0}^{n-1} |(x_k)^{(i)}(a) - \overline{x}_k{}^{(i)}(a)| + \gamma \sum_{i=0}^{n-1} |(x_k)^{(i)}(b) - \overline{x}_k{}^{(i)}(b)|\Big] ds.
\end{aligned} \tag{45}$$

Now, by taking supremum in the above inequalities, we obtain

$$\begin{aligned}
\|x_{k+1} - \overline{x}_{k+1}\|_B &\le M_{\mathcal{F}}|\mu_1 - \mu_2|(b-a) + M_{\mathcal{F}} \int_a^b \Big[\alpha + \beta + \gamma\Big] \|y_k - \overline{y}_k\|_B ds \\
&= M_{\mathcal{F}}|\mu_1 - \mu_2|(b-a) + M_{\mathcal{F}}\Big[\alpha + \beta + \gamma\Big](b-a)\|y_k - \overline{y}_k\|_B,
\end{aligned} \tag{46}$$

and

$$\begin{aligned}
\|y_k - \overline{y}_k\|_B &\le (1-\xi_k)\|x_k - \overline{x}_k\|_B + \xi_k\Big[M_{\mathcal{F}}|\mu_1 - \mu_2|(b-a) + M_{\mathcal{F}} \int_a^b \Big(\alpha + \beta + \gamma\Big)\|x_k - \overline{x}_k\|_B ds\Big] \\
&= (1-\xi_k)\|x_k - \overline{x}_k\|_B + \xi_k\Big[M_{\mathcal{F}}|\mu_1 - \mu_2|(b-a) + M_{\mathcal{F}}\Big(\alpha + \beta + \gamma\Big)(b-a)\|x_k - \overline{x}_k\|_B\Big] \\
&= \xi_k\Big(M_{\mathcal{F}}|\mu_1 - \mu_2|(b-a)\Big) + \Big[(1-\xi_k) + \xi_k M_{\mathcal{F}}\Big(\alpha + \beta + \gamma\Big)(b-a)\Big)\Big]\|x_k - \overline{x}_k\|_B \\
&= \xi_k\Big(M_{\mathcal{F}}|\mu_1 - \mu_2|(b-a)\Big) + \Big[1 - \xi_k\Big(1 - M_{\mathcal{F}}\big(\alpha + \beta + \gamma\big)(b-a)\Big)\Big]\|x_k - \overline{x}_k\|_B,
\end{aligned} \tag{47}$$

respectively.

Therefore, using (47) in (46) and using hypothesis (H_2), and $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N}$, the resulting inequality become

$$\begin{aligned}
\|x_{k+1} - \overline{x}_{k+1}\|_B &\le \Big[1 - \xi_k\Big(1 - M_{\mathcal{F}}\big(\alpha + \beta + \gamma\big)(b-a)\Big)\Big]\|x_k - \overline{x}_k\|_B \\
&\quad + \xi_k\Big(M_{\mathcal{F}}|\mu_1 - \mu_2|(b-a)\Big) + 2\xi_k\Big(M_{\mathcal{F}}|\mu_1 - \mu_2|(b-a)\Big) \\
&\le \Big[1 - \xi_k\Big(1 - M_{\mathcal{F}}\big(\alpha + \beta + \gamma\big)(b-a)\Big)\Big]\|x_k - \overline{x}_k\|_B \\
&\quad + \xi_k\Big(1 - M_{\mathcal{F}}\big(\alpha + \beta + \gamma\big)(b-a)\Big)\frac{3\Big(M_{\mathcal{F}}|\mu_1 - \mu_2|(b-a)\Big)}{\Big(1 - M_{\mathcal{F}}\big(\alpha + \beta + \gamma\big)(b-a)\Big)}.
\end{aligned} \tag{48}$$

We denote

$$\begin{aligned}
&\beta_k = \|x_k - \overline{x}_k\|_B, \\
&\mu_k = \xi_k\Big(1 - M_{\mathcal{F}}\big(\alpha + \beta + \gamma\big)(b-a)\Big) \in (0, 1),
\end{aligned}$$

$$\gamma_k = \frac{3\Big(M_{\mathcal{F}}|\mu_1 - \mu_2|(b-a)\Big)}{\Big(1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)} \geq 0.$$

