

# ***M*- Projective Curvature Tensor over $(\kappa, \mu)$ - Contact Riemannian Manifolds**

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**Abstract:** -

In 1995, the concept of  $(\kappa, \mu)$ -contact Riemannian manifolds was introduced by Blair, Koufogiorgos, and Papantoniou [5]. Subsequently, a comprehensive investigation into the classification of contact metric  $(\kappa, \mu)$ -spaces was conducted by Boeckx, E. [7] in 2000. Blair explored the  $(\kappa, \mu)$ -nullity condition in the context of contact Riemannian manifolds and provided various motivations for its study. The current paper focuses on the examination of flatness conditions concerning the *M*-projective curvature tensor within the framework of  $(\kappa, \mu)$ -contact Riemannian manifolds.

## **1. Introduction-**

In 1958, Boothby and Wong first introduced the concept of odd-dimensional manifolds with contact and almost contact structures, primarily approaching it from a topological perspective. Subsequently, in 1961, Sasaki and Hatakeyama re-examined these structures using tensor calculus techniques.

Alternatively, in the work of Pokhariyal and Mishra, a tensor field  $W^*$  is introduced on a Riemannian manifold as

$$W^*(S, T, U, V) = \mathcal{R}(S, T, U, V) - \frac{1}{2(n-1)} \times [\rho(T, U)g(S, V) - \rho(S, U)g(T, V) + g(T, U)\rho(S, V) - g(S, U)\rho(T, V)], \quad (1)$$

Where  $W^*(S, T, U, V) = g(W^*(S, T)U, V)$  and  $\mathcal{R}(S, T, U, V) = g(\mathcal{R}(S, T)U, V)$ . The tensor field  $W^*$  is referred to as the *M*-projective curvature tensor. Subsequently, Ojha conducted a comprehensive investigation of the properties of this tensor in both Sasakian and Kähler manifolds.

The category of  $(\kappa, \mu)$ -contact Riemannian manifolds encompasses both Sasakian and non-Sasakian manifolds. Boeckx [7] provided a comprehensive categorization of  $(\kappa, \mu)$ -contact Riemannian manifolds. These manifolds retain their properties under *D*-homothetic transformations.

In an earlier study [6], Blair, Kim, and Tripathi commenced an inquiry into the concircular curvature tensor of contact Riemannian manifolds. The examination of the pseudo-projective curvature tensor on a contact Riemannian manifold was recorded in [5]. More contemporarily, the investigations carried out by [14] and [15] delved into exploring the quasi-conformal curvature tensor and the E-Bochner curvature tensor on a  $(\kappa, \mu)$ -contact Riemannian manifold, respectively. In addition to the well-known Riemannian curvature tensor, the Weyl conformal curvature tensor, and the concircular curvature tensor, the *M*-projective curvature tensor

emerges as a pivotal tensor within the realm of differential geometry. This curvature tensor effectively bridges the gap between the conformal curvature tensor, the conharmonic curvature tensor, and the concircular curvature tensor on one side, while connecting with the  $H$ -projective curvature tensor on the other.

Recently, the  $M$ -projective curvature tensor has been a subject of study for various researchers, including Chaubey, Ojha [13], Singh [11], and others.

Expanding upon prior research, our current study investigates the symmetry and flatness characteristics of  $(\kappa, \mu)$ -contact Riemannian manifolds in the context of the  $M$ -projective curvature tensor. In Section 3, we review and deduce our initial findings. Subsequently, in Segment 4, we analyze  $M$ -projectively flat  $(\kappa, \mu)$ -contact Riemannian manifolds. Segment 5 centers on exploring  $\zeta$ - $M$ -projectively Sasakian flat  $(\kappa, \mu)$ -contact Riemannian manifolds, where we establish the requisite and sufficient conditions for the manifestation of  $\zeta$ - $M$ -projective Sasakian flatness in an  $(\kappa, \mu)$ -contact Riemannian manifold.

## 2. Contact Riemannian Manifold-

An almost contact structure on an  $(2n+1)$ -dimensional differentiable manifold  $M$  is defined by the existence of a tensor field  $\mathcal{F}$  of type  $(1, 1)$ , a vector field  $\zeta$ , and a 1-form  $\eta$  such that

$$\mathcal{F}^2 = -I + \eta \otimes \zeta, \quad \eta(\zeta) = 1 \quad (2)$$

$$\mathcal{F}\zeta = 0, \quad \eta \circ \mathcal{F} = 0 \quad (3)$$

Take into account a consistent Riemannian metric  $g$  in conjunction with an almost contact structure  $(\mathcal{F}, \zeta, \eta)$

$$g(\mathcal{F}S, \mathcal{F}T) = g(S, T) - \eta(S)\eta(T) \quad (4)$$

Subsequently, when  $M^{2n+1}$  undergoes a transformation, it transforms into an almost contact Riemannian manifold by acquiring an almost contact metric structure represented as  $(\mathcal{F}, \zeta, \eta, g)$ . By observing equations (2) and (4), it becomes evident that

$$g(S, \mathcal{F}T) = -g(\mathcal{F}S, T), \quad g(S, \zeta) = \eta(S), \quad (5)$$

for all vector fields  $S$  and  $T$ .

