BS-ALGEBRAS AND ITS FUZZY IDEAL

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Abstract: In this paper, a new algebra called BS-Algebras which is a generalization of B-Algebras is introduced. We include the concept of fuzzy ideal in BS-Algebras. Some new characterizations were given.

Keywords: BS-Algebras, Fuzzy ideal, Homomorphism, Cartesian Product.

1 Introduction

After the introduction of fuzzy subsets by L.A.Zadeh[4], several researches explored on the generalization of the notion of fuzzy subset.In 1966, Imai and Iseki introduced two classes of abstract algebras viz. BCK-algebras and BCI-algebras[2]. J.Neggers and H.S. Kim introduced the notion of B-algebras[3] which is a generalisation of BCK-algebras. We introduce the notion of BS-algebras which is a generalisation of B-algebras. In this paper, we study the concepts of BS-Algebras with its examples. We apply the concept of fuzzy ideal in BS-Algebras and establish some of their basic properties. We investigate some algebraic nature of fuzzy ideal in BS-algebras. The homomorphic behaviour of fuzzy ideal of BS-algebras have been obtained. Finally, Fuzzy ideal of BS-algebras is also applied in Cartesian product.

2 Preliminaries

Definition 2.1. *[3]* A B-algebra is a non empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms

(i) $x * x = 0$

(ii) $x * 0 = x$

(iii) $(x * y) * z = x * (z * (0 * y)) \; \forall \; x, y, z \in X$

Definition 2.2. Let μ_1 and μ_2 be two fuzzy sets of X. Then its Intersection is denoted by $\mu_1 \cap \mu_2$ is defined by $\mu_1 \cap \mu_2 = min\{\mu_1(x), \mu_2(x)\} \forall x \in X$

Definition 2.3. Let $A = \{\mu_1(x) : x \in X\}$ and $B = \{\mu_2(x) : x \in X\}$ be two fuzzy sets on X. The cartesian product $A \times B = \{\mu_1 \times \mu_2(x, y) : x, y \in X\}$ is defined by $(\mu_1 \times \mu_2)(x, y) = min\{\mu_1(x), \mu_2(y)\}$ where $\mu_1 \times \mu_2 : X \times X \to [0, 1] \forall x, y \in X$.

Definition 2.4. *[1]* A fuzzy set μ of BS-algebra X is called a doubt fuzzy bi-ideal of X if

i) $\mu(1) \leq \mu(x)$

ii) $\mu(y * z) \leq max\{\mu(x), \mu(x * (y * z))\} \; \forall \; x, y, z \; \in \; X$

3 BS-Algebras

Definition 3.1. A BS-algebra is a non empty set with a constant 1 and a binary operation $*$ satisfying the following axioms

(i) $x * x = 1$

(ii) $x * 1 = x$

(iii) $1 * x = x$

(iv) $(x * y) * z = x * (z * (1 * y)) \; \forall \; x, y, z \in X$

A binary relation $\leq$ on X can be defined by $x \leq y$ if and only if $x * y = 1$

Example 3.2. i) Let $X = \{1, a, b, c\}$ be a set with the following table

*	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Then $(X, *, 1)$ is a BS-algebra.

ii) Let $X = \{1, a, b\}$ be a set with the following table

*	1	a	b
1	1	a	b
a	a	1	a
b	b	a	1

Then $(X, *, 1)$ is a BS-algebra.

