Rascal Triangle

Suryaprakash Vishwakarma
Department of Mathematics
B.K. Birla College
Kalyan, India
fundamantalsurya@gmail.com

Seema Purohit
Department of CS, IT and Mathematics
B. K. Birla College
Kalyan, India
supurohit@gmail.com

ABSTRACT

In Combinatorics use of Pascal Triangle techniques and identities is well known when it comes to deriving Binomial Coefficients, Fibonacci Numbers, Interesting Numbers Patterns. Literature shows though not much efforts are being done to generate the Pascal like pattens, a decade ago in 2010, Pascal like triangles are generated by three middle school students using their own alternate ways, without knowing what Pascal Triangle is about. They coined the name “Rascal Triangle” for the new number patterns generated by them. In this Chapter, researchers tried to regenerate the generalizations of Rascal Triangle using recursion and generating functions as well as arithmetic progression. It is demonstrated that these Generalized Rascal Triangles are characterized by arithmetic sequences on all rows / columns as well as Rascal-like multiplication and addition rules.

Keywords—Pascal Triangle, number pattern construction, Rascal Triangle, intersection, summation

I. INTRODUCTION

Combinatorics is an important branch of mathematics which helped in providing solutions to many problems in Computer Science and other fields. Combinatorial problems involve finding a grouping, patterns, ordering, or assignment of a discrete, finite set of objects or numbers that satisfies given conditions.

Combinatorics also forms an important chapter in the history of Indian mathematics, way back in the 2nd century B.C.E. with the formal theory of Sanskrit meters formulated by Piṅgala[1]. The recursive algorithms proposed by him are considered as the first example of recursion in Indian mathematics. The calculation of the binomial coefficients, use of repeated partial sums of sequences and the formula for summing a geometric series by Pingala has become an integral part of Indian mathematics. Unfortunately, there are very few efforts made to preserve, modify, adapt, or surpass Pingala’s Algorithms till today. Albrecht Weber’s criticism about Halāyudha’s attribution of the construction, which now is known as Pascal’s triangle, still have the traces to the construction given by Piṅgala[1].

Pascal's triangle is named after Blaise Pascal, who published an important treatise on the subject in the 17th century. However, Pascal was not the discoverer of the triangle, it was already known in medieval Europe, China and discovered much earlier in Asia rather in India. It is Pascal who developed many uses for it and was the first one to organise all the information together in his 1653 treatise and his work on the binomial coefficients led to Newton's discovery of the general binomial theorem for fractional and negative powers [2].

The number pattern of Pascal Triangle has been a subject of interest not only for many mathematicians but also for students, who are fascinated by the symmetric triangular structure. There are even attempts by them to construct the triangular pattern differently which looks like Pascal Triangle and hence coined the name Rascal Triangle.

II. LITERATURE REVIEW

A. Origin of Rascal Triangle

In 2010, middle school students Alif Anggaro, Eddy Liu and Angus Tulloch [2] were asked to determine the next row of numbers in the following triangular array, by their teacher:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ ?&&?&&?&&?&&? \end{array}$$

Figure 1 A triangular array

Instead of providing the expected answer

1 4 6 4 1

from Pascal’s Triangle, they produced the row

1 4 5 4 1.

They did this by using the rule that the outside numbers are 1s and the inside numbers are determined by the diamond formula

South = East ・West + 1 / North

where North, South, East and West form a diamond in the triangular array as in Figure 2.

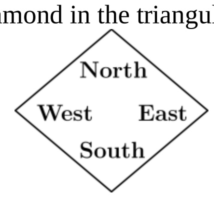

Figure 2 North, South, East and West entries in a triangular array

Continuing with this rule Anggaro, Liu and Tulloch[2] created a number triangle they called the Rascal Triangle.

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 5 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 7 & & 7 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 9 & & 10 & & 9 & & 6 & & 1 & & \\
 & 1 & & 7 & & 11 & & 13 & & 13 & & 11 & & 7 & & 1 & \\
1 & & 8 & & 13 & & 16 & & 17 & & 16 & & 13 & & 8 & & 1 \\
 & & & & & & & & \vdots & & & & & & & &
\end{array}
$$

Figure 3 The Rascal Triangle.

Because the diamond formula involves division, their instructor challenged Anggaro, Liu and Tulloch, to prove that it would always result in an integer. They did this by observing that the diagonals in the Rascal Triangle formed arithmetic sequences.