The assumption $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Now, it can be easily see that (48) satisfies all the conditions of Lemma 1 and hence we have

$$\begin{aligned} &0 \leq \limsup_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \gamma_k \\ &\Rightarrow 0 \leq \limsup_{k\to\infty} \|x_k - \overline{x}_k\|_B \leq \limsup_{k\to\infty} \frac{3\Big(M_{\mathcal{F}}|\mu_1 - \mu_2|(b-a)\Big)}{\Big(1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)} \\ &\Rightarrow 0 \leq \limsup_{k\to\infty} \|x_k - \overline{x}_k\|_B \leq \frac{3\Big(M_{\mathcal{F}}|\mu_1 - \mu_2|(b-a)\Big)}{\Big(1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)\Big)}. \end{aligned} \tag{49}$$

By (i), we have $\lim_{k\to\infty} x_k = x$. Using this fact and the assumption $\lim_{k\to\infty} \overline{x}_k = \overline{x}$, we get from (49) that

$$\|x - \overline{x}\|_B \leq \frac{3\Big[M_{\mathcal{F}}|\mu_1 - \mu_2|(b-a)\Big]}{1 - M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)}. \tag{50}$$

□

Remark: The result dealing with the property of a solution called "dependence of solutions on parameters". Here the parameters are scalars. Notice that the initial conditions do not involve parameters. The dependence on parameters are an important aspect in various physical problems.

4. Example

We consider the following integral equation:

$$x(t) = \frac{t+e^{-t}}{3} + \int_0^1 \frac{3t-2s}{5}\Big[\frac{s-\sin(x(s))}{2} + \frac{x(0)+x(1)}{3}\Big]ds, \ t \in [0,1]. \tag{51}$$

Comparing this equation with proposed equation (1) for $n = 1$, we get

$$\begin{aligned} &\mathcal{G} \in C(I = [0,1], \mathbb{R}), \quad \mathcal{G}(t) = \frac{t+e^{-t}}{3}; \quad \mathcal{F} \in C(I \times I \times \mathbb{R}^3, \mathbb{R}), \\ &\mathcal{F}(t,s,x(s),x(0),x(1)) = \frac{3t-2s}{5}\Big[\frac{s-\sin(x(s)))}{2} + \frac{x(0)+x(1)}{3}\Big]. \end{aligned}$$

Now, we have

$$\begin{aligned} &\Big|\mathcal{F}(t,s,x,x(0),x(1)) - \mathcal{F}(t,s,y(s),y(0),y(1))\Big| \\ &\qquad = \Big|\frac{3t-2s}{5}\Big[\frac{s-\sin(x(s)))}{2} + \frac{x(0)+x(1)}{3}\Big] - \frac{3t-2s}{5}\Big[\frac{s-\sin(y(s)))}{2} + \frac{y(0)+y(1)}{3}\Big]\Big| \\ &\qquad \leq \Big|\frac{3t-2s}{5}\Big|\Big[\Big|\frac{s-\sin(x(s)))}{2} - \frac{s-\sin(y(s))}{2}\Big| + \Big|\frac{x(0)+x(1)}{3} - \frac{y(0)+y(1)}{3}\Big|\Big] \\ &\qquad \leq \Big|\frac{3t-2s}{5}\Big|\Big[\frac{1}{2}\Big|\sin(x(s)) - \sin(y(s))\Big| + \frac{1}{3}\Big|x(0)-y(0)\Big| + \frac{1}{3}\Big|x(1)-y(1)\Big|\Big]. \end{aligned} \tag{52}$$

Taking sup norm, we obtain

$$\Big\|\mathcal{F}(t,s,x,x(0),x(1)) - \mathcal{F}(t,s,y(s),y(0),y(1))\Big\| \leq \sup_{t,s\in I}\Big|\frac{3t-2s}{5}\Big|\Big(\frac{1}{2}+\frac{1}{3}+\frac{1}{3}\Big)\Big\|x-y\Big\|$$

$$\begin{aligned}&\leq \frac{3}{5}\Big(\frac{7}{6}\Big)\Big\|x-y\Big\|\\ &= \frac{7}{10}\Big\|x-y\Big\|,\end{aligned} \tag{53}$$

where $M_{\mathcal{F}} = \dfrac{3}{5}, \quad \alpha = \dfrac{1}{2}, \quad \beta = \dfrac{1}{3}, \quad \gamma = \dfrac{1}{3}.$

