DECOMPOSABLE OF SINGLE-TIME AND MULTI-TIME GEOMETRIC DYNAMICS ON RIEMANN-KAEHLERIAN MANIFOLDS

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Abstract:

Samuelson (1970), has studied the law conservation of capital-output ratio. After that, Isvoranu and Udriste (2006), locate fluid flow versus Geometric Dynamics and achieved from metrics to dynamics and flows and winds. Also, Gay-Balmaz, Holm and Ratiu (2009) stumble on Geometric dynamics of optimization. In this paper, the author calculated decomposable single-time and multi-time dynamics on Riemann-Kaehlerian manifolds.

Keywords: Decomposable dynamics, Dynamical systems, Riemann manifolds, Kaehlerian manifolds and partial differential equations.

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1. Introduction: `

The single-time dynamics we identify with an ordinary differential equation related to Newton second law. A second-order elliptic partial differential equation elucidates a dynamic that occurs across multiple time instances. Any ordinary differential equation is malformed addicted to a one-flow or any partial differential equation is malformed addicted to an m-flow in any satisfactorily huge dimension. The spatial geometry transforms the mono-flow dependency into a geodesic movement within a gyroscope-influenced field of forces. Similarly, the dual spatial geometry alters the multi-flow into harmonically mapped distortions under the influence of the gyroscopic field of forces. [Udriste (2005); Udriste and Bejenaru (2012)].

The equations of mechanics might appear differently in terms of their form: $\dot{x}(t) = X\big(x(t)\big)$, The ordinary differential equations of the form $F(x(t), \dot{x}\,(t), \ddot{x}(t), \dddot{x}(t)) = 0$. Occasionally termed "jerk equations" involving third-order derivatives have been shown to accurately represent the fundamental setup where solutions display chaotic behaviour in a mathematically precise fashion. It has been proven that these jerk equations correspond to a system of three initial nonlinear first ODE, capturing the most minimal configuration for chaotic dynamics.

$$\dot{x}(t) = y(t),\ \dot{y}(t) = z(t),\ \dot{z}(t) = \emptyset\big(x(t), y(t), z(t)\big).$$

This pertains to a Lagrangian system existing on the jet space defined by coordinates $(t, x, y, z, \dot{x}, \dot{y}, \dot{z})$, along with its associated geometric dynamics in relation to the Riemannian metric $g_{ij}(x)$.

$$2L_1 = (\dot{x}(t) - y(t))^2 + (\dot{y}(t) - z(t))^2 + (\dot{z}(t) - \emptyset(x(t), y(t), z(t)))^2$$

$$(\dot{z} - \emptyset)\emptyset_x + \dot{x} - \dot{y} = 0, (\dot{z} - \emptyset)\emptyset_y + \dot{y} - \dot{z} = 0,$$

$$(\dot{z} - \emptyset)\emptyset_z + D_t(\dot{z} - \emptyset)) = 0.$$

Further usually, given a set of *n* Lagrangians:

$$L^i\big(t, x(t), \dot{x}(t)\big), \quad i = \overline{1, n}, \; x(t) = \big(x^1(t), \dots, x^n(t)\big), \; t \in I \subset R,$$

$$\mathcal{L} = \frac{1}{2} g_{ij}\big(x(t)\big) L^i\big(t, x(t), \dot{x}(t)\big) L^j\big(t, x(t), \dot{x}(t)\big).$$

The solution to a system of ordinary differential equations is given by the Extremely Euler-Lagrange method.

$$\frac{1}{2}\frac{\partial g_{il}}{\partial x^k} L^i L^j + g_{ij} L^i \frac{\partial L^j}{\partial x^k} - D_t\left(g_{ij} L^i \frac{\partial L^j}{\partial x^k}\right) = 0.$$

If the Lagrangian L^i is connected to ordinary differential equations $L^i\big(t, x(t), \dot{x}(t)\big) = 0$, then the extremals have solutions to the dynamics and that equation is decomposable [Mihlin (1983); Stefanescu and Udriste (1993); Furi (1995); Treanta and Udriste (2013)].