In particular, the k^{th} entry on the r^{th} diagonal running from right to left is given by $1 + rk$, where $r = 0$ corresponds to the outside diagonal consisting of all 1s, and $k = 0$ corresponds to the first entry on each diagonal. Thus, if North = $1+(r-1)(k-1)$, then East = $1+r(k-1)$, West = $1+(r-1)k$ and South = $1+rk$,

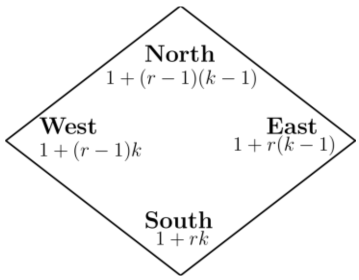

Figure 4 Algebraic Representation of North, East, West and South

and a straightforward calculation verifies that South = (East ・West + 1) / North

In the Spring of 2015 semester, a Mathematics for Liberal Arts class taught by, Julian Fleron, they discovered that the Rascal Triangle can also be generated by the rule that the outside numbers are 1s and the inside numbers are determined by the formula: South = East +West −North + 1.

This formula also follows from the arithmetic progressions along the diagonals [2]. Thus, the Rascal Triangle has the property that for any diamond containing 4 entries, the South entry satisfies two equations:

South = East ・West + 1 / North ………… (1)

South = East +West −North + 1 ………… (2)

The fact that both Equations (1) and (2) can be used to generate the Rascal Triangle was intriguing and this has led to the assertion that Rascal Triangle uniquely defined by either one of the two equations; so, the process of deducing equation (1) from equation (2) or vice versa started.

In addition, taking forward the idea of Julian Fleron and using some mathematical ideas discovered during liberal arts classes, some interesting patterns in the Rascal Triangle started emerging [3].

Philip K. Hotchkiss in 2019, [4] developed his own patterns and recursive relations to derive the Generalised Rascal Number Triangles, by defining the Major and Minor diagonals. He developed his own formulae and along with the number patterns discovered diamond structures hidden in the Rascal Triangles. Starting with the definition of Generalise Rascal Triangle, he proposed and demonstrated the algebraic properties of Rascal Numbers. Interesting properties of Rascal Triangle including Arithmetic diagonals implies a Generalized Rascal Triangle, Uniqueness of the Rascal Triangle, Odd/Even Diamond Patterns, modified Ashley Rule, T – Meg Rule are derived to further extend the scope of Generalised Rascal Triangle.

III. PROPOSED RASCAL TRAINGLE CONSTRUCTION

The Rascal triangle is classified into row, column and diagonals as shown in figure 5 which are denoted by j, i and d (diagonal of LHS) and d'(diagonal of RHS) respectively.

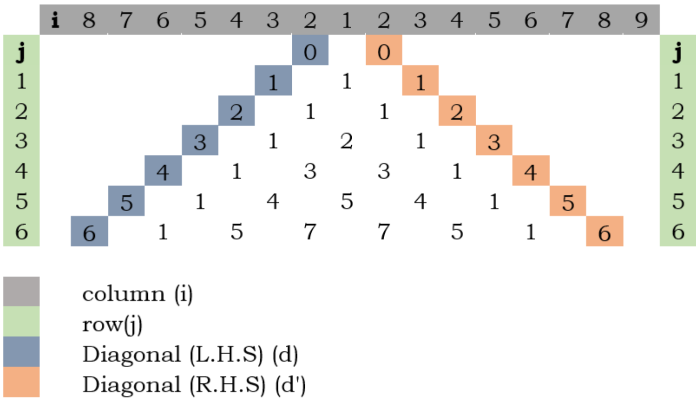

Figure 5 Rascal Triangle

Construction of the triangle is done by two different methods which are Row wise construction and Column wise construction.

A. Row wise construction

To derive a general formula for n^{th} entry in j^{th} row use the following:
Convention: Total no. of elements present in the j^{th} Row is denoted as Row number (j).
Notation: Denote the elements in the Row wise construction of the triangle by $E_R(j,n)$ (n^{th} element of Row j) and construct all entries of j^{th} Row using the following recursive definition.
$E(j,1) = 1\,, \forall j \in \mathbb{N}$
$E(j,k+1) = E(j,k) + j - 2k\,, \forall k \in \mathbb{N}$
Illustration and Hand Simulation for finding the generating function, $E_R(j,n)$
Take $j\ =\ 7$, That is element of 7^{th} row

1	6	9	10	9	6	1

Now we are going to see the relation in the given table to find a general formula,

Table 1: Hand Simulation of Recursive Definition

n	Element	In general,		
1	1	=1	=1	=1
2	6	=1+(7-2)	=1+(j-2)	=j-2+1
3	9	=6+(7-4)	=j-2+1+(j-4)	=2j-6+1
4	10	=9+(7-6)	=2j-6+1+(j-6)	=3j-12+1
5	9	=10+(7-8)	=3j-12+1+(j-8)	=4j-20+1
6	6	=9+(7-10)	=4j-20+1+(j-10)	=5j-30+1
7	1	=6+(7-12)	=5j-30+1+(j-12)	=6j-42+1