The fact that the tangent sphere bundle of a Euclidean Riemannian manifold possesses a contact metric structure with the property  $\mathcal{R}(S, T)\zeta = 0$  is widely acknowledged. Conversely, in the context of a Sasakian manifold, the subsequent assertion is valid:

$$\mathcal{R}(S, T)\zeta = \eta(T)S - \eta(S)T. \quad (6)$$

Blair et al. extended the concepts of  $\mathcal{R}(S, T)\zeta = 0$  and the Sasakian case by investigating the  $(\kappa, \mu)$ -nullity condition on a contact Riemannian manifold. They introduced the  $(\kappa, \mu)$ -nullity distribution  $N(\kappa, \mu)$  ([3,5]) to characterize this condition on the contact Riemannian manifold.

$$N(\kappa, \mu): \mathcal{P} \rightarrow N_{\mathcal{P}}(\kappa, \mu) = \{U \in \mathcal{T}_{\mathcal{P}}M: \mathcal{R}(S, T)U = (\kappa I + \mu h)[g(T, U)S - g(S, U)T]\} \quad \dots(7)$$

For any pair of vectors  $S$  and  $T$  belonging to the tangent space  $\mathcal{T}M$ , where  $(\kappa, \mu)$  are elements of the  $R^2$ , a Riemannian manifold  $M^{2n+1}$  possessing  $\zeta$  in the set  $N(\kappa, \mu)$  is referred to as a

manifold with  $(\kappa, \mu)$  characteristic. Specifically, on a manifold with  $(\kappa, \mu)$  attributes, the following holds true

$$\mathcal{R}(S, T)\zeta = \kappa[\eta(T)S - \eta(S)T] + \mu[\eta(T)hS - \eta(S)hT]. \quad (8)$$

On a  $(\kappa, \mu)$ -manifold, where  $\kappa \leq 1$ , the structure becomes Sasakian with  $h = 0$  and  $\mu$  remaining indeterminate when  $\kappa = 1$ . When  $\kappa < 1$ , the  $(\kappa, \mu)$ -nullity condition uniquely prescribes the curvature of  $M^{2n+1}$ . Essentially, for a  $(\kappa, \mu)$ -manifold, the properties of being a Sasakian manifold, a  $K$ -contact manifold,  $\kappa = 1$ , and  $h = 0$  are all interchangeable and equivalent.

In a  $(\kappa, \mu)$ -manifold, the following relations hold:

$$\begin{aligned} h^2 &= (\kappa - 1)^2 \mathcal{F}^2, \quad \kappa \leq 1, \\ \mathcal{R}(\zeta, S)T &= \kappa[g(S, T)\zeta - \eta(T)S] + \mu[g(hS, T)\zeta - \eta(T)hS], \\ \rho(S, \zeta) &= 2n\kappa\eta(S), \\ \rho(S, T) &= [2(n - 1) - n\mu]g(S, T) + [2(n - 1) + \mu]g(hS, T) \\ &\quad + [2(1 - n) + n(2\kappa + \mu)]\eta(S)\eta(T), \quad n \geq 1, \\ \rho(\mathcal{F}S, \mathcal{F}T) &= \rho(S, T) - 2n\kappa\eta(S)\eta(T) - 2(2n - 2 + \mu)g(hS, T), \end{aligned} \quad (9)$$

Where  $\rho$  is the Ricci tensor of type  $(0, 2)$ ,  $Q$  is the Ricci operator, that is,  $g(QS, T) = \rho(S, T)$ .