Let $u(x, t)$ be the density of the diffusive material at location $x \in R^n$ and time $t \in R$. Let $g^{ij}(u(x, t), x), i, j = \overline{1, n},$ be the collective spreading coefficient for density u at location x. The spreading partial differential equations are:

$$\frac{\partial u}{\partial t}(x, t) = \frac{\partial}{\partial x^i}\Big(g^{ij}(u(x, t), x\Big)\frac{\partial u}{\partial x^j}(x, t).$$

If the behaviour relies on both density and diffusion coefficient variations, The diffusion equation exhibits nonlinearity; otherwise, it remains linear. Moreover, in cases where $g^{ij}(u(x, t), x$ forms a symmetric positive definite matrix, the equation characterizes anisotropic diffusion. [Arnold (1969); Chorin and Marsden (2000); Udriste and Teleman (2004)].

The equivalent first-order non-linear partial differential equations is diffusion partial differential equations

$$\frac{\partial u}{\partial x^j} = v_j, \quad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x^i}\big(g^{ij} v_j\big),$$

The limitation on evolution (x, t) occurs in an $(n + 1)$ - dimensional context. A Riemannian metric $h^{ij}(u(x, t), x)$ gives rise to a Lagrangian that minimizes the sum of squares.

$$2L_2 = h^{ij}\left(\frac{\partial u}{\partial x^i} - v_i\right)\left(\frac{\partial u}{\partial x^j} - v_j\right) + \left(\frac{\partial u}{\partial t} - \frac{\partial}{\partial x^i}(g^{ij} v_i)\right)^2,$$

On the jet space of coordinates $(x, t, u, v, u_x, u_t, v_x, v_t)$. Then Euler-Lagrange equations are

$$\frac{1}{2}\frac{\partial h^{ij}}{\partial u}\left(\frac{\partial u}{\partial x^i}-v_i\right)\left(\frac{\partial u}{\partial x^j}-v_j\right)+\left(\frac{\partial u}{\partial t}-\frac{\partial}{\partial x^i}(g^{ij}v_j)\right)\left(-\frac{\partial}{\partial x^i}\left(\frac{\partial g^{ij}}{\partial u}v_j\right)\right)$$

$$-D_{x^i}\left(h^{ij}\left(\frac{\partial u}{\partial x^j}-v_j\right)\right)-D_t\left(\frac{\partial u}{\partial t}-\frac{\partial}{\partial x^i}(g^{ij}v_j)\right)=0,$$

$Or\ g^{lm}D_{x^m}\left(\frac{\partial u}{\partial t}-\frac{\partial}{\partial x^i}(g^{ij}v_j)\right)=0.$

Again, consider an orientable manifold denoted as T, with the coordinates $t=(t^1,\ldots,t^m)$ and let M be another manifold with coordinate $x=(x^1,\ldots,x^n)$. Utilizing a set of m smooth vector fields, denoted as $X_\alpha(t,x)$ of class c^∞ on $\mathrm{T}\times M$, we can represent the spreading by means of pfaff equations.

$$dx^i(t)-X^i_\alpha(t,x)dt^\alpha=0,\quad i=\overline{1,n},\ \ \alpha=\overline{1,m}.$$

Constructing a least squares Lagrangian for non-decomposable dynamics involves utilizing the metric tensors $h_{\alpha\beta}(t), g_{ij}(t)$ along with the components $\frac{\partial x^i}{\partial t^\alpha}(t)-X^i_\alpha(t,x)$ of the pullbacks.

$$L=\frac{1}{2}g_{ij}h^{\alpha\beta}\left(\frac{\partial x^i}{\partial t^\alpha}(t)-X^i_\alpha(t,x)\right)\left(\frac{\partial x^j}{\partial t^\beta}(t)-X^j_\beta(t,x)\right)\sqrt{det(h_{\alpha\beta})}>0$$

Further generally being given n, m Lagrangians from:

$$L^i_\alpha\left(t,x(t),x_\gamma(t)\right), i=\overline{1,n},\ \ \alpha=\overline{1,m}.\ x(t)=\left(x^i(t),\ldots.x^n(t)\right), t=(t^1,\ldots,t^m)\in I\subset \mathrm{T},$$

Subsequently, the corresponding Lagrangian density for least squares in related to the Riemannian metrics $g_{ij}(x)h^{\alpha\beta}(t),\ h^{\alpha\beta}(t)$ *is* formulated as follows:

$$\mathcal{L}=\frac{1}{2}g_{ij}\big(x(t)\big)h^{\alpha\beta}(t)L^i_\alpha\left(t,x(t),x_\gamma(t)\right)L^j_\beta\left(t,x(t),x_\gamma(t)\right).$$