Take any element from the above table,

Let's take $6j - 42 + 1 \ldots (n = 7)$
$E_R(j, 7) = 6j - 42 + 1$
$= 6j - 6 * 7 + 1 = (7 - 1)j - (7 - 1)7 + 1$
Thus, more generally, we observe that, n^{th} element in j^{th} row can be given by:
$E_R(j, n) = (n - 1)j - (n - 1)(n) + 1$
$E_R(j, n) = (n - 1)(j - n) + 1 \quad \ldots\ldots\ldots\ldots\ldots \quad (1 \le n \le j)$
This is the formula to find n^{th} element of Row j, so by changing the value of n, one can find the all the elements of the particular Row j, and then by changing the value of j and n, the entire Rascal triangle can be constructed.
Theorem 1 For given j, k
$E_R(j, n) = (n - 1)(j - n) + 1 \ \forall n \in \mathbb{N}$
Proof: (by method of induction)
Let P(n): $E_R(j, n) = (n - 1)(j - n) + 1 \cdots (1 \le n \le j) \cdots\cdots (a)$
Step 1 Take $\boldsymbol{n = 1}$ $(\boldsymbol{a})$
We know that, $L.H.S = E_R(j, 1) = 1 \cdots\cdots (from\ definitiom)$
Now,
$R.H.S. = (1 - 1)(j - 1) + 1 = 1$
$L.H.S. = R.H.S.$
Hence (a) is true for $n = 1$ → P(1) is proved
Let us assume that (a) is true for $n = k$. That is,
$E_R(j, k) = (k - 1)(j - k) + 1 \cdots (1 \le k \le j) \cdots\cdots (b)$ holds
To Prove P(k+1) to be true that is, $\boldsymbol{E_R(j, k + 1) = ((k + 1) - 1)(j - (k + 1)) + 1}$
$E_R(j, k + 1) = k(j - k - 1) + 1 \cdots\cdots (c)$
Consider LHS:

$$L.H.S. = E_R(j, k + 1) = E(k, j) + j - 2k \cdots\cdots \text{(By definition of the Rascal triangle)}$$
$$= (k - 1)(j - k) + 1 + j - 2k \cdots\cdots (from\ (b))$$
$$= (K - 1)j - (k - 1)k + 1 + j - 2k$$
$$= (k\,j) - j - k^2 + k + 1 + j - 2k$$
$$= (k\,j) - k^2 + k + 1 - 2k$$
$$= (k\,j) - k^2 - k + 1$$
$$= k(j - k - 1) + 1$$
$$= RHS\ of\ (c)$$

B. Column wise construction

To construct the Column wise triangle construction
Convention: Denote $E_C(i, n)\ as\ n^{th}$ element of Column i.
More generally, all entries of i^{th} Column can be constructed using the following recursive definition.
$E_C(i, 1) = 1, \forall i \in \mathbb{N}$
$E_C(i, k + 1) = E(i, k) + i + 2(k - 1), \forall k \in \mathbb{N}$
This is recursive definition for the entries of i^{th} Column of the triangle. In the column, there is no limit for last value, so it goes on up to infinity.
Finding: general formula for i^{th} Column,
Notation: $E_C(i,\ n) \to n^{th}$ element of Column i.
Take $i = 2$, That is element of 2nd Column,

1	3	7	13	21	31	43	⋯⋯⋯

To find a general formula $E_C(i,\ n)$,

Table 2: Hand Simulation of Recursive Definition

n	Element	In general,		
1	1	= 1	= 1	=1
2	3	= 1+2	= 1+(i+0)	=i+1
3	7	= 3+(2+2)	= i+1+(i+2)	=2i+2+1
4	13	= 7+(2+4)	= 2i+2+1+(i+4)	=3i+6+1
5	21	= 13+(2+6)	= 3i+6+1+(i+6)	=4i+12+1
6	31	= 21+(2+8)	= 4i+12+1+(i+8)	=5i +20+1
7	43	= 31+(2+10)	= 5i+20+1+(i+10)	=6i+30+1
⋮	⋮	⋮	⋮	⋮

Take any element from the above table,
Let's take $6i + 30 + 1 \ldots (n = 7)$

$E_C(i, 7) = 6i + 30 + 1$
$\quad = 6\text{i} + 6 * 5 + 1$
$\quad = (7-1)\text{i} + (7-1)(7-2) + 1$
Thus, in general we can observe that n^{th} entry in i^{th} column can be given as:
$E_C(i, n) = (n-1)i + (n-1)(n-2) + 1$
$E_C(i, n) = \boldsymbol{(n-1)(i+n-2)+1} \cdots\cdots\cdots (n \in N)$
This is the formula to find n^{th} element of column i, so by changing the value of n,one can find all the elements of the particular Column i, and then by changing the value of i and n, we can construct the whole triangle.