(iii) Let $X = \{1, a, b, c, d, e\}$ be a set with the following table

*	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	b	d	e	c
b	b	a	1	e	c	d
c	c	d	e	1	b	a
d	d	e	c	a	1	b
e	e	c	d	b	a	1

If we put $y = x$ in $(x * y) * z = x * (z * (1 * y))$,

then we have $(x * x) * z = x * (z * (1 * x)) \rightarrow$ (I)

$\Rightarrow 1 * z = x * (z * (1 * x))$

If we put $z = x$ in (I) then we obtain

$1 * x = x * (x * (1 * x)) \rightarrow (II)$

Using (i) and (I) and $z = 1$ it follows that $1 = x * (1 * (1 * x)), \rightarrow (III)$

we observe that the four axioms (i),(ii),(iii) and(iv) are independent.

iv) Let $X = \{1, a, b\}$ be a set with the following table

*	1	a	b
1	1	a	1
a	a	1	a
b	1	a	1

Then the axioms (i) and (iv) hold, but not (ii) and (iii) since $b * 1 = 1 \neq b$

v) The set $X = \{0, 1, 2\}$ with the following table

*	1	a	b
1	1	a	b
a	a	a	a
b	b	a	b

The axioms (ii), (iii) and(iv) satisfies but not (i) since $a*a = a \neq 1$ and $b*b = b \neq 1$

vi) Let $X = \{1, a, b, c\}$ be a set with the following table

*	1	a	b	c
1	1	a	b	c
a	a	1	1	1
b	b	1	1	a
c	c	1	1	1

Then $(X, *, 1)$ satisfies the axioms (i),(ii) and (iii) but not (iv)

since $(b * c) * 1 = a \neq b = b * (1 * (1 * c))$

Theorem 3.3. *If* $(X, *, 1)$ *is a BS-algebra, then* $y*z = y*(1*(1*z))$ *for any* $y, z \in X$

Proof. This comes from the axioms $x * 1 = x$ and $(x * y) * z = x * (z * (1 * y))$

Now, $y * z = (y * z) * 1$ (by (ii))

$= y * (1 * (1 * z))$ (by (iv)) ■

Theorem 3.4. *If* $(X, *, 1)$ *is a BS-algebra, then* $(x * y) * (1 * y) = x$ *for any* $x, y \in X$

Proof. From (iv) with $z = 1 * y$ we have $(x * y) * (1 * y) = x * ((1 * y) * (1 * y))$

From axiom(i) $(x * y) * (1 * y) = x * 1$

From axiom(ii),it follows that $(x * y) * (1 * y) = x$ ■

Theorem 3.5. *If* $(X, *, 1)$ *is a BS-algebra, then* $x * z = y * z \Rightarrow x = y$ *for any* $x, y, z \in X$

Proof. If $x * z = y * z$,then $(x * z) * (1 * z) = (y * z) * (1 * z)$

and by previous theorem, it follows that $x = y$ ■

Theorem 3.6. *If* $(X, *, 1)$ *is a BS-algebra, then* $x * (y * z) = (x * (1 * z)) * y$ *for any* $x, y, z \in X$

Proof. Using (iv) we obtain

$$(x * (1 * z)) * y = x * (y * (1 * (1 * z)))$$

$$= x * (y * z)\text{(by thm(3.3))}$$

■

Theorem 3.7. *Let* $(X, *, 1)$ *be a BS-algebra.Then for any* $x, y \in X$

(i) $x * y = 1 \Rightarrow x = y$

(ii)$1 * x = 1 * y \Rightarrow x = y$

(iii)$1 * (1 * x) = x$

Proof. (i) Since $x * y = 1 \Rightarrow x * y = y * y$, by theorem (3.5), it follows that $x = y$

(ii)If $1 * x = 1 * y$, then

$$\begin{aligned} 1 = x * x &= (x * x) * 1 \\ &= x * (1 * (1 * x)) \\ &= x * (1 * (1 * y)) \\ &= (x * y) * 1 \\ 1 &= x * y \end{aligned}$$

By (i), $x = y$

(iii) For any $x \in X$, we obtain

$$\begin{aligned} 1 * x &= (1 * x) * 1 (by (ii)) \\ &= 1 * (1 * (1 * x))(by (iv)) \end{aligned}$$

