ON APPROXIMATION OF FUNCTION $\tilde{f} \in H_w$ CLASS BY *(C, 2)(E, 1)* MEANS OF CONJUGATE SERIES OF FOURIER SERIES.

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ABSTRACT- We studied on "degree of approximation of function belonging to Hölder metric by *(C, 2) (E, 1)* mean" has been discussed by Rathore, Shrivastava and Mishra. Since (E, 1*)* includes *(E, q)* method, so for obtaining more generalized result we replace *(E, q)* by *(E, 1)* mean. The Euler mean (E, 1) contains the summability method of generalized *Borel, Euler, Taylor etc.* In this chapter we obtain on "approximation of function $\tilde{f} \in H_w$ class by *(C, 2)(E, 1)* means of conjugate series of Fourier series" has been proved.

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1. INTRODUCTION

In this direction we studied on approximation of *f* belong to many classes also Hölder metric by Cesa̧ro mean, Nörlund mean, Euler mean has been discussed by several investigator like respectively Alexits [2], Khan [6], Chandra [3], Mohapatra and Chandra [11], Das, Ghosh and Ray[4], etc. Further in this field several researchers like Lal and Kushwaha [8], Lal and Singh [9], Rathore and Shrivastava [14], Nigam [12], Albayrak, Koklu and Bayramov [1], Rathore, Shrivastava and Mishra ([15], [16],), Kushwaha [7], Singh and Mahajan [18], Mishra and Khatri [10] etc. Recently Rathore, Shrivastava and Mishra [17] have been determined. We extend the result on "approximation of function $\tilde{f} \in H_w$ class by *(C, 2)(E, 1)* mean of conjugate series of Fourier series, has been proved.

2. DEFINITION AND NOTATIONS

Let *f (x)* be periodic and integrable in the sense of Lebesgue on [-π, π]. Then f(x) is defined by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx) \cong \sum_{n=0}^{\infty} A_n(x) \qquad (2.1)$$

The conjugate series of (2.1) is

$$\sum_{n=1}^{\infty}(b_n cosnx - a_n sinnx) \cong \sum_{n=1}^{\infty} B_n(x) \qquad (2.2)$$

with n^{th} partial sum $\widetilde{S_n}$ *(f; x)*

Let $w(t)$ and $w^*(t)$ denote two given moduli of continuity such that

$(w(t))^{\beta/\alpha}$ =O($w^*(t)$) as t→ 0^+ for 0< β≤ $\alpha \le 1$

If $C_{2\pi}$ denote the Banach spaces of all 2 π −periodic continuous function under "sup'' norm for 0<α≤ 1 and constant K the function H_w is

$$H_w = \{f \epsilon C_{2\pi}: |f(x)-f(y)| \le K\, w\, |x-y|\}. \qquad (2.3)$$

with the norm $\|.\|_{w^*}$ defined by

$$\|f\|_{w^*} = \|f\|_c + \operatorname{Sup}_{x,y} \Delta^{w^*} [f(x, y)], \qquad (2.4)$$

where

$$\|f\|_c = \operatorname{Sup}_{-\pi \le x \le \pi} |f(x)|. \qquad (2.5)$$

and

$$\Delta^{w^*}\{f(x, y)\} = \frac{|f(x)-f(y)|}{w^*(|x-y|)}, \qquad (x \ne y). \qquad (2.6)$$

the convention that Δ^0 f(x, y)=0. If there exit positive constant B and K such that w |x-y| $\leq B\ |x-y|^{\alpha}$ and w^*|x-y| $\leq K\ |x-y|^{\beta}$ then

$$H_\alpha = \{f\epsilon\ C_{2\pi}: |f(x)-f(y)| \leq K|x-y|^\alpha, 0<\alpha\leq 1\}. \text{ (see Prössdorf's[13])} \quad (2.7)$$

the metric induced (2.5) by the norm $\|.\|_\alpha$ on the H_α is called the Hölder metric. If can be seen that $\|f\|_\beta \leq (2\pi)^{\alpha-\beta}\ \|f\|_\alpha$ for $0\leq\beta<\alpha\leq1$.Thus $\{(H_\alpha, \|.\|_\alpha)\}$ is a family of Banach spaces which decreases as α increase.

