ON APPROXIMATION OF FUNCTION $\tilde{f} \in H_w$ CLASS BY (C, 2)(E, 1) MEANS OF CONJUGATE SERIES OF FOURIER SERIES.

H. L. RATHORE

Department of Mathematics,Govt. College Pendra, Bilaspur, Chhattisgarh, 495119 India.

Email:hemlalrathore@gmail.com

ABSTRACT-A theorem on degree of approximation of function belonging to Hölder metric by (C, 2) (E, q) mean, has been discussed by Rathore, Shrivastava and Mishra [17]. Since (E, 1*)* includes (E, q) method, so for obtaining more generalized result to replace (E, q) by (E, 1) mean. The Euler mean (E, 1) contains the summability method of generalized Borel, Euler, Taylor etc. In this chapter we obtain on approximation of function $\tilde{f} \in H_w$ class by (C, 2)(E, 1) means of conjugate series of Fourier series, has been proved.

2010 MSC: 42B05, 42B08
Keywords & Phrases Hölder metric, Fourier series, (C, 2) summability, (E, 1*)* summability, (C, 2)(E, 1) summability, Lebesgue integral.

1. INTRODUCTION

In this direction, on approximation of *f* belong to many classes also Hölder metric by Cesa̦ro mean, Nörlund mean, Euler mean has been discussed by several investigator like respectively Alexits [2], Khan [6], Chandra [3], Mohapatra and Chandra [11], Das, Ghosh and Ray[4], etc. Further in this field several researchers like Lal and Kushwaha [8], Lal and Singh [9], Rathore and Shrivastava [14], Nigam [12], Albayrak, Koklu and Bayramov [1], Rathore, Shrivastava and Mishra ([15], [16]), Kushwaha [7], Singh and Mahajan [18], Mishra and Khatri [10] etc. Recently Rathore, Shrivastava and Mishra [17] have been determined. We extend the result on approximation of function $\tilde{f} \in H_w$ class by (C, 2)(E, 1) mean of conjugate series of Fourier series, has been proved.

2. DEFINITION AND NOTATIONS

Let *f (x)* be periodic and integrable in the sense of Lebesgue on [-π, π]. Then f(x) is defined by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \cong \sum_{n=0}^{\infty} A_n(x) \tag{2.1}$$

The conjugate series of (2.1) is

$$\sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx) \cong \sum_{n=1}^{\infty} B_n(x) \tag{2.2}$$

with n^{th} partial sum $\widetilde{S_n}$ *(f; x)*

Let $w(t)$ and $w^*(t)$ denote two given moduli of continuity such that
$(w(t))^{\beta/\alpha} = O(w^*(t))$ as $t \to 0^+$ for $0 < \beta \le \alpha \le 1$

If $C_{2\pi}$ denote the Banach spaces of all 2π −periodic continuous function under "sup'' norm for $0<\alpha \le 1$ and some positive constant K, the function H_w is defined by

$$H_w = \{f \in C_{2\pi} : |f(x)-f(y)| \le K\, w\, |x-y|\}. \tag{2.3}$$

with the norm $\|.\|_{w^*}$ defined by

$$\|f\|_{w^*} = \|f\|_c + \sup_{x,y} \Delta^{w^*}[f(x,y)], \tag{2.4}$$

where

$$\|f\|_c = \sup_{-\pi \le x \le \pi} |f(x)|. \tag{2.5}$$

and

$$\Delta^{w^*}\{f(x, y)\}= \frac{|f(x)-f(y)|}{w^*(|x-y|)}, \qquad (x \neq y). \tag{2.6}$$

The convention that Δ^0 f(x, y) =0.If there exit positive constant B and K such that w |x-y| $\leq B\ |x-y|^{\alpha}$ and w^*|x-y| $\leq K\ |x-y|^{\beta}$ then

$$H_\alpha = \{f \epsilon\ C_{2\pi}: |f(x)-f(y)| \leq K|x-y|^\alpha,\ 0<\alpha \leq 1\}. \text{ (see Prössdorf's[13])} \tag{2.7}$$