If we consider a subset $\mathrm{T}\subset R^m$, the solutions that extremize the Euler-Lagrange partial differential equation system can be described as follows:

$$\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}h^{\alpha\beta}L^i_\alpha L^j_\beta+g_{ij}h^{\alpha\beta}L^i\frac{\partial L^j_\beta}{\partial x^k}-D_\gamma\left(g_{ij}h^{\alpha\beta}L^i_\alpha\frac{\partial L^j}{\partial x^k_\gamma}\right)=0.$$

If the Lagrangian L^i_α is associated to the partial differential equation $L^i_\alpha(t,x(t),x_\gamma(t))=0$, The solutions of that equation are given by the extremals, and the decomposability property holds for the dynamics [Lovelock and Rund (1975)].

2. SINGLE-TIME GEOMETRIC DYNAMICS ON RIEMANN-KAEHLERIAN MANIFOLDS:

Consider a differentiable manifold denoted as M, and let I be a nontrivial interval contained with the real number, R. we define a non-autonomous first order differential equation on the manifold M as the mapping of a non-autonomous C^{∞} vector field X : $V \to R^n$, where V represents an open subset of R × M. This vector field remains consistently aligned with M for all values of t in the real numbers, for any all $t \in$ R, the map $X_t: V_t \to R^n$ holds, defined as $X_t(x) = X(t,x)$, signifying a tangent vector field on the open subset $V_t = \{x \in M \mid (t,x) \in V\}$ of M. [Furi (1995]

$$\dot{x} = X(t,x), (t,x) \in V. \tag{2.1}$$

The solution to differential Equation (2.1) corresponds to a function $x: I \to M$ that belongs to the class C^1 . the function has the property that for every t in the interval I, the point $(t, x(t)) \in V$. additionally, the derivative of $x(t)$, denoted as $\dot{x}(t) = X(t, x(t))$ for all t in I. Specifically considering the Cauchy problem, a resolution to the ordinary differential equation (2.1) is sought, which adheres to the original condition $x(t_0) = x_0$. Under these circumstances, it can be concluded Then the solution of this Cauchy problem not only exists but is also unique.

Let $F: R \times TM \to R^n$ be a C^{∞} map. An equality of the type:

$$\dot{x} = F(t,x,\dot{x}), (t,x,\dot{x}) \in R \times TM. \tag{2.2}$$

Designated as a second-order differential equation on M, given the condition of a connected vector field. [Furi (1995)]:

$$G: R \times TM \to R^n \times R^n, G(t,x,y) = (y, F(t,x,y))$$

is tangent to TM, i.e., $(y, F(t,x,y)) \in T_{(x,y)}TM \;\; \forall\, (t,x,y) \in R \times TM$. A solution of the differential Equation (2.2) is a C^2 curve $x: I \to R^n$, in such a way that $x(t) \in M$ and $\ddot{x}(t) = F(t, x(t), \dot{x}(t))$, identically on I. In the case of The Cauchy problem, a solution of the ordinary differential equation (2.2) which satisfies the preliminary conditions $x(t_0) = x_0$, $\dot{x}(t_0) = v$. The solution to this initial value problem exists and is singular. When employing the constituents, the equations (2.1) and (2.2) are referred to as first-order and second-order systems of ordinary differential equations, respectively.

Let's begin with the triple (M, g, X), where M represents a manifold with a dimension of $n, g(x) = \left(g_{ij}(x)\right)$, for i, j ranging from 1 to n, is a smooth, time-dependent vector field in the manifold M. Consider that the Levi-Civita connection ∇ associated with the (M, g) pair is described by the components G^i_{jk}, where i, j, k take value from 1 to n.

Definition 2.1 Use the notations:

$$F_j = \left(F^i_j\right),\; F^i_j = \nabla_j X^i - g^{ih} g_{kj} \nabla_h X^k,\;\; f = \frac{1}{2} g(X,X).$$

A function $F: R \times TM \to R^n$ is consider to be produced by the pair (X, g) if it can be expressed in the following manner:

$$F = -G_{jk}\dot{x}^j\dot{x}^k + F_j\dot{x}^j + \nabla \mathrm{f} + \frac{\partial}{\partial t}X.$$

If F arises from the combination of X and g, then the differential equation (2.2) represents a form of geometric dynamics occurring at a specific time, akin to motion along a geodesic path within a gyroscopic force field. Drawing an analogy to the way force systems are simplified in mechanics, involving resultants and momentum, the breakdown of the solution set leads back to the concepts of flow and motion within the gyroscopic force field. [Udriste (2000); Udriste (2004); Udriste (2005); Isvoranu, and Udriste (2006) and Udriste and Bejenaru (2012)].