Theorem 2 For given i,
$E_C(i, n) = (n-1)(i+n-2) + 1 \; \forall \, n \in \mathbb{N}$
Proof (by method of induction)
$Let\ P(n)$: $E_C(i, n) = \boldsymbol{(n-1)(i+n-2)+1} \cdots (n \in N) \cdots\cdots\cdots (a)$
Evaluate (a) for $n = 1$
R.H.S. $= E(I, 1) = 1 \cdots\cdots\cdots (from\ definition)$
Now,.$L.H.S. = (1-1)(i+1-2) + 1 = 1$
Hence, (a) is true for $n = 1$ → P(1) is true
Let us assume that (a) is true for $n = k$. That is,
$E_C(i, k) = (k-1)(i+k-2) + 1 \cdots (k \in N) \cdots\cdots\cdots (b)$ holds
To prove that P(k+1) is true, that is
$$E_C(i, k+1) = (k+1-1)(i+k+1-2) + 1 = k(i+k-1) + 1 \cdots\cdots\cdots (c)$$
Consider LHS:
$E_C(i, k+1) = E(i, k) + i + 2(k-1) \cdots\cdots$ (by the definition of rascal triangle)
$\quad = E(i, k) + i + 2(k-1)$
$\quad = (k-1)(i+k-2) + 1 + i + 2(k-1)$
$\quad = (k-1)i + (k-1)(k-2) + 1 + i + 2(k-1)$ ………..from (1)
$\quad = i(k-1+1) + (k-1)(k-2+2) + 1$
$\quad = ik + (k-1)k + 1$
$E(k+1\ i) = k(i+k-1) + 1 = RHS\ of\ (c)$

IV. ROW AND COLUMN WISE INTERSECTION

As we know in the Rascal Triangle, we have some different type of notation or representation like Row(j), Column(i), Diagonal from LHS is (d) and Diagonal from RHS is(d'). Here, we are going to introduce a new concept "INTERSECTIONAL ELEMENT" which is an element where Row, Column, and diagonal meets. Here, we can clearly observe that when we go along any 2 of them, then they intersect at a point (on an element) under certain conditions. The intersections can happen in three ways,
A. Row and Column.
B. Row and Diagonal.
C. Column and Diagonal.

A. Row and column.

In Rascal Triangle on close observation it indicates that when we go along Row and Column simultaneously then they intersect each other on an element denoted by $E_{CR}(i, j)$ and here we are going to find that element.

j	i	8	7	6	5	4	3	2	1	2	3	4	5	6	7	8	9	j
1									1									1
2								1		1								2
3							1		2		1							3
4						1		3		3		1						4
5					1		4		5		4		1					5
6				1		5		7		7		5		1				6
7			1		6		9		10		9		6		1			7
8		1		7		11		13		13		11		7		1		8

Figure 6: Intersecting Rows and Columns of Rascal Triangle

Note: when i and j have same pairity and $i \leq j$, only then th element exist, on different pairity element does not exist.

Where i and j intersect each other for that element n is same for both $E_C(i,n) and\ E\_R(j,n)$

$E_C(i,n) = E\_R(j,n)$

$\therefore (n-1)(i+n-2)+1 = (n-1)(j-n)+1$

$\therefore (i+n-2) = (j-n)$

$\therefore 2n = j - i + 2$

$\therefore n = \left(j - \frac{i}{2}\right) + 1 \cdots\cdots\cdots (\blacksquare)$

Now,

$E_{CR}(i,j) = \frac{E_C(i,n)+E_R(j,n)}{2}$

$= \frac{[\,(n-1)(i+n-2)+1\,] + [\,(n-1)(j-n)+1\,]}{2}$

$= \frac{(n-1)(i+n-2+j-n)+2}{2}$

$E_{CR}(i,j) = \frac{(n-1)(i+j-2)+2}{2}$

$E_{CR}(i,j) = \frac{\left(\left[\frac{j-i}{2}\right] + 1 - 1\right)(i+j-2) + 2}{2} \cdots\cdots\cdots (from\ \blacksquare)$

$E_{CR}(i,j) = \frac{\left(\frac{j-i}{2}\right)(i+j-2) + 2}{2}$

$E_{CR}(i,j) = \frac{(j-i)(j+i-2) + 4}{4} \cdots\cdots (\blacksquare\blacksquare)$

We can find this $R_{CR}(i,j)$(intersectional element) with two more ways which are given as follows:

A.1. With the help of Row $\boldsymbol{E_R(j,n)}$

Note $\boldsymbol{j \geq i}$.

We know that..