Furthermore, the  $(\kappa, \mu)$ -manifold exhibits the following property:

$$\begin{aligned} \eta(\mathcal{R}(S, T)U) &= \kappa[g(T, U)\eta(S) - g(S, U)\eta(T)] \\ &\quad + \mu[g(hT, U)\eta(S) - g(hS, U)\eta(S)] \end{aligned} \quad (10)$$

In the context of Riemannian manifold, the  $M$ -projective curvature tensor  $W^*$  can be stated as follows [8].

$$W^*(S, T)U = \mathcal{R}(S, T)U - \frac{1}{2(n-1)} \times [\rho(T, U)S - \rho(S, U)T + g(T, U)QS - g(S, U)QT], \quad \dots(11)$$

Given arbitrary vector fields  $S, T$ , and  $U$ , where  $\rho$  represents the Ricci tensor of type  $(0, 2)$  and  $Q$  denotes the Ricci operator, we have the relation  $g(QS, T) = \rho(S, T)$ .

**Lemma 2.1.** [1] In  $(\kappa, \mu)$ -contact Riemannian manifolds that are not Sasakian, the conditions that follow are mutually equivalent:

- (i)  $\eta$ -Einstein manifold,

$$(ii) Q\mathcal{F} = \mathcal{F}Q$$

**Definition 2.1.** An  $M$  manifold with a  $(k, \mu)$ -contact metric structure is referred to as  $\eta$ -Einstein when the Ricci operator  $Q$  fulfills the conditions

$$Q = aI + b\eta \otimes \zeta \quad (12)$$

Where  $a$  and  $b$  represent smooth functions defined on the manifold. Notably, when  $b$  is set to zero,  $M$  qualifies as an Einstein manifold.

In the case where an  $(2n+1)$ -dimensional non-Sasakian  $(k, \mu)$ -contact Riemannian manifold  $(M^{2n+1}, g)$  is  $\eta$ -Einstein, the expression for the non-zero Ricci tensor  $\rho$  takes the following form:

$$\rho(S, T) = ag(S, T) + b\eta(S)\eta(T). \quad (13)$$

**Lemma 2.2.** On a non-Sasakian  $(k, \mu)$ -contact Riemannian manifold  $(M^{2n+1}, g)$ ,  $a + b = 2n\kappa$

Proof. In view of (2)-(5) and (13), we have

$$QS = aS + b\eta(S)\zeta, \quad (14)$$

such that Ricci operator  $Q$  is defined by

$$\rho(S, T) = g(QS, T). \quad (15)$$

Again, contracting (14) with respect to  $S$  and using (2)-(5), we have

$$r = (2n + 1)a + b. \quad (16)$$

Now, putting  $\zeta$  instead of  $S$  and  $T$  in (13) and then using the equations in (2)-(5) and (9) we get

$$a + b = 2n\kappa. \quad (17)$$

Equations (16) and (17) give

$$a = \left(\frac{r}{2n} - \kappa\right) \text{ and } b = \left((2n + 1)\kappa - \frac{r}{2n}\right). \quad (18)$$

Equation (18) prove the statement of the Lemma 2.2.

### 3. The $M$ -projective curvature tensor $W^*$ for an $(\kappa, \mu)$ - contact Riemannian manifolds-

The curvature tensor  $W^*$  associated with  $M$ -projective geometry on a  $(\kappa, \mu)$ -contact Riemannian manifold is expressed as

$$W^*(S, T)\zeta = -\frac{\kappa}{(n-1)}[\eta(T)S - \eta(S)T] + \mu[\eta(T)hS - \eta(S)hT] - \frac{1}{2(n-1)}[\eta(T)QS - \eta(S)QT], \quad (19)$$

$$\eta(W^*(S, T) \zeta) = 0, \quad (20)$$

$$\begin{aligned} W^*(\zeta, T)U &= -W^*(T, \zeta)U = -\frac{\kappa}{(n-1)} [g(T, U)\zeta - \eta(U)T] + \mu[g(hT, U)\zeta - \eta(U)hT] \\ &\quad -\frac{1}{2(n-1)} [\rho(T, U)\zeta - \eta(U)QT], \end{aligned} \quad (21)$$

$$\begin{aligned} \eta(W^*(\zeta, T)U) &= -\eta(W^*(T, \zeta)U) \\ &= -\frac{\kappa}{(n-1)} [g(T, U) - \eta(T)\eta(U)] + \mu[g(hT, U) - \eta(U)\eta(hT)] \\ &\quad -\frac{1}{2(n-1)} [\rho(T, U) - 2n\kappa\eta(T)\eta(U)], \end{aligned} \quad (22)$$

$$\begin{aligned} \eta(W^*(S, T)U) &= -\frac{\kappa}{(n-1)} [g(T, U)\eta(S) - g(S, U)\eta(T)] + \mu[g(hT, U) - \eta(U)\eta(hT)] \\ &\quad -\frac{1}{2(n-1)} [\rho(T, U)\eta(S) - \rho(S, U)\eta(T)]. \end{aligned} \quad (23)$$

#### 4. $M$ -Projectively Flat $(\kappa, \mu)$ -Contact Riemannian Manifolds-

The class of  $(\kappa, \mu)$ -contact Riemannian manifolds known as  $M$ -projectively flat manifolds is a distinctive category within contact Riemannian manifold where the geometry is such that the curvature tensor satisfies certain conditions related to the  $M$ -projective flatness property. The parameters  $\kappa$  and  $\mu$  are involved in the definition of the curvature conditions and can affect the geometry of the manifold.