Theorem 2.1 If $F: R \times TM \to R^n$ is generated by the pair (X, g), then the set of maximal solutions of ODE (2.2) is decomposable into a subset corresponding to the initial values

$$x(t_0) = x_0, \;\; \dot{x}(t_0) = X(\,t_0, x(t_0)),$$

results which are reducible to solutions of the ordinary differential equation (2.1), and a subset of solutions corresponding to the preliminary values

$$x(t_0) = x_0, \qquad \dot{x}(t_0) = W \neq \lambda X\big(t_0, x(t_0)\big), \qquad \lambda > 0,$$

transversal to the solutions of the ordinary differential equation (2.1). converse is also true.

Proof. According to the theorem of existence and uniqueness, any solution x(t) derived from a second-order continuous or first-order ordinary differential equation (ODE) system possesses the following characteristic:

$$\dot{x}(t_0) = X\big(t_0, x(t_0)\big) \;\Rightarrow\; \dot{x}(t) = X\big(t, x(t)\big), \forall\, t \in I.$$

A Riemannian metric g and a flow X together give rise to a Lagrangian of least squares nature.

$$L(t, x, \dot{x}) = \frac{1}{2} g\big(\dot{x} - X(t, x), \dot{x} - X(t, x)\big).$$

The Euler-Lagrange ordinary differential equations represent a seamless extension of the trajectory in a geometric sense. These equations effectively describe a separable and dynamic motion along geodesic paths within fields of forces akin to gyroscopic effects. These geodesic paths are interwoven with additional trajectories that are influenced by the spatial geometry.

Theorem 2.2 If the function $F: TM \to R^n$ is generated by X and g, X is an autonomous vector field then the set of maximal solutions of ordinary differential equation (2.2) divides into three parts i. e. Curves $[x(t), H\big(x(t)\big)] = const = 0; > 0; < 0$.

Proof. We have from Hamiltonian:

$$H(t, x, \dot{x}) = \frac{1}{2} g\big(\dot{x} - X(t, x), \qquad \dot{x} - X(t, x)\big) = \frac{1}{2}\Big(g(\dot{x}, \dot{x}) - g(X, X)\Big) = H(x, \dot{x}),$$

The curves $x(t)$ with $H\big(x(t)\big) = const = 0$ are solutions of ordinary differential equation (2.1). The solutions with $H\big(x(t)\big) = const \neq 0$, are transversal to solutions of ordinary differential equation (2.1).

For any given ordinary differential equation, the resulting flow within the phase space creates a geometric dynamic when combined with the inherent phase space geometry. However, complications arise when dealing with a flow that is subject to constraints.

Let's examine the elements (M, X, g, Γ) where M represents a Riemannian manifold, X stands for a flow within M, g corresponds to a fundamental tensor field, and Γ represents a symmetric connection. Together, these components (X, g, Γ) give rise to an expanded geometric motion on M, which is defined by systems of ordinary differential equations (ODEs).

$$\ddot{x}^i(t) = \left(\delta_k^i \delta_j^l - g_{kj} g^{li}\right) X^k_{,\,l} \dot{x}^j(t) + \frac{\partial X^i}{\partial t} g_{kj} g^{li} X^k_{,\,l} X^j.$$

In Riemannian manifold $\left((0,\infty),\ g(x) = 1\right)$, take the flow $\dot{x} = 1$. We assign the least squares lagrangian $L_1 = (\dot{x} - 1)^2$, with Euler-Lagrange equation $\ddot{x} = 0$. On any other Riemannian manifold $\left((0,\infty),\ g(x)\right)$, we find the least squares Lagrangian $L_2 = g(x)(\dot{x} - 1)^2$, with Euler-Lagrange equation $\ddot{x} = \frac{g'(x)}{2g(x)}(1 - \dot{x})(1 + \dot{x})$. Here, $\Gamma(x) = \frac{g'(x)}{2g(x)}$ is a linear connection. We have the option to broaden the preceding ordinary differential equation into a system of ordinary differential equations.

$$\ddot{x}^i(t) = a_0^i\left(x(t)\right) + a_j^i\left(x(t)\right)\dot{x}^j(t) + b_{jk}^i\left(x(t)\right)\dot{x}^j(t)\dot{x}^k(t), i, j, \qquad k = 1, \ldots, n,$$

with possible disorder in velocities.