Therefore, we the estimate

$$M_{\mathcal{F}}\Big(\alpha+\beta+\gamma\Big)(b-a) = \frac{3}{5}\Big(\frac{1}{2}+\frac{1}{3}+\frac{1}{3}\Big)(1-1) = \frac{7}{10}\times 1 = \frac{7}{10} < 1. \tag{54}$$

We define the operator $T : B = C(I,\mathbb{R}) \to B = C(I,\mathbb{R})$ by

$$(Tx)(t) = \frac{t+e^{-t}}{3} + \int_0^1 \frac{3t-2s}{5}\Big[\frac{s-\sin(x(s))}{2} + \frac{x(0)+x(1)}{3}\Big]ds,\ t\in[0,1]. \tag{55}$$

Since all conditions of Theorem 2 are satisfied and so by its conclusion, the sequence $\{x_k\}$ associated with the normal $S-$iterative method (4) for the operator T in (55) converges to a unique solution $x \in B$.

Further, we also have for any $x_0 \in B$

$$\begin{aligned}\|x_{k+1}-x\|_B &\leq \frac{\Big[M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\Big]^{k+1}}{e^{\Big[1-M_{\mathcal{F}}(\alpha+\beta+\gamma)(b-a)\Big]\sum_{i=0}^{k}\xi_i}}\|x_0-x\|\\ &\leq \frac{\Big[\frac{7}{10}\Big]^{k+1}}{e^{\Big[1-\frac{7}{10}\Big]\sum_{i=0}^{k}\xi_i}}\|x_0-x\|\\ &\leq \frac{\Big(\frac{7}{10}\Big)^{k+1}}{e^{\Big(\frac{3}{10}\Big)\sum_{i=0}^{k}\xi_i}}\|x_0-x\|\\ &\leq \frac{\Big(\frac{7}{10}\Big)^{k+1}}{e^{\Big(\frac{3}{10}\Big)\sum_{i=0}^{k}\xi_i}}\|x_0-x\|\\ &\leq \frac{\Big(\frac{7}{10}\Big)^{k+1}}{e^{\Big(\frac{3}{10}\Big)\sum_{i=0}^{k}\frac{1}{1+i}}}\|x_0-x\|,\end{aligned} \tag{56}$$

where we have chosen $\xi_i = \frac{1}{1+i} \in [0,1]$. The estimate obtained in (56) is called a bound for the error (due to truncation of computation at the $k-$th iteration).

Next, we consider the perturbed integral equation:

$$\overline{x}(t) = \frac{t+2e^{-t}}{3} + \int_0^1 \frac{3t-2s}{5}\Big[\frac{s-\sin(\overline{x}(t))}{2} + \frac{\overline{x}(0)+\overline{x}(1)}{3} - s + \frac{1}{7}\Big]ds,\ t\in[0,1]. \tag{57}$$

Similarly, comparing it with the equation (19) for $n = 1$, we have

$$\mathcal{H}(t) = \frac{t+2e^{-t}}{3}, \quad \mathcal{L}\Big(t,s,\overline{x}(s),\overline{x}(0),\overline{x}(1)\Big) = \frac{3t-2s}{5}\Big[\frac{s-\sin(\overline{x}(t))}{2} + \frac{\overline{x}(0)+\overline{x}(1)}{3} - s + \frac{1}{7}\Big].$$

One can easily show on the same lines as above that the mapping $\overline{T} : B = C(I,\mathbb{R}) \to B = C(I,\mathbb{R})$ defined by

$$(\overline{T}x)(t) = \frac{t+2e^{-t}}{3} + \int_0^1 \frac{3t-2s}{5}\Big[\frac{s-\sin(\overline{x}(t))}{2} + \frac{\overline{x}(0)+\overline{x}(1)}{3} - s + \frac{1}{7}\Big]ds,\ t\in[0,1]. \tag{58}$$

In perturbed integral equation, all conditions of Theorem 2 are also satisfied and so by its conclusion, the sequence $\{\overline{x}_k\}$ associated with the normal $S-$iterative method (4) for the operator $\overline{T}$ in (58) converges to a unique solution $\overline{x} \in B$.