By (ii) part of this theorem, we have $x = 1 * (1 * x)$ ■

Theorem 3.8. *If* $(X, *, 1)$ *is a BS-algebra,then* $(x * y) * y = x * y^2$ *for any* $x, y \in X$

Proof. From (iv),

$$\begin{aligned} (x * y) * y &= x * (y * (1 * y)) \\ &= x * (y * y) \\ &= x * y^2 \end{aligned}$$

■

Theorem 3.9. *If* $(X, *, 1)$ *is a BS-algebra,then* $(1 * y) * (x * y) = x$ *for any* $x, y \in X$

Proof. From theorem (3.6),

$$\begin{aligned}(1*y)*(x*y) &= ((1*y)*(1*y))*x \\ &= 1*x \\ &= x\end{aligned}$$

■

Definition 3.10. A BS-algebra $(X,*,1)$ is said to be commutative if $x*(1*y) = y*(1*x)$ for any $x,y \in X$

Note 3.11. The BS-algebra in example:3.2 (i) is commutative while the algebra in example:3.2 (iii)is not commutative since
$c*(1*d) = b \neq a = d*(1*c)$

Theorem 3.12. *If $(X,*,1)$ is a commutative BS-algebra,then $(1*x)*(1*y) = y*x$ for any $x,y \in X$*

Proof. Since X is commutative,

$$\begin{aligned}(1*x)*(1*y) &= y*(1*(1*x)) \\ &= y*x \quad \text{(by thm(3.3))}\end{aligned}$$

■

Theorem 3.13. *If $(X,*,1)$ is a commutative BS-algebra,then $x*(x*y) = y$ for any $x,y \in X$*

Proof. By theorem (3.6),

$$\begin{aligned}\text{Now, } x*(x*y) &= (x*(1*y))*x \\ &= (y*(1*x))*x \text{ (since X is commutative)} \\ &= y*(x*x) \\ &= y*1 \\ &= y\end{aligned}$$

■

Corollary 3.14. *If* $(X, *, 1)$ *is a commutative BS-algebra,then the left cancellation law holds (i.e)* $(x * y) = x * y' \Rightarrow y = y'$

Proof. From previous theorem, $y = x * (x * y) = x * (x * y') = y'$ ■

Theorem 3.15. *If* $(X, *, 1)$ *is a commutative BS-algebra,then* $(1 * x) * (x * y) = y * x^2$ *for any* $x, y \in X$

Proof.

$$\begin{aligned} \text{Now,}(1 * x) * (x * y) &= ((1 * x) * (1 * y)) * x \text{ (by theorem (3.6))} \\ &= (y * x) * x \text{ (by theorem (3.12))} \\ &= y * x^2 \text{ (by theorem (3.8))} \end{aligned}$$

■

4 Fuzzy Ideal

Definition 4.1. *A fuzzy set A in X is called a fuzzy ideal of X if it satisfies*
i) $\mu(1) \geq \mu(y)$
ii) $\mu(y) \geq min\{\mu(y * x), \mu(x)\}$

Example 4.2. Let $X = \{1, a, b, c\}$ be a set with the following table

*	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Then $(X, *, 1)$ is a BS-algebra. Define a fuzzy set μ in X by $\mu(1) = \mu(b) = 0.7$ and $\mu(a) = \mu(c) = 0.3$ is the fuzzy ideal of X.

Theorem 4.3. *If a fuzzy set A in X is a fuzzy ideal, then $\forall\ x \in X, \mu(1) \geq \mu(x)$*

Proof. straight forward ■

Theorem 4.4. *Let μ_1 and μ_2 be two fuzzy ideals of a BS-algebra X. Then $\mu_1 \cap \mu_2$ is also a fuzzy ideal of BS-algebra X.*

Proof.

$$\begin{aligned}\text{Now,}(\mu_1 \cap \mu_2)(1) &= (\mu_1 \cap \mu_2)(x * x) \\ &\geq min\{(\mu_1 \cap \mu_2)(x), (\mu_1 \cap \mu_2)(x)\} \\ &= (\mu_1 \cap \mu_2)(x)\end{aligned}$$