The metric induced (2.5) by the norm $\|.\|_\alpha$ on the H_α is called the Hölder metric. If can be seen that $\|f\|_\beta \leq (2\pi)^{\alpha-\beta}\ \|f\|_\alpha$ for $0\leq\beta<\alpha\leq1$.Thus $\{(H_\alpha, \|.\|_\alpha)\}$ is a family of Banach spaces which decreases as α increase.
The series $\sum_{n=0}^{\infty} u_n$ is said to be (C, 2) summable to S. If the *(C, 2)* transform of S_n is defined as (see Hardy [5])

$$t_n^{\widetilde{(C,2)}}(f:x) = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)\,\widetilde{S_k} \to S \quad \text{as } n\to\infty \tag{2.8}$$

The $t_n^{\widetilde{(E,1)}}(f:x)$ denotes the transform of $\widetilde{(E,1)}$ is defined as

$$t_n^{\widetilde{(E,1)}}(f:x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\widetilde{S_k} \to S, \text{ as } n\to\infty$$

and

$$t_n^{\widetilde{(C,2)(E,1)}}(f:x) = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)\sum_{v=0}^{k}\binom{k}{v}\widetilde{S_v} \to S \quad \text{as } n\to\infty \tag{2.9}$$

The conjugate function $\widetilde{f(x)}$ is defined by

$$\widetilde{f(x)} = -\frac{1}{2\pi}\int_0^{\pi} \varphi(t)\ \cot\frac{t}{2}\,dt$$
$$= \lim_{h\to0}\left(-\frac{1}{2\pi}\int_h^{\pi} \varphi(t)\ \cot\frac{t}{2}\,dt\right) \tag{2.10}$$

The degree of approximation $E_n(f)$ be

$$E_n(f) = min\,\|T_n - f\|_p\,, \tag{2.11}$$

$T_n(x)$ denotes a trigonometric polynomial of degree n by (see Zygmund [20]).

Using the following notation

$$\Phi_x(t) = f(x+t) + f(x-t) - 2f(x) \tag{2.12}$$

and

$$\varphi(t) = \Phi_x(t) - \Phi_y(t). \tag{2.13}$$

3. **Known Theorem.**

Theorem (see [18]). Let w (t) defined in (2.3) be such that

$$\int_t^{\pi}\frac{w(u)}{u^2}\,du = O\,(H(t),\ H(t) \geq 0, \tag{3.1}$$

$$\int_0^t H(u)du = O\,(t\,H(t), \text{ as } t\to 0^+ \tag{3.2}$$

then, for $0< \beta\leq \alpha \leq 1$ and f ϵ H_α we have

$$\|t_n^{C^1.E^1}(f) - f(x)\|_{w^*} = O\left(\left((n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right)^{1-\beta/\alpha}\right) \tag{3.3}$$

4. **MAIN THEOREM**

On approximation of function $\tilde{f} \in H_w$ class by (*C*, 2)(*E, 1*) mean of conjugate of Fourier series, has been established.

Theorem: If $\tilde{f} \in H_w$ and $0\leq\beta<\alpha\leq1$ then

$$\|\,t_n^{\widetilde{(C,2)(E,1)}}(f;x) - \tilde{f}(x)\|_{w^*} = O\left\{\frac{w(|x-y|)^{\beta/\alpha}}{w^*(|x-y|)}(\log(n+1))^{\beta/\alpha}\left[(n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}\right\} \tag{4.1}$$

where $t_n^{(C,\widetilde{2)(E},1)}$ is the $(C,\widetilde{2)(E},1)$ mean of $S_n(f;x)$.