Now, we have the following estimates:

$$\Big|\mathcal{G}(t)-\mathcal{H}(t)\Big| = \Big|\frac{t+e^{-t}}{3}-\frac{t+2e^{-t}}{3}\Big| = \Big|\frac{t+2e^{-t}-t-2e^{-t}}{3}\Big| = \frac{e^{-t}}{3} \le \frac{1}{3} = \epsilon_1,\ t\in I=[0,1], \tag{59}$$

$$\begin{aligned}
&\Big|\mathcal{F}(t,s,x,x(0),x(1))-\mathcal{L}(t,s,x(s),x(0),x(1))\Big| \\
&= \Big|\frac{3t-2s}{5}\Big[\frac{s-\sin(x(s)))}{2}+\frac{x(0)+x(1)}{3}\Big]-\frac{3t-2s}{5}\Big[\frac{s-\sin(x(t))}{2}+\frac{x(0)+x(1)}{3}-s+\frac{1}{7}\Big]\Big| \\
&= \Big|\frac{3t-2s}{5}\Big|\Big|s-\frac{1}{7}\Big| \\
&\le \frac{1}{7} = \overline{\epsilon}_1.
\end{aligned} \tag{60}$$

Consider the sequences $\{x_k\}_{k=0}^{\infty}$ with $x_k \to x$ as $k\to\infty$ and $\{\overline{x}_k\}_{k=0}^{\infty}$ with $\overline{x}_k \to \overline{x}$ as $k\to\infty$ generated normal $S-$iterative method associated with operators T in (55)and $\overline{T}$ in (58), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2}\le\xi_k$ for all $k\in\mathbb{N}$. Then we have from Theorem 3 that

$$\begin{aligned}
\|x-\overline{x}\|_B &\le \frac{3\Big[M_{\mathcal{F}}\big((b-a)\big)\overline{\epsilon}_1+\epsilon_1\Big]}{1-M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)} \\
\Rightarrow \|x-\overline{x}\|_B &\le \frac{3\Big[\frac{3}{5}\big((1-0)\big)\frac{1}{7}+\frac{1}{3}\Big]}{1-\frac{3}{5}\big(\frac{1}{2}+\frac{1}{3}+\frac{1}{3}\big)(1-0)} \\
\Rightarrow \|x-\overline{x}\|_B &\le \frac{\frac{44}{35}}{\frac{3}{10}} = \frac{44}{35}\times\frac{10}{3} = \frac{88}{21}.
\end{aligned} \tag{61}$$

This shows that the closeness of solutions and dependency of solutions on functions involved therein.

Finally, we shall prove the dependency of solutions on real parameters.

We consider the following integral equations involving real parameters:

$$x(t) = \frac{t+e^{-t}}{3}+\int_0^1 \frac{3t-2s}{5}\Big[\frac{s-\sin(x(s))}{2}+\frac{x(0)+x(1)}{3}\Big]ds+\mu_1,\ t\in[0,1] \tag{62}$$

and

$$x(t) = \frac{t+e^{-t}}{3}+\int_0^1 \frac{3t-2s}{5}\Big[\frac{s-\sin(x(s))}{2}+\frac{x(0)+x(1)}{3}\Big]ds+\mu_2,\ t\in[0,1]. \tag{63}$$

Hence by making similar arguments and from Theorem 4, one cam has

$$\begin{aligned}
\|x-\overline{x}\|_B &\le \frac{3\Big[M_{\mathcal{F}}|\mu_1-\mu_2|(b-a)\Big]}{1-M_{\mathcal{F}}\big(\alpha+\beta+\gamma\big)(b-a)} \\
\Rightarrow \|x-\overline{x}\|_B &\le \frac{3\Big[\frac{3}{5}|\mu_1-\mu_2|(1-0)\Big]}{\frac{3}{10}} \\
\Rightarrow \|x-\overline{x}\|_B &\le \frac{3\frac{3}{5}|\mu_1-\mu_2|}{\frac{3}{10}} \\
\Rightarrow \|x-\overline{x}\|_B &\le \frac{3\frac{3}{5}|\mu_1-\mu_2|}{\frac{3}{10}} = \frac{9}{5}\times\frac{10}{3}|\mu_1-\mu_2|
\end{aligned}$$