Therefore, $(\mu_1 \cap \mu_2)(1) \geq (\mu_1 \cap \mu_2)(x)$
Also,

$$\begin{aligned}(\mu_1 \cap \mu_2)(y) &= min\{\mu_1(y), \mu_2(y)\} \\ &\geq min\{min\{\mu_1(x), \mu_1(y * x)\}, min\{\mu_2(x), \mu_2(y * x)\}\} \\ &= min\{min\{\mu_1(x), \mu_2(x)\}, min\{\mu_1(y * x), \mu_2(y * x)\}\} \\ &= min\{(\mu_1 \cap \mu_2)(x), (\mu_1 \cap \mu_2)(y * x)\}\end{aligned}$$

$(\mu_1 \cap \mu_2)(y) \geq min\{(\mu_1 \cap \mu_2)(x), (\mu_1 \cap \mu_2)(y * x)\}$
Hence $\mu_1 \cap \mu_2$ is a fuzzy ideal of X.

■

Theorem 4.5. *Let μ be a fuzzy ideals of BS-algebra X. If $x * y \leq z$, then $\mu(x) \geq min\{\mu(y), \mu(z)\}$*

Proof. Let $x, y, z \in X$ such that $x * y \leq z$. Then $(x * y) * z = 1$, and thus

$$
\begin{aligned}
\mu(x) &\geq min\{\mu(x*y), \mu(y)\} \\
&\geq min\{min\{\mu((x*y)*z), \mu(z)\}, \mu(y)\} \\
&= min\{min\{\mu(1), \mu(z)\}, \mu(y)\} \\
&= min\{\mu(z), \mu(y)\}
\end{aligned}
$$

Therefore, $\mu(x) \geq min\{\mu(z), \mu(y)\}$ ■

Theorem 4.6. *Let A be a fuzzy ideals of a BS-algebras X. If $x \leq y$, then $\mu(x) \geq \mu(y)$ (i.e)order reversing*

Proof. Let $x, y \in X$ such that $x \leq y$. Then $x * y = 1$ and thus

$$
\begin{aligned}
\mu(x) &\geq min\{\mu(x*y), \mu(y)\} \\
&= min\{\mu(1), \mu(y)\} \\
&= \mu(y) \\
\mu(x) &\geq \mu(y)
\end{aligned}
$$

Hence, μ is order reversing. ■

Theorem 4.7. *Let A be a fuzzy ideal of X. Then $(...((x*a_1)*a_2)*a_n = 0$ for any $x, a_1, a_2...a_n \in X$, implies $\mu(x) \geq min\{\mu(a_1), \mu(a_2)...\mu(a_n)\}$*

Proof. Using induction on n and by previous two theorems we can easily prove the theorem ■

Theorem 4.8. *Let B be a crisp subset of X. Suppose that $A = \mu(x)$ is a fuzzy set in X defined by $\mu(x) = \lambda$ if $x \in B$ and $\mu(x) = \tau$ if $x \notin B$ $\forall$ $\lambda, \tau \in [0, 1]$ with $\lambda \geq \tau$. Then A is a fuzzy ideal of X if and only if B is a ideal of X*

Proof. Assume that A is a fuzzy ideal of X. Let $x \in B$.
Let $x, y \in X$ be such that $y * x \in B$ and $x \in B$. Then $\mu(y * x) = \lambda = \mu(x)$, and hence $\mu(y) \geq min\{\mu(x), \mu(y * x)\} = \lambda$
Thus $\mu(y) = \lambda$ (i.e)$y \in B$

Therefore B is a ideal of X.