$E_R(j,n) = (n-1)(j-n) + 1 \cdots\cdots\cdots (a)$

$E_{CR}(i,j) = \left(\frac{j-i}{2} + 1 - 1\right)\left(j - \left[\frac{j-i}{2} + 1\right]\right) + 1 \cdots\cdots\cdots (from\ \blacksquare)$

$E_{CR}(i,j) = \left(\frac{j-i}{2} + 1 - 1\right)\left(j - \frac{j-i}{2} - 1\right) + 1$

$E_{CR}(i,j) = \left(\frac{j-i}{2}\right)\left(\frac{2j-j+i-2}{2}\right) + 1$

$E_{CR}(i,j) = \frac{\mathbf{(j-i)(j+i-2)+4}}{\mathbf{4}} = \blacksquare\blacksquare$

A.2. With the help of column $\boldsymbol{E_C(i,n)}$

Note $\boldsymbol{j \geq i}$.

We know that,

$E_C(i,n) = (n-1)(i+n-2) + 1$ ……[a]

$E_{CR}(i,j) = \left(\frac{j-i}{2} + 1 - 1\right)\left(i + \frac{j-i}{2} + 1 - 2\right) + 1$

$E_{CR}(i,j) = \left(\frac{j-i}{2}\right)\left(\frac{2i+j-i-2}{2}\right) + 1$

$E_{CR}(i,j) = \frac{(j-i)(j+i-2) + 4}{4} = \blacksquare\blacksquare$

B. Row and Diagonal:

We observe that for any diagonal, the difference between any two successive entries is constant. If this common difference is d, then we denote that diagonal by d.

if $j > d$ then $E_{RD}(j,d)$ will exist...($j \in \mathbb{N}$, $d \in \mathbb{W}$)

Intersectional element of row and diagonal, denoted by $E_{RD}(j,d)$ and we can prove this with the help of $E_R(j,n)$ as given below:

Where Row j and Diagonal d intersect each other for that element n is always same for both $E_R(j,n) and\ E_D(d,n)$

$E_R(j,n) = E_D(d,n)$

$\therefore (n-1)(j-n)+1 = (n-1)d+1$

$\therefore d = j - n \cdots\cdots\cdots (\beta)$

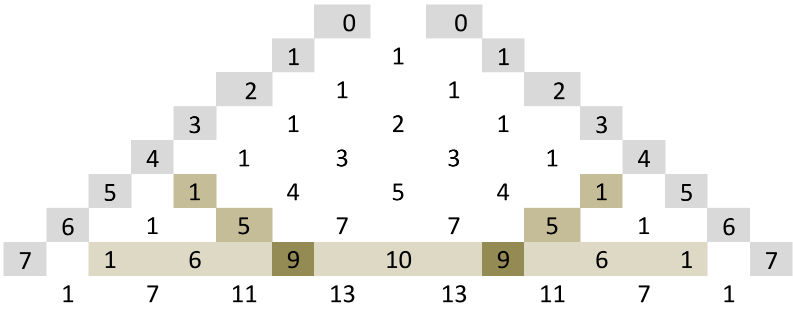

Figure 7: Row and Diagonal of Rascal Triangle

With the help of $E_R(j,n)$

We know that,

$E_R(j,n) = (n-1)(j-n)+1$

$E_{RD}(j,d) = (j-d-1)(j-[j-d])+1 \cdots\cdots\cdots (from\ \beta)$ … from (a)

$E_{RD}(j,d) = (j-d-1)(j-j+d)+1$

$E_{RD}(j,d) = (j-d-1)(j-j+d)+1$

$E_{RD}(j,d) = \boldsymbol{(j-d-1)d+1}$

C. Column and diagonal

Intersectional element if row and diagonal, denoted by E_{CD}. hear we will get two different intersectional elements according to the side of the diagonal and column.

1. Both are on same side $E_{CD}(i,d)$
2. Both are on opposite side $E_{CD'}(i,d)$

C.1. Both are on same side $E_{CD}(i,d)$

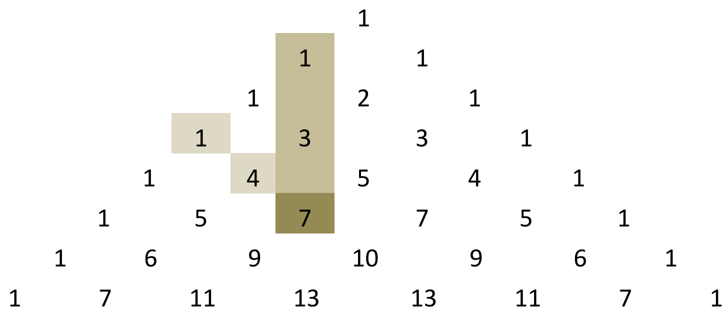

Figure 8: Column and Diagonal on same side of Rascal Triangle

if $i \leq d+1$ then $E_{CD}(i,d)$ will exist...$(j \in \mathbb{N},\ d \in \mathbb{W})$

since diagonal are in arithmetic progression.