$$\Rightarrow \|x - \overline{x}\|_B \leq 6|\mu_1 - \mu_2|. \quad (64)$$

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References

[1] R. Agarwal, D. O'Regan, and D. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. Journal of Nonlinear and Convex Analysis, 8(2007), pp. 61-79.
[2] Y. Atalan and V. Karakaya, Iterative solution of functional Volterra-Fredholm integral equation with deviating argument, Journal of Nonlinear and Convex Analysis, 18(4)(2017), pp. 675-684.
[3] Y. Atalan and V. Karakaya, An example of data dependence result for the class of almost contraction mappings, Sahand Communications in Mathematical Analysis(SCMA)Vol.17, No.1(2020),pp. 139-155.
[4] Yunus Atalan, Faik Gürsoy and Abdul Rahim Khan, Convergence of S-iterative method to a solution of Fredholm integral equation and data dependency, Facta Universitatis, Ser. Math. Inform. Vol. 36, No 4(2021), 685-694.
[5] V. Berinde, and M. Berinde, The fastest Krasnoselskij iteration for approximating fixed points of strictly pseudo-contractive mappings, Carpathian J. Math, 21(1-2)(2005), pp. 13-20.
[6] V. Berinde, Existence and approximation of solutions of some first order iterative differential equations, Miskolc Mathematical Notes, Vol. 11(2010), No. 1, pp. 13-26.
[7] B. Cahlon and D. Westreich, Complete continuity of integro-differential operators with discontinuous kernels and collectively compact approximations, J. Math. Anal. Appl., 71(1979), pp. 313-332.
[8] C. E. Chidume, Iterative approximation of fixed points of Lipschitz pseudocontractivemaps, Proceedings of the American Mathematical Society, vol. 129, no. 8, pp. 2245-2251, 2001.
[9] R. Chugh, V. Kumar and S. Kumar, Strong Convergence of a new three step iterative scheme in Banach spaces, American Journal of Computational Mathematics 2(2012), pp. 345-357.
[10] C. Corduneanu, Principles of Differential and Integral Equations, Chelsea, New York, 1971.
[11] M. Dobritoiu, An integral equation with modified argument, Stud. Univ. Babes–Bolyai Math. XLIX (3)2004, pp. 27-33.
[12] M. Dobritoiu, System of integral equations with modified argument, Carpathian J. Math, 24(2008), No. 2, pp. 26-36.
[13] M. Dobritoiu, A nonlinear Fredholm integral equation, TJMM, 1(2009), No. 1-2, pp. 25-32.
[14] M. Dobritoiu, A class of nonlinear integral equations, TJMM, 4(2012), No. 2, pp.117-123.
[15] F. Gürsoy and V. Karakaya, Some Convergence and Stability Results for Two New Kirk Type Hybrid Fixed Point Iterative Algorithms. Journal of Function Spaces doi:10.1155/2014/684191, 2014.
[16] F. Gürsoy, V. Karakaya and B. E. Rhoades, Some Convergence and Stability Results for the Kirk Multistep and Kirk-SP Fixed Point Iterative Algorithms. Abstract and Applied Analysis doi:10.1155/2014/806537, 2014.
[17] N. Hussain, A. Rafiq, Damjanović, B. and R. Lazović, On rate of convergence of various iterative schemes. Fixed Point Theory and Applications 2011(1), 1-6.
[18] S. Ishikawa, Fixed points by a new iteration method, Proceedings of the American Mathematical Society, vol. 4491974), pp. 147-150.
[19] S. M. Kang, A. Rafiq and Y. C. Kwun, Strong convergence for hybrid $S-$ iteration scheme, Journal of Applied Mathematics, Vol. 2013, Article ID 705814, 4 Pages, http://dx.doi.org/10.1155/2013/705814.
[20] S. H. Khan, A Picard-Mann hybrid iterative process. Fixed Point Theory and Applications (1)2013, pp. 1-10.
[21] B. G. Pachpatte, On Fredholm type integrodifferential equation, Amkang Journal of Mathematics, Vol.39, No. 1(2008), pp. 85-94.
[22] B. G. Pachpatte, On higher order Volterra-Fredholm integrodifferential equation, Fasciculi Mathematici, No. 47(2007), pp. 35-48.
[23] D. R. Sahu, Applications of the S-iteration process to constrained minimization problems and split feasibility problems, Fixed Point Theory, vol. 12, no. 1, pp. 187-204, 2011.
[24] D. R. Sahu and A. Petrus-el, Strong convergence of iterative methods by strictly pseudocontractive mappings in Banach spaces, NonlinearAnalysis.Theory,Methods & Applications, vol. 74, no. 17, pp. 6012-6023, 2011.
[25] S. Soltuz and T. Grosan, Data dependence for Ishikawa iteration when dealing with contractive-like operators, Fixed Point Theory Appl, 2008, 242916(2008). https://doi.org/10.1155/2008/242916.

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