**STUDY OF STRUCTURE AND OPERATORS ON ALMOST KAEHLERIAN MANIFOLDS**

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**Abstract:**

Kodaira and Spencer (1957) have studied on the variation of almost complex structure. Hsiung (1966) has defined and studied structures and operators on almost Hermition manifolds. Also, Ogawa (1970) has studied operators on almost Hermition manifolds. In this paper, we have defined and studied structure and operators on almost Kaehlerian spaces and several theorems have been derived. We have also been demonstrated within nearly Kaehlerian spaces that for the structure to be integrable, it is both necessary and sufficient that the square of the difference between and ( )2 = 0. Additionally, when the operator vanishes across the entire space, then the space can be classified as Kaehlerian.

**Keywords:** Almost complex structure, almost Hermition spaces, almost Kaehlerian spaces, Kaehlerian spaces.

**MSC:** 53C55, 53B35.

**1 Introduction:**

Consider as a Riemannian space, where its fundamental metric tensor is denoted as, and In this context, Greek indices and so on, range from 1 to n, which is the dimension of the space. Let represent the generalized Kronecker’s delta, and signify . We define as the algebra of differential p-forms on Consequently, the operators of exterior differentiation ***d****:* , and the adjoint operator can be expressed for a *p*-form *u* =as follows:

(1.1) =

(1.2) =

where Represents the covariant differentiation concerning the Riemannian connection, the exterior co-differentiation *:*is specified by

(1.3)

can be expressed locally as

(1.4)

Let be the Laplace-Beltrami operator defined by

Subsequently, utilizing equations (1.1) and (1.3), it is straightforward to confirm that in the case of a degree form

(1.5) = +

+

holds, where ) represents the curvature (or Ricci) tensor linked to the Riemann connection. In the notation , the index replaces the index , while in indicates that the subscript is deleted.

If a Riemannian space admits an almost complex structure satisfying

(1.6)

then it is called an almost Hermitian space. If in an almost Kaehler space, the Nijenhuis tensor satisfies the condition + then we deduce from it = 0, i.e. + = 0 and the space is an almost Tachibana space. Thus, we have 3 = = 0. Consequently, the space is a Kaehler space i.e., an almost Kaehler space is a Kaehler space, if and only if the Nijenhuis tensor equation is satisfied.

Let represent complexified tangent space of the manifold Consider as the space of complexified differential *p*-forms, which are essentially complex-valued functions defined on For non-negative integers r, s we introduce the projection mapping denoted by where *p* = r + s as follows:

(1.7)

1,0

and its conjugate

(1.8)

0,1  1,0

which will be abbreviated to respectively. Then for a *p*-form **u** of , we define:

(1.9) =

r,s r,s

=

A *p*-form u of is called of type (r, s) if it satisfies ( ) = *u*.

r,s

Now, here following two Lemmas given by [Kodaira and Spencer (1957)], Ogawa (1970),

**Lemma (1.1):** In an almost complex space, for any set of functions we have

(1.10) **==**

**(p-*v, v*)**

and

(1.11)

holds for any *p*-form ,

Now we define the operators of type (1, 0) and of type (2,) in accordance with [Kodaira and Spencer (1957)] given by

(1.12) ,

r+s=p r+1,s r,s

(1.13) .

r+s=p r+2,s-1 r,s

Here we denote the conjugate operator of ) by .

**Lemma (1.2):** In an almost complex space, on , we have

(1.14) ,

r+3,s-2 r,s

where, r + s = p.

From Lemmas (1.1) and (1.2), we have [2] [Kodaira and Spencer (1957)] given by

(1.15)

The definitions of complex counterparts of the real operators as per the framework established by Kodaira-Spencer in their (1957) work [2], can be stated as follows:

(1.16)

(1.17)

On the other hand, Hsiung (1966) defined them by the following operators

(1.18) ,

r+s=p r+1,s

(1.19) ,

r+s=p r ,s

for a *p*-form ). After then we shall show that the relation

(1.20)

is valid.