That is, $E_D(d,n) = a+(n-1)d$.

Since where diagonal and column intersect each other for that element n is always same for both column and diagonal. And we are talking about the same element,

$\therefore E_C(i,n) = E_D(d,n)$

$\therefore (n-1)(i+n-2)+1 = (n-1)d+1$

$\therefore i+n-2 = d$

$\therefore n = d-i+2 \cdots\cdots\cdots \delta$

With the help of $E_C(i,n)$

We know that,

$E_C(i,n) = (n-1)(i+n-2)+1$

$E_{CD}(i,d) = (d-i+2-1)(i+[d-i+2]-2)+1 \cdots\cdots\cdots (from\ (\delta))$

$E_{CD}(i,d) = (d-i+2-1)(i+d-i+2-2)+1$

$\boldsymbol{E_{CD}(i,d) = (d-i+1)d+1}$

C.2. Both are on opposite side $E_{CD'}(i,d)$

$E_{CD'}(i,d)$ always exist...$(\forall j \in \mathbb{N}, and\ d \in \mathbb{W})$

Hear the value of ‘n’ is not same for $E_D(d,n)$ and $E_C(i,n)$, therefor hear we have two different value of ‘n’ for $E_D(d,n)$ and n' for $E_C(i,n')$.

We can observe the diagonal are in arithmetic progression.

That is, $E_D(d,n) = a+(n-1)d$.

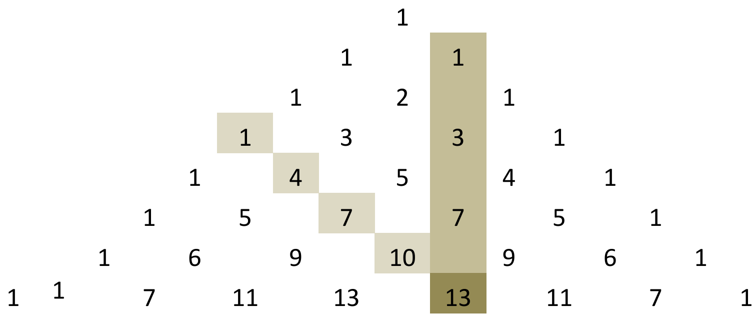

Figure 9: Column and Diagonal on opposite sides of Rascal Triangle

With the help of $\boldsymbol{E_C(i,n)}$
the relation between 'n', 'i' and 'd' for $E_C(i,n)$
$n = d + 1 \cdots\cdots\cdots$ (b)
We know that…
$E_C(i,n) = (n-1)(i+n-2)+1$
$E_{CD'}(i,d) \; = (d+1-1)(i+d+1-2)+1$
$E_{CD'}(i,d) \; = d(i+d-1)+1$
$E_{CD'}(i,d) \; = \boldsymbol{(i+d-1)d+1}$

V. RASCAL TRIANGLE SUMMATIONS

Using the recursive formulae and generating function
$E_R(j,n) \; = \; (n-1)(j \; - \; n) \; + \; 1 \; \forall n \in \mathbb{N}$
$E_C(i,n) = \; (n-1)(\, i+n- \; 2)+1 \; \forall \; n \in \mathbb{N}$
Summations can be found as follows:
A. Sum of first $\boldsymbol{n}$ element of Column $\boldsymbol{i}$.

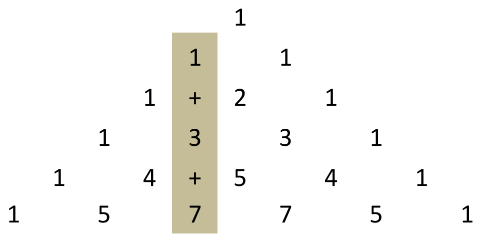

Figure 10: Column sum

We know that,
$E_C(i,n) \; = (n-1)(i+n-2)+1 \cdots\cdots (i,n)$
$E_C(i,n) \; = (n-1)i+(n-1)(n-2)+1 \cdots\cdots (i,n)$

Table 3: sum of column elements

1	0i+0(-1)+1	=0i+0+1
2	1i+(1)(0)+1	=1i+0+1
3	2i+(2)(1)+1	=2i+2+1
4	3i+(3)(2)+1	=3i+6+1
5	4i+(4)(3)+1	=4i+12+1
6	5i+(5)(4)+1	=5i+20+1
⋮	⋮	⋮

$$S(n\ \ i) = \{0i + 1i + 2i + 3i + 4i + \cdots + (n-1)i\} + \{0 + 0 + 2 + 6 + 12 + \cdots (n-1)(n-2)\} + \{1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \cdots + n\}$$

$$= \{i(1 + 2 + 3 + 4 + 5 + \cdots + (n-1)\} + \{(1^1 + 1) + (2^2 + 2) + (3^3 + 3) + \cdots + (n-1)(n-2)\} + \{1 + 1 + 1 + 1 + 1 + 1 + \cdots + n\}$$