The $\sum_{n=0}^{\infty} u_n$ is said to be (C, 2) summable to S. If the *(C, 2)* transform of S_n is defined as(see Hardy [5])

$$t_n^{\widetilde{(C,2)}}(f:x) = \frac{2}{(n+2)(n+1)} \sum_{k=0}^{n} (n-k+1)\, \widetilde{S_k} \to S \quad \text{as } n\to\infty \quad (2.8)$$

The $t_n^{\widetilde{(E,1)}}(f:x)$ denotes the transform of $\widetilde{(E,1)}$ is defined as

$$t_n^{\widetilde{(E,1)}}(f:x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \widetilde{S_k} \to S, \text{ as } n\to\infty$$

and

$$t_n^{\widetilde{(C,2)(E,1)}}(f:x) = \frac{2}{(n+2)(n+1)} \sum_{k=0}^{n} (n-k+1) \sum_{v=0}^{k} \binom{k}{v} \widetilde{S_v} \to S \quad \text{as } n\to\infty \quad (2.9)$$

The conjugate function $\widetilde{f(x)}$ is defined by

$$\widetilde{f(x)} = -\frac{1}{2\pi}\int_0^{\pi} \varphi(t)\ \cot\frac{t}{2}\, dt$$
$$= \lim_{h\to0}\left(-\frac{1}{2\pi}\int_h^{\pi} \varphi(t)\ \cot\frac{t}{2}\, dt\right) \quad (2.10)$$

"The degree of approximation $E_n(f)$ be

$$E_n(f) = min\, \|T_n - f\|_p\ , \quad (2.11)$$

$T_n(x)$ denotes a polynomial of degree n" by (see Zygmund[20]).

We shall use following notation

$$\Phi_x(t) = f(x+t)+f(x-t) -2f(x) \quad (2.12)$$

and

$$\varphi(t) = \Phi_x(t) - \Phi_y(t). \quad (2.13)$$

3. Known Theorem.

Theorem 1 (see [18]). Let w(t) defined in (2.3) be such that

$$\int_t^{\pi} \frac{w(u)}{u^2}\, du = O\,(H(t), H(t) \geq 0, \quad (3.1)$$

$$\int_0^t H(u)du = O(t\, H(t), \text{ as } t\to 0^+ \quad (3.2)$$

then, for $0< \beta\leq \alpha \leq 1$ and f ϵ H_α ,
we have

$$\|t_n^{C^1.E^1}(f)\text{-} f(x)\|_{w^*} = O\left(\left((n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right)^{1-\beta/\alpha}\right) \quad (3.3)$$

4. MAIN THEOREM

"On approximation of function $\tilde{f} \in H_w$ class by (*C*, 2)(*E, 1*) mean of conjugate of Fourier series" has been established.

Theorem: "If $\tilde{f} \in H_w$ and $0\leq\beta<\alpha\leq1$ then

$$\|\ t_n^{\widetilde{(C,2)(E,1)}}(f;x) - \tilde{f}(x)\|_{w^*}$$
$$= O\left\{\frac{w(|x-y|)^{\beta/\alpha}}{w^*(|x-y|)} (\log(n+1))^{\beta/\alpha} \left[(n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}\right\} \quad (4.1)$$

where $t_n^{(C,\widetilde{2)(E},1)}$ is the $(C,\widetilde{2)(E},1)$ mean of $S_n(f;x)$".

5. **Lemmas**: We require lemmas

Lemma 1. Let $\left|\widetilde{M_n(t)}\right| = \frac{1}{\pi(n+2)(n+1)}\left|\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin^t/_2}\right\}\right]\right|$

Apply $|\sin\frac{t}{2}| \geq \frac{t}{\pi}$ and $|\cos\left(v+\frac{1}{2}\right)t| \leq 1$, for $0 \leq t \leq \frac{\pi}{(n+1)}$

$$= \frac{1}{\pi(n+2)(n+1)}\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\frac{|\cos\left(v+\frac{1}{2}\right)t|}{|\sin^t/_2|}\right\}\right]$$
$$= \frac{1}{t(n+2)(n+1)}\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\right\}\right]$$
$$= \frac{1}{t(n+2)(n+1)}\sum_{k=0}^{n}(n-k+1) \qquad \left(\because \sum_{v=0}^{k}\binom{k}{v} = 2^k\right)$$
$$= \frac{(n+1)}{t(n+2)(n+1)} - \frac{1}{t(n+2)(n+1)}\sum_{k=0}^{n}k$$
$$= \frac{1}{t(n+2)} - \frac{n(n+1)}{2t(n+2)(n+1)}$$
$$= \frac{1}{t(n+2)} - \frac{n}{2t(n+2)}$$
$$= O\left(\frac{1}{t}\right) \tag{5.1}$$