**Theorem 4.1.** A  $(\kappa, \mu)$ -contact Riemannian manifold  $M^{2n+1}$  that is  $M$ -projectively flat exhibits the property of being an Einstein manifold.

Proof. Let  $W^*(S, T, U, V) = 0$ . Subsequently, utilizing equation (11), we derive the following outcome:

$$\begin{aligned} \mathcal{R}(S, T, U, V) &= \frac{1}{2(n-1)} [\rho(T, U)g(S, V) - \rho(S, U)g(T, V) + g(T, U)\rho(S, V) \\ &\quad -g(S, U)g(T, V)] \end{aligned} \quad (24)$$

Considering  $e_i$  as an orthonormal basis of the tangent space at any point, if we set  $T = U = e_i$  in the given equation and then sum up over  $i$ , where  $1 \leq i \leq 2n + 1$ , we arrive at the same result,

$$\rho(S, T) = -rg(S, T), \quad (25)$$

Where  $r$ -Scalar curvature of the manifold and  $r = 2n(2n - 2 + \kappa - n\mu)$ .

This indicates that  $M^{2n+1}$  is an Einstein manifold. This completes the proof.

### 5. $\zeta$ - $M$ - Projectively Sasakian Flat $(\kappa, \mu)$ -Contact Riemannian Manifolds-

$\zeta$ - $M$ -Projectively Sasakian flat  $(\kappa, \mu)$ -contact Riemannian manifolds likely refer to a specific class of contact Riemannian manifolds that satisfy curvature conditions related to  $M$ -projective flatness and these manifolds also have a distinguished Reeb vector field ( $\zeta$ ) and Sasakian geometry. This indicates a very specialized and intricate geometric structure where various curvature conditions, contact structures, and vector fields are intertwined.

**Definition 5.1.** An  $(2n+1)$ -dimensional (with  $n > 1$ )  $(\kappa, \mu)$ -contact Riemannian manifold is classified as  $\zeta$ - $M$ -projectively Sasakian flat when the condition  $W^*(S, T)\zeta = 0$  holds for all  $S$  and  $T$  belonging to the tangent space  $\mathcal{T}M$ .

**Theorem 5.1.** An  $(2n+1)$ -dimensional ( $n > 1$ )  $(\kappa, \mu)$ -contact Riemannian manifold exhibits  $\zeta$ - $M$ -projective Sasakian flatness if and only if it possesses the characteristic of being an  $\eta$ -Einstein manifold.

Proof. Let  $W^*(S, T)\zeta = 0$ . Then, in view of (11), we have

$$\mathcal{R}(S, T)\zeta = \frac{1}{2(n-1)} [\rho(T, \zeta)S - \rho(S, \zeta)T + g(T, \zeta)QS - g(S, \zeta)QT] \quad (26)$$

Due to the presence of (5), (8), and (9), the equation above can be simplified to

$$\begin{aligned} & \kappa[\eta(T)S - \eta(S)T] + \mu[\eta(T)hS - \eta(S)hT] \\ &= \frac{n\kappa}{n-1} [\eta(T)S - \eta(S)T] + \frac{1}{2(n-1)} [\eta(T)QS - \eta(S)QT] \end{aligned} \quad (27)$$

which by putting  $T = \zeta$ , gives

$$QS = 2\kappa [-S + (n+1)\eta(S)\zeta] + 2(n-1)\mu(hS) \quad (28)$$

In the case of Sasakian manifolds,  $\kappa = 1$ , (and hence  $h = 0$ )

Now, taking the inner product of above equation with  $V$ , we get

$$\rho(S, V) = 2[-g(S, V) + (n+1)\eta(S)\eta(V)] \quad (29)$$

Furthermore, it can be demonstrated that a  $(\kappa, \mu)$ -contact Riemannian manifold represents an  $\eta$ -Einstein manifold. Conversely, assume that condition (29) is fulfilled. As a result of the implications of (28) and (19), we can deduce  $W^*(S, T)\zeta = 0$ . Thus, the proof is concluded.

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