$S(i, n) = i[\sum_{p=1}^{n-1} q\ ] + [\sum_{q=1}^{n-2}(q^2 + q)]$ …….(a)

Now take,

$$i(\sum_{p=1}^{n-1} q)\ = 1 + 2 + 3 + 4 + 5 + \cdots (n-1)$$

$$i(\sum_{p=1}^{n-1} q)\ = \frac{n(n-1)}{2} \quad \dots \dots (b)$$

Now take,

$$\sum_{q=1}^{n-2}(q^2 + q) = \sum_{q=1}^{n-2} q^2 + \sum_{q=1}^{n-2} q$$

For ‘n’…

$$\sum_{q=1}^{n}(q^2 + q) = \sum_{q=1}^{n} q^2 + \sum_{q=1}^{n} q$$

$$= (1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2) + (1 + 2 + 3 + 4 + \cdots + n$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)(2n+1) + 3n(n+1)}{6}$$

$$= \frac{n(n+1)(2n+4)}{6}$$

$$= \frac{2n(n+1)(n+2)}{6}$$

$$= \frac{n(n+1)(n+2)}{3}$$

But hear $n = n - 2$

$$\sum_{q=1}^{n-2}(q^2 + q) = \frac{(n-2)(n-2+1)(n-2+2)}{3}$$

$$\sum_{q=1}^{n-2}(q^2 + q) = \frac{n(n-1)(n-2)}{3} \quad \dots \dots \dots \dots (c)$$

From eq (a),(b) and (c)…..

$$\boldsymbol{S(i, n) = i\left(\frac{n(n-1)}{2}\right) + \left(\frac{n(n-1)(n-2)}{3}\right) + n}$$

B. Sum of first n element of Row j.

```
            1
         1     1
      1     2     1
   1     3     3     1
1  +  4  +  5  +  4     1
1     5     7     7     5     1
```

Figure 11: Row sum first n elements

We know that ,

$E_R(j, n) = (n-1)(j-n) + 1$

$E_R(j, n) = (n-1)j - n(n-1) + 1$ ……… $(1 \le n \le j)$

Table 4: Row sum

n	elements€	
1	0j – (1)(0) + 1	=0j – 0 + 1
2	1j – (2)(1) + 1	=1j – 2 + 1
3	2j – (3)(2) + 1	=2j – 6 + 1
4	3j – (4)(3) + 1	=3j – 12 + 1
5	4j – (5)(4) + 1	=4j – 20 + 1
6	5j – (6)(5) + 1	=5j – 30 + 1

$S(j,n) = \{\ 0j + 1j + 2j + 3j + \cdots + (n-1)j\ \} - \{0 + 2 + 6 + 12 + \cdots + n(n-1)\} + \{1 + 1 +\ 1 + 1 + \cdots + n\}$
$S^j_n = S(n\ \ j) = j\{\ 0 + 1 + 2 + 3 + \cdots + (n-1)\ \} - \{(0^2+0) + (1^2+1) + (2^2+2)(3^2+3) + \cdots + n(n-1)\} + \{1 + 1 +\ 1 + 1 + \cdots + n\}$
$S^i_n = S(n\ \ j) = j\ \sum_{q=0}^{n-1} q - \sum_{q=0}^{n-1}(q^2+q) + n$ ……….(a)
Now take,

$$\sum_{q=0}^{n-1} q\ = 0 + 1 + 2 + 3 + \cdots + (n-1)$$

$$\sum_{q=0}^{n-1} q\ = \left[\frac{n(n-1)}{2}\right]\ \ \dots\dots\dots(b)$$

Now ,

$$\sum_{q=0}^{n-1}(q^2+q) = \sum_{q=0}^{n-1} q^2 + \sum_{q=0}^{n-1} q\ \ \dots\dots\dots.(b)$$

$$\sum_{q=1}^{n} q^2 =\ \frac{n(n-1)(2n+1)}{6}$$

$$\therefore\ \sum_{q=1}^{n-1} q^2 =\ \frac{(n-1)n(2[n-1]+1)}{6}$$

$$\sum_{q=1}^{n-1} q^2 =\ \frac{(n-1)n(2n-1)}{6}\ \ \dots\dots\dots\dots\dots\dots(1)$$

$$\sum_{q=0}^{n-1} q\ = \frac{n(n-1)}{2}\ \dots\dots\dots(2)$$

From eq (b), (1) and (2)……..