Consider a differentiable manifold of dimension n, denoted as M, and let I⊂R represent a nontrivial interval. In the case where the ordinary differential equation system (2.2) corresponds to a Euler-Lagrange system on the manifold M, with respect to a regular Lagrangian function $L(t, x, \dot{x})$, then there exists a fundamental tensor field $g = \left(g_{ij}\right)$ on TM s.t.:

$$g_{ij}(t, x, \dot{x}) = \frac{1}{2} \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}(t, x, \dot{x}), i, j = 1, \ldots., n.$$

On the other hand, when provided with $g_{ij}(t, x, \dot{x})$, the task is to find $L(t, x, \dot{x})$. Under these circumstances, by employing two consecutive line integrals of the second kind, we are able to express

$$L(t, x, \dot{x}) = \int_{\gamma \dot{x}_0 \dot{x}} \int_{\gamma \dot{x}_0 \dot{x}} g_{ij}(t, x, \dot{x})\, d\dot{x}^i d\dot{x}^j + a_i(t, x)\dot{x}^i + b(t, x).$$

The pair (M, g) is called a Lagrangian manifold.

3. MULTI-TIME GEOMETRIC DYNAMICS ON RIEMANN-KAEHLERIAN MANIFOLDS.

We start with an operator $\left((T, h), (M, g),\ X_n\right)$, where:

(i) (T, h) represents an oriented Riemannian manifold with a dimension of m. It possesses local coordinates denoted as $t = (t^\alpha)$, where α ranges from 1 to m. The manifold is equipped with a metric tensor denoted as $h_{\alpha\beta}$, along with Christoffel symbols given by $H_{\beta\gamma}^\alpha$.

(ii) Consider the Riemannian manifold (M, g) with dimension n, where $x = (x^i)$ represents the local coordinates, g_{ij} is the metric tensor, and G^i_{jk} denotes the Christoffel symbols.

(iii) $X_\alpha(t,x) = (X^i_\alpha(t,x)), \alpha = 1, \ldots, m; i = 1, \ldots, n$ are C^∞ vector fields on M, dependent on (t,x) which describe the first order PDE system:

$$\frac{\partial x}{\partial t^\alpha}(t) = X_\alpha(t, x(t)). \tag{3.1}$$

Theorem 3.1 The existence of a unique solution for the Cauchy problem, which includes the partial differential equation system (3.1) and the initial condition $x(t_0) = x_0$, is guaranteed only when the system is completely integrable. This equivalence holds:

$$h^{\alpha\beta} \frac{\partial^2 x}{\partial t^\alpha \partial t^\beta}(t) = F\left(t, x(t), x_\gamma(t)\right), (t, x, x_\gamma) J^1 \in (T, M) \tag{3.2}$$

is called a second order elliptical partial differential equation on M.

Proof: Consider a hypersurface Γ: G(t) = 0 within T, which passes through the point t_0 . Let Λ(t) represent a unit vector field along Γ that intersects it in a transversal manner. Let $\varphi_0(t)$ and $\varphi_1(t)$ be vector functions with n components defined on Γ. The first function belongs to class C^1, while the second function is of class C^0. We are concerned with the Cauchy problem associated with partial differential equation (3.2) [Mihlin (1983)] and find in a unilateral neighbourhood of Γ, the solution of the PDE (3.2) satisfying the Cauchy conditions:

$$x(t)|\Gamma = \varphi_0(t), D_\Lambda x(t)|\Gamma = \varphi_1(t). \tag{3.3}$$

Hence the solution of this Cauchy problem exists and it is unique.