Conversely, Suppose that B is a ideal of X. Let $y \in X$

Let $x, y \in X$. If $y * x \in B$ and $x \in B$, then $y \in B$

Hence,$\mu(y) = \lambda = min\{\mu(y * x), \mu(x)\}$

If $y * x \notin B$ and $x \notin B$, then clearly,$\mu(y) \geq min\{\mu(x), \mu(y * x)\}$

If exactly one of $y * x$ and x belong to B, then exactly one of $\mu(y * x), \mu(x)$ is equal to τ. Therefore, $\mu(y) \geq \tau = min\{\mu(x), \mu(y * x)\}$

Consequently,A is a fuzzy ideal of X. ∎

Theorem 4.9. *A fuzzy set μ is a ideal of X then the set $U(\mu : t)$ is ideal of X for every $t \in [0, 1]$*

Proof. Suppose that μ is a fuzzy ideal of X. For $t \in [0, 1]$.

Let $x, y \in X$ be such that $y * x \in U(\mu : t)$ and $x \in U(\mu : t)$.

Then $\mu(y) \geq min\{\mu(x), \mu(y * x)\}$.

Then $y \in U(\mu : t)$. Hence $U(\mu : t)$ is a ideal of X.

∎

Definition 4.10. *Let f be a mapping from the set X into the set Y. Let B be a fuzzy set in Y. Then the inverse image of B is defined as $f^-1(\mu)(x) = \mu(f(x))$. The set $f^{-1}(B) = \{f^{-1}(\mu)(x) : x \in X\}$ is a fuzzy set.*

Theorem 4.11. *Let $f : X \to Y$ be a homomorphism of BS-algebra. If B is a fuzzy ideal of Y, then the pre-image $f^{-1}(B)$ of B under f in X is a fuzzy ideal of X*

Proof. For all $x \in X$, $f^{-1}(\mu)(x) = \mu(f(x)) \leq \mu(1) = \mu(f(1)) = f^{-1}(\mu)(1)$

Therfore, $f^{-1}(\mu)(x) \leq f^{-1}(\mu)(1)$

Let $x, y \in X$. Then

$$\begin{aligned} f^{-1}(\mu)(x) &= \mu(f(x)) \\ &\geq min\{\mu(f(x) * f(y)), \mu(f(y))\} \\ &\geq min\{\mu(f(x * y)), \mu(f(y))\} \\ &= min\{f^{-1}(\mu)(x * y), f^{-1}(\mu)(y)\} \end{aligned}$$

Therefore, $f^{-1}(\mu)(x) \geq min\{f^{-1}(\mu)(x * y), f^{-1}(\mu)(y)\}$

Hence $f^{-1}(B) = \{f^{-1}(\mu)(x) : x \in X\}$ is a fuzzy ideal of X. ■

Theorem 4.12. *Let $f : X \to Y$ be a onto homomorphism of BS-algebra. Then B is a fuzzy ideal of Y, if $f^{-1}(B)$ of B under f in X is a fuzzy ideal of X*

Proof. For any $u \in Y$, there exists $x \in X$ such that $f(x) = u$

Then $\mu(u) = \mu(f(x)) = f^{-1}(\mu)(x) \leq f^{-1}(\mu)(1) = \mu(f(1)) = \mu(1)$

Therefore, $\mu(u) \leq \mu(1)$

Let $u, v \in Y$. Then $f(x) = u$ and $f(y) = v$ for some $x, y \in X$.

$$\begin{aligned}
\text{Thus,}\mu(u) = \mu(f(x)) &= f^{-1}(\mu)(x) \\
&\geq min\{f^{-1}(\mu)(x * y), f^{-1}(\mu)(y)\} \\
&= min\{\mu(f(x * y)), \mu(f(y))\} \\
&= min\{\mu(f(x) * f(y)), \mu(f(y))\} \\
&= min\{\mu(u * v), \mu(v)\}
\end{aligned}$$

Therefore, $\mu(u) \geq min\{\mu(u * v), \mu(v)\}$

Then B is a fuzzy ideal of Y ■

Theorem 4.13. *Let A and B be fuzzy ideals of X, then $A \times B$ is a fuzzy ideal of $X \times X$*