$$\sum_{q=0}^{n-1}(q^2+q) = \frac{(n-1)n(2n-1)}{6} + \frac{n(n-1)}{2}$$

$$\sum_{q=0}^{n-1}(q^2+q) = \frac{n(n-1)}{2}\ (\frac{2n-1}{3} + 1)$$

$$\sum_{q=0}^{n-1}(q^2+q) = (\frac{n(n-1)}{2})\ (\frac{2n+2}{3})$$

$$\sum_{q=0}^{n-1}(q^2+q) = \frac{2n(n-1)(n+1)}{6}$$

$$\sum_{q=0}^{n-1}(q^2+q) = \frac{n(n-1)(n+1)}{3}\dots\dots\dots(c)$$

$$\boldsymbol{S(j,n) = j\left(\frac{n(n-1)}{2}\right) - \left(\frac{n(n-1)(n+1)}{3}\right) + n} \;\ldots\ldots [from\ eq(a),(b) and (c)]$$

C. Sum of all element of Row $\boldsymbol{j}$.

Note… *In any row 'j' , the value of 'j' and the total no. of element present in that row is always same ….*
$\therefore (n = j)$

1

1 1

1 2 1

1 3 3 1

1 + 4 + 5 + 4 + 1

1 5 7 7 5 1

Figure 12: Row sum

We know that…

$$S(j,n) = j\left(\frac{n(n-1)}{2}\right) - \left(\frac{n(n-1)(n+1)}{3}\right) + n$$

$$s(j,j) = j\left(\frac{j(j-1)}{2}\right) - \left(\frac{j(j-1)(j+1)}{3}\right) + j \ldots\ldots (j = n)$$

$$\mathbb{S}(j) = j(j-1)\left(\frac{j}{2} - \frac{j+1}{3}\right) + j$$

$$\mathbb{S}(j) = \frac{j(j-1)(3j-2j-2)}{6} + j$$

$$\mathbb{S}(j) = \frac{\boldsymbol{j(j-1)(j-2)}}{\boldsymbol{6}} + j$$

D. Sum of all elements from Row 1 to Row $\boldsymbol{j}$.

1

1 + 1

1 + 2 + 1

1 + 3 + 3 + 1

1 + 4 + 5 + 4 + 1

1 5 7 7 5 1

1 6 9 10 9 6 1

Figure 13: Total Row sum

We know that,

$$\mathbb{S}(j) = \frac{\boldsymbol{j(j-1)(j-2)}}{\boldsymbol{6}} + j$$

Table 5: Total Row sum

n	$\mathbb{S}(j)$	
1	$\frac{(1)(0)(-1)}{6} + 1$	$= \frac{0}{6} + 1$
2	$\frac{(2)(1)(0)}{6} + 2$	$= \frac{0}{6} + 2$
3	$\frac{(3)(2)(1)}{6} + 3$	$= \frac{6}{6} + 3$
4	$\frac{(4)(3)(2)}{6} + 4$	$= \frac{24}{6} + 4$
5	$\frac{(5)(4)(3)}{6} + 5$	$= \frac{60}{6} + 5$

now add all the above values …..

$$\$(j) = \left(\frac{0}{6} + \frac{0}{6} + \frac{6}{6} + \frac{24}{6} + \cdots \frac{n(n-1)(n-2)}{6}\right) + (1 + 2 + 3 + \cdots + j)$$

$$\$(j) = \frac{1}{6}\big(0 + 0 + 6 + 24 + \cdots n(n-1)(n-2)\big) + (1 + 2 + 3 + \cdots + j)$$
$$\$(j) = \frac{1}{6}\{(0^3 - 0) + (1^3 - 3) + (2^3 - 3) + \cdots + n(n-1)(n-2)\} + \{1 + 2 + 3 + \cdots + n\}$$
$$\$(j) = \frac{1}{6}\{(0^3 + 1^3 + 2^3 + 3^3 + \cdots (n-1)^3) - (0 + 1 + 2 + 3 + \cdots + (n-1)\} + \{1 + 2 + \ 3 + \cdots + n\}$$
$$\$(j) = \frac{1}{6}\left(\frac{j^2(j-1)^2}{4} - \frac{j(j-1)}{2}\right) + \frac{j(j+1)}{2}$$

VI. RESEARCH FINDINGS AND CONCLUSION

In sections III to V researchers have tried

- to derive the Pascal like Rascal Identities
- to give a new step by step approach for generating Rascal Triangle by using arithmetic sequences, row and column wise recursion formulae as well as generating functions.
- to devise the formulae for finding Intersectional Elements of Rows and Columns
- to derive the summation formulae for rows as well as columns.
- to find total row and column sums.

While deriving the above findings, researchers have identified some interesting, beautiful number patterns such ones, counting numbers, triangular numbers, symmetries, horizontal and vertical sums, primes which will definitely open new directions for further future applications in combinatorics and probability theory.

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