From, Cauchy conditions at function $x(t)$ on the Cauchy surface Γ, initially,

$$\left.\frac{\partial x}{\partial t^\alpha}\right|_\Gamma = \frac{\partial \varphi_0}{\partial t^\alpha}(t), \alpha = 1, \ldots, m-1$$

and then the equalities:

$$\varphi_1(t) = D_\Lambda x(t)|_\Gamma = \frac{\partial x}{\partial t^\alpha}(t)\Lambda^\alpha(t),$$

Together with $\Lambda^m \neq 0, give$

$$\frac{\partial x}{\partial t^m}(t)|_\Gamma = \frac{1}{\Lambda^m(t)}\left[\varphi_1(t) - \sum_{\alpha=1}^{m-1} \frac{\partial \varphi_0}{\partial t^\alpha}(t)\, \Lambda^\alpha(t)\right].$$

The preliminary conditions (3.3) are equivalent either to the preliminary conditions:

$x(t)|_\Gamma = \varphi_0(t), \frac{\partial x}{\partial t^m}(t)|_\Gamma = W_m(t)$ and

$x(t)|_\Gamma = \varphi_0(t), \frac{\partial x}{\partial t^\alpha}(t)|_\Gamma = W_\alpha(t), \alpha = 1, \ldots, m.$

Regarding the complete set of integrability conditions and the compatibility condition with respect to $\varphi_{0.}$

Definition 3.2 Using the vector field X_α, the metric tensor $h_{\alpha\beta,}\, g_{ij}$, and the Christoffel symbols $H^{\alpha}_{\beta\gamma}$, G^{i}_{jk}, we define:

$$F^{i}_{j\alpha} = \nabla_j X^i_\alpha - g^{ih} g_{kj} \nabla_h X^k_\alpha, \quad f = \frac{1}{2} h^{\alpha\beta} g_{ij} X^i_\alpha X^j_\beta$$

and

$$\nabla_j X^i_\alpha = \frac{\partial X^i_\alpha}{\partial X^j} + G^i_{jk} X^k_\alpha, \quad D_\beta X^i_\alpha = \frac{\partial X^i_\alpha}{\partial X^j} - H^{\gamma}_{\alpha\beta} X^i_\gamma.$$

The function $F: J^1(T, M) \to R^n$ is said to be generated by the operator (X_α, h, g) if it is of the form:

$$F = h^{\alpha\beta} \left(-G_{jk} x^j_\alpha x^k_\beta + H^{\gamma}_{\alpha\beta} x_\gamma + F_{j\alpha} x^j_\beta + g_{kj} (\nabla X^k_\alpha) X^j_\beta + D_\beta X_\alpha \right).$$

Theorem 3.2 If $F: J^1(T, M) \to R^n$ is created by the triplet (X_α, h, g) then the set of maximal solutions of partial differential equation (3.2) is decomposable into a subset corresponding to the preliminary values:

$$x(t)|_\Gamma = \varphi_0(t), \frac{\partial x}{\partial t^\alpha}(t)|_\Gamma = X_\alpha\big(t, x(t)\big),$$

solutions which are reducible to solutions of partial differential equation (3.1), and a subset of solutions corresponding to the preliminary values:

$$x(t)|_\Gamma = \varphi_0(t), \frac{\partial x}{\partial t^\alpha}(t)|_\Gamma = W_\alpha(t) \notin K^+\{X_\alpha(t, x(t))\},$$

A transversal intersecting the solutions of Partial Differential Equation (3.1) demonstrates the same truth in reverse.

Proof. Let solution $x = x(t)$ of any second order continuation of the first order partial differential equation system has the property: $X_\alpha(t_0) = X_\alpha(t_0, x(t_0))$ implies $X_\alpha(t) = X_\alpha\big(t, x(t)\big), \forall\, t \in T$. Any m-flow X_α and two Riemannian metrics h and g determine a least squares Lagrangian density:

$$L\big(t, x, x_\gamma\big) = \tfrac{1}{2} h^{\alpha\beta} g_{ij}\big(x^i_\alpha - x^i_\alpha(t, x)\big)\left(x^j_\beta - x^j_\beta(t, x)\right).$$

The Euler-Lagrange Partial differential equations denote a continuation of the m-flow and just a decomposable dynamic. Once again, a typical partial differential equation generates a multi-dimensional flow within the phase space. This, in conjunction with the geometry of the phase space, results in a dynamic with a distinct geometric interpretation. This principle holds for all partial differential equations; however, complications arise when dealing with a multi-dimensional flow subject to constraints.

Let's explore the tuple (T, h, H), where T signifies a Kahlerian manifold, h represents a fundamental tensor field, and H indicates a symmetric connection. Introducing the operator (M, X_α, g, G), where M stands for a Kahlerian manifold, X_α denotes an m-flow on M, g is a fundamental tensor field, and G is a symmetric connection (derivation). The pentad $(X_\alpha; h, H; g, G)$ gives rise to an expanded geometric dynamic on T×M.

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