Lemma2. Let $\left|\widetilde{M_n(t)}\right| = \frac{1}{\pi(n+2)(n+1)}\left|\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin^t/_2}\right\}\right]\right|$

Using $|\sin\frac{t}{2}| \geq \frac{t}{\pi}$ and $|\sin t| \leq 1$ for $\frac{\pi}{(n+1)} \leq t \leq \pi$

$$= \frac{1}{t(n+2)(n+1)}\left|\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\cos\left(v+\frac{1}{2}\right)t\right\}\right]\right|$$
$$= \frac{1}{t^2(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1) \qquad \text{(see [9])}$$

$$= \frac{(n+1)}{t^2(n+1)(n+2)} - \frac{n(n+1)}{2t^2(n+1)(n+2)}$$
$$= \frac{1}{t^2(n+2)} \tag{5.2}$$

Lemma 3. (see [18]). If w(t) satisfies condition (3.1) and (3.2), we get

$$\int_0^u t^{-1}w(t)dt = O(u\,H(u)), \qquad \text{as } u\to 0^+. \tag{5.3}$$

Lemma 4 Let $\Phi_x(t)$ defines (2.13) for $\tilde{f} \in H_w$

$$|\Phi_x(t) - \Phi_y(t)| \leq 2M\,w|x-y| \tag{5.4}$$

also
$$|\Phi_x(t) - \Phi_y(t)| \leq 2M\,w|t| \tag{5.5}$$

It is easy to verify.

6. **PROOF OF THE MAIN THEOREM**

Using (see [19]) and Riemann – Lebesgue theorem, then

$$\widetilde{S_n}(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin\frac{t}{2}}\cos\left(n+\frac{1}{2}\right)t\,dt \tag{6.1}$$

If $t_n^{(\widetilde{E,1})}$ denotes $(\widetilde{E,1})$ transform of $\widetilde{S_n}(f;x)$ then

$$t_n^{(\widetilde{E,1})}(f;x) - \tilde{f}(x) = \frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin^t/_2}\sum_{k=0}^{n}\binom{n}{k}\cos\left(k+\frac{1}{2}\right)t\,dt, \tag{6.2}$$

If $t_n^{(C,\widetilde{2)(E},1)}$ denotes $(C,\widetilde{2)(E},1)$ transform of $\widetilde{S_n}(f;x)$,

We write

$$t_n^{(C,\widetilde{2)(E},1)}(f;x)-\tilde{f}(x) = \frac{1}{\pi(n+1)(n+2)} \sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\int_0^{\pi}\frac{\phi_x(t)}{\sin^{t}/_2}\left\{\sum_{v=0}^{k}\binom{k}{v}\cos\left(v+\frac{1}{2}\right)t\right\}\right] \quad (6.3)$$

Writing $I_n(x) = t_n^{(C,\widetilde{2)(E},1)}(f;x) - \tilde{f}(x)$ we have

$$|I_n(x)| = |t_n^{(C,\widetilde{2)(E},1)}(f;x) - \tilde{f}(x)|$$

$$\leq \left|\frac{1}{\pi(n+2)(n+1)}\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\int_0^{\pi}\frac{\phi_x(t)}{\sin^{t}/_2}\left\{\sum_{v=0}^{k}\binom{k}{v}\cos\left(v+\frac{1}{2}\right)t\right\}\right]\right| dt \quad (6.4)$$

$$|I_n(x) - I_n(y)|$$

$$=\left|\frac{1}{\pi(n+2)(n+1)}\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\int_0^{\pi}\frac{\phi_x(t)-\phi_y(t)}{\sin^{t}/_2}\left\{\sum_{v=0}^{k}\binom{k}{v}\cos\left(v+\frac{1}{2}\right)t\right\}\right]\right| dt \quad (6.5)$$

$$=\frac{1}{\pi(n+2)(n+1)}\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\int_0^{\pi}\frac{|\phi_x(t)-\phi_y(t)|}{\sin^{t}/_2}\left\{\sum_{v=0}^{k}\binom{k}{v}\cos\left(v+\frac{1}{2}\right)t\right\}\right] dt$$

$$=\frac{1}{\pi(n+2)(n+1)}\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\int_0^{\pi}\frac{|\phi(t)|}{\sin^{t}/_2}\left\{\sum_{v=0}^{k}\binom{k}{v}\cos\left(v+\frac{1}{2}\right)t\right\}\right] dt$$

$$=\int_0^{\pi}|\phi(t)|\,|M_n(t)|\,dt \qquad \text{using Lemma 1}$$

$$=\left[\int_0^{\pi/_{n+1}}+\int_{\pi/_{n+1}}^{\pi}\right]|\phi(t)|\,|M_n(t)|\,dt$$

$$= I_1 + I_2 \quad (6.6)$$

Now using (5.5) and Lemma3

$$|I_1| = \int_0^{\pi/_{n+1}}|\phi(t)||M_n(t)|\,dt$$

$$= O(1)\int_0^{\pi/(n+1)} t^{-1}\, w(t)dt$$

$$= O\left((n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right). \quad (6.7)$$