**2. Operators on Almost Kaehlerian Manifolds:**

We have studied the following properties of the operators

**Lemma (2.1):** In an almost Kaehlerian space, the operator is askew-derivation and satisfies

(2.1) =

**Proof:** Ogawa (1967) gives that is a skew-derivation and that for any *p*-form *u*

=, **=**

holds, where ***n*** is the dimension of the space. Since ***n*** is even, therefore

(2.1) is proof.

**Lemma (2.2):** In an almost Kaehlerian space, the operator is a derivation and satisfies for any *p*-form ,

(2.2) =

(2.3)

**Proof:** From directive calculation with respect to an orthonormal local coordinate system for any *p*-form, we have

= .

Since n is even, we have, and thus (2.2) is proved.

Now, we have

=

+

=

+

.

Hence it follows that

=

=

=

Now, we have consider the following relation

+

=

Then, we have

+

=

+

=

=

Thus, the operator is a derivation. From this, we have the following:

**Corollary (2.1):** In almost Kaehlerian space, the operator is a skew-derivation.

**Corollary (2.2):** In almost Kaehlerian space, the relation

(2.4) = +

holds.

**Theorem (2.3):** In almost Kaehlerian spacer, we have

(2.5) =

where denotes the inner product with respect to a 1-form)

**Proof:** We have the definition of**,** fora*p*-form*u***,**

**,**

Where, we write Therefore we have

=

=

= [ i (

Similarly, we have proof of the following:

**Theorem (2.4):** In an almost Kaehlerian space, we have

(2.6)

(2.7)

(2.8)

(2.9)

(2.10) ,

(2.11)

**3. Structure on almost kaehlerian spaces:**

**Theorem (3.1):** In an almost Kählerian space, the structure's integrability is both a necessary and sufficient condition when:

= 0.

**Proof.** We have the intergability condition of the almost complex structure is defined by given by [2] Kodaira and spencer (1957), Then by equation (2.10)

+ (d

Considering that the imaginary components disappear due to the implication of *Corollary* (2.2), we derive the result: Which is real operator.

The operator which delineates a Kählerian structure through an almost Hermitian structure, demonstrates Kählerian characteristics only when the operator ceases to have an effect. As functions as a skew- derivation, its second operation, , acts as a derivation. Consequently, when nullifies its impact on forms of degrees 0 and 1,its influence dissipates across forms of all degrees. Taking into consideration a 0-form *f*  and a 1-form the following relationship holds:

Where indicates that the terms are summed cyclically with respect to I, i, j, k. Consequently, the condition can be expressed equivalently through the following relationships:

(3.1)

(3.2)

**Theorem (3.2):** In analmost Kaehlerian space, the operator consistently equals zero.

**Proof:** Since the complex structure is a covariant constant in an Kaehlerian space, we have from (3.1) , and therefore = ,

which gives (3.2) holds.

**Theorem (3.3):** In an almost Kaehlerian space, when if signifies that the structure is almost semi-Kaehlerian.

**Proof:** We have, Transvecting (3.1) with, then = 0.

Contracting ***l* and h** and noting prove the theorem.

**Theorem (3.4):** If in an almost Kaehlerian space, then we have

(3.3)

(3.4)

(3.5)

**Proof:** Here, from equation (3.2), we get

(3.6)

Taking the sum of terms of (3.6) cyclically with respect to the indices **i, j, k**, we have

= 0.

gives (3.3). Contraction of **i and** in (3.3) yields

(3.7)

And, from equation (3.6) we get

Which can reduced to (3.4) by contracting with Also, from (3.7) and (3.4), then we get the relation (3.5).

**Theorem (3.5):** If in an almost Kaehlerian space, then we have

(3.8)

**Proof:** We have,Differentiating (3.1) by , then

From (3.4) and (3.5) and noting above equation, we get

Here, the second and third terms on the right- hand side are reduced to and , respectively and thus we have (3.8).

**Theorem (3.6):** If the operator vanishes everywhere in an almost Kaehlerian space, it implies that the space is Kaehlerian.

**Proof:** Here, firstly we prove that

(3.9)

Then, by virtue of (3.1) we find

the above equation and (3.8) and (3.5) gives

Now contracting above equation with and noting *theorem* (3.5)*,* we obtain (3.9). From equation (3.9) follows immediately Which means proving the structure to be Kaehlerian.

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