Proof. For any $(x, y) \in X \times X$, we have

$$\begin{aligned}
(\mu_1 \times \mu_2)(1, 1) &= min\{\mu_1(1), \mu_2(1)\} \\
&\geq min\{\mu_1(x), \mu_2(y)\} \ \forall\ x, y \in X \\
&= (\mu_1 \times \mu_2)(x, y)
\end{aligned}$$

Therefore, $(\mu_1 \times \mu_2)(1, 1) \geq (\mu_1 \times \mu_2)(x, y)$

Let $(x_1, y_1), (x_2, y_2) \in X \times X$.Then

$$\begin{aligned}(\mu_1 \times \mu_2)(x_1, y_1) &= min\{\mu_1(x_1), \mu_2(y_1)\} \\ &\geq min\{min\{\mu_1(x_1 * x_2), \mu_1(x_2)\}, min\{\mu_2(y_1 * y_2), \mu_2(y_2)\}\} \\ &= min\{min\{\mu_1(x_1 * x_2), \mu_2(y_1 * y_2)\}, min\{\mu_1(x_2), \mu_2(y_2)\}\} \\ &= min\{(\mu_1 \times \mu_2)((x_1 * x_2), (y_1 * y_2)), (\mu_1 \times \mu_2)(x_2, y_2)\}\end{aligned}$$

Therefore, $(\mu_1 \times \mu_2)(x_1, y_1) \geq min\{(\mu_1 \times \mu_2)((x_1 * x_2), (y_1 * y_2)), (\mu_1 \times \mu_2)(x_2, y_2)\}$
Hence, $A \times B$ is a fuzzy ideal of $X \times X$ ■

Theorem 4.14. *Let A and B be fuzzy sets in X such that $A \times B$ is a fuzzy ideal of $X \times X$,then*
i) Either $\mu_1(1) \geq \mu_1(x)$ or $\mu_2(1) \geq \mu_2(x)$ $\forall$ $x \in X$
ii) If $\mu_1(1) \geq \mu_1(x)$ $\forall$ $x \in X$, then either $\mu_2(1) \geq \mu_1(x)$ or $\mu_2(1) \geq \mu_2(x)$
iii)If $\mu_2(1) \geq \mu_2(x)$ $\forall$ $x \in X$, then either $\mu_1(1) \geq \mu_1(x)$ or $\mu_1(1) \geq \mu_2(x)$

Proof. (i) Assume that $\mu_1(x) > \mu_1(1)$ and $\mu_2(y) > \mu_2(1)$ for some $x, y \in X$.

$$\begin{aligned}\text{Then}(\mu_1 \times \mu_2)(x, y) &= min\{\mu_1(x), \mu_2(y)\} \\ &> min\{\mu_1(1), \mu_2(1)\} \\ &= (\mu_1 \times \mu_2)(1, 1)\end{aligned}$$

$$\Rightarrow (\mu_1 \times \mu_2)(x, y) > (\mu_1 \times \mu_2)(1, 1) \ \forall \ x, y \in X, \text{which is a contradiction.}$$

Hence (i) is proved.
(ii) Again assume that $\mu_2(1) < \mu_1(x)$ and $\mu_2(1) < \mu_2(y)$ $\forall$ $x, y \in X$

$$\begin{aligned}\text{Then}(\mu_1 \times \mu_2)(1, 1) &= min\{\mu_1(1), \mu_2(1)\} \\ &= \mu_2(1) \\ \text{Now,}(\mu_1 \times \mu_2)(x, y) &= min\{\mu_1(x), \mu_2(y)\} \\ &> \mu_2(1) \\ &= (\mu_1 \times \mu_2)(1, 1), \text{which is a contradiction.}\end{aligned}$$

Hence (ii) is proved

(iii) The proof is similar to (ii) ∎

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