Now

$$|I_2| = \int_{\pi/_{n+1}}^{\pi}|\phi(t)||M_n(t)|dt \qquad \text{using (5.5) and Lemma 2}$$

$$= O(1)\int_{\pi/_{(n+1)}}^{\pi} t^{-2}\, w(t)\,\mathrm{dt}$$

$$= O\left((n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right). \quad (6.8)$$

Now using (5.4), Lemma 1, we get

$$I_1 = O\left(\frac{1}{n+2}\right)\int_0^{\pi/(n+1)} t^{-1}\, w(|x-y|)dt$$

$$= O(w(|x-y|))\int_0^{\pi/(n+1)} t^{-1}\,dt$$

$$= O(\log(n+1)\, w(|x-y|)) \quad (6.9)$$

Now using (5.4) and Lemma2

$$I_2 = O\left(\frac{1}{n+2}\right)\int_{\pi/_{(n+1)}}^{\pi} t^{-2}\, w(|x-y|)\,\mathrm{dt}$$

$$= O(w(|x-y|)). \quad (6.10)$$

We have

$|I_k| = |I_k|^{1-\beta/\alpha}\, |I_k|^{\beta/\alpha}$. when $k = 1, 2$ (6.11)

By using (6.7) and (6.9) respectively in the first and the second factor on the right of the above identify (6.11) for k = 1 we obtain that

$$|I_1| = O\left(\left[(n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha} . [\log(n+1)\, w(|x-y|)]^{\beta/\alpha}\right) \quad (6.12)$$

Again using (6.8) and (6.10) in the first and second factor on the right of the identify (6.11) for k = 2 we have

$$|I_2| = O\left(\left[(n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha} . [w(|x-y|)]^{\beta/\alpha}\right) \quad (6.13)$$

Thus from (2.6), (6.12) and (6.13) we get

$$\sup_{x\neq y} \left| \Delta^{w^*} I_n(x, y) \right| = \sup_{\substack{(x,y)\\ x\neq y}} \frac{|I_n(x)-I_n(y)|}{w^*(|x-y|)}$$

$$= O\left\{\frac{w(|x-y|)^{\beta/\alpha}}{w^*(|x-y|)} (\log(n+1))^{\beta/\alpha} \left[(n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}\right\} \quad (6.14)$$

Using the fact that $\tilde{f} \epsilon H_w \Rightarrow \phi_x(t) = O(w(t))$

we obtain

$$\| I_n \|_c = \sup_{-\pi\le x\le\pi} \| t_n^{(C,2)\widetilde{(E,1)}}(f;x) - \tilde{f}(x) \|$$

$$= O\left\{(n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right\}. \quad (6.15)$$

Combining the result of (6.14) and (6.15), we get

$$\| t_n^{(C,2)\widetilde{(E,1)}}(f;x) - \tilde{f}(x) \|_{w^*} = O\left\{\frac{w(|x-y|)^{\beta/\alpha}}{w^*(|x-y|)} (\log(n+1))^{\beta/\alpha} \left[(n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}\right\} \quad (6.16)$$

Completes the proof of main theorem

7. Corollaries:

The corollaries can be derived from main theorem.

Corollary7. 1: "If $\beta = 0$ and $\tilde{f} \in Lip(\alpha, p)$, $0 < \alpha \le 1$ then

$$\| t_n^{(C,2)\widetilde{(E,1)}}(f;x) - \tilde{f}(x) \|_c = O\left\{\frac{1}{(n+1)^{\alpha}}\right\} \quad \text{for } 0 < \alpha < 1.$$

$$= O\left(\frac{\log(n+1)}{(n+1)}\right), \text{ for } \alpha = 1$$

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Conclusion

The summability method F(a, q) includes method of summability like Borel, *(E, 1), (E, q), (e, c)* and *[F, d_n]* then by using the result of main theorem we can derive more generalizing result and also the result of J. K.

Kushwaha [6] can be derived directly.

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