5. **Lemmas**: Using the following lemmas

Lemma 1 Let $|\widetilde{M_n}(t)| = \frac{1}{\pi(n+2)(n+1)} \left| \sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin^{t}/_{2}}\right\}\right]\right|$

Apply $|\sin\frac{t}{2}| \geq \frac{t}{\pi}$ and $|\cos\left(v+\frac{1}{2}\right)t| \leq 1$, for $0 \leq t \leq \frac{\pi}{(n+1)}$

$$= \frac{1}{\pi(n+2)(n+1)} \sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\frac{|\cos\left(v+\frac{1}{2}\right)t|}{|\sin^{t}/_{2}|}\right\}\right]$$
$$= \frac{1}{t(n+2)(n+1)} \sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\right\}\right]$$
$$= \frac{1}{t(n+2)(n+1)} \sum_{k=0}^{n}(n-k+1) \qquad \left(\because \sum_{v=0}^{k}\binom{k}{v} = 2^k\right)$$
$$= \frac{(n+1)}{t(n+2)(n+1)} - \frac{1}{t(n+2)(n+1)}\sum_{k=0}^{n} k$$
$$= \frac{1}{t(n+2)} - \frac{n(n+1)}{2t(n+2)(n+1)}$$
$$= \frac{1}{t(n+2)} - \frac{n}{2t(n+2)}$$
$$= O\left(\frac{1}{t}\right) \tag{5.1}$$

Lemma2 Let $|\widetilde{M_n}(t)| = \frac{1}{\pi(n+2)(n+1)} \left| \sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin^{t}/_{2}}\right\}\right]\right|$

Using $|\sin\frac{t}{2}| \geq \frac{t}{\pi}$ and $|\sin t| \leq 1$ for $\frac{\pi}{(n+1)} \leq t \leq \pi$

$$= \frac{1}{t(n+2)(n+1)} \left| \sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\cos\left(v+\frac{1}{2}\right)t\right\}\right]\right|$$
$$= \frac{1}{t^2(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1) \qquad \text{(see [9])}$$

$$= \frac{(n+1)}{t^2(n+1)(n+2)} - \frac{n(n+1)}{2t^2(n+1)(n+2)}$$
$$= \frac{1}{t^2(n+2)} \tag{5.2}$$

Lemma 3 (see [18]) If w (t) satisfies condition (3.1) and (3.2), we get

$$\int_0^u t^{-1}w(t)dt = O(u\,H(u)), \qquad \text{as } u \to 0^+. \tag{5.3}$$

Lemma 4 Let $\Phi_x(t)$ defines (2.13) for $\tilde{f} \in H_w$

$$|\Phi_x(t) - \Phi_y(t)| \leq 2M\,w|x-y| \tag{5.4}$$

also $$|\Phi_x(t) - \Phi_y(t)| \leq 2M\,w|t| \tag{5.5}$$

It is easy to verify.

6. PROOF OF THE MAIN THEOREM

Using (see [19]) and Riemann – Lebesgue theorem, then

$$\widetilde{S_n}(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin\frac{t}{2}}\cos\left(n+\frac{1}{2}\right)t\,dt \tag{6.1}$$

If $t_n^{\widetilde{(E,1)}}$ denotes $\widetilde{(E,1)}$ transform of $\widetilde{S_n}(f;x)$ then

$$t_n^{\widetilde{(E,1)}}(f;x) - \tilde{f}(x) = \frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin^{t}/_{2}}\sum_{k=0}^{n}\binom{n}{k}\cos\left(k+\frac{1}{2}\right)t\,dt, \tag{6.2}$$

If $t_n^{(C,\widetilde{2)(E},1)}$ denotes $(C,\widetilde{2)(E},1)$ transform of $\widetilde{S_n}(f;x)$, Now

$$t_n^{(C,\widetilde{2)(E},1)}(f;x) - \tilde{f}(x) = \frac{1}{\pi(n+1)(n+2)} \sum_{k=0}^{n} \left[\frac{(n-k+1)}{2^k} \int_0^{\pi} \frac{\phi_x(t)}{\sin^{t}/_{2}} \left\{\sum_{v=0}^{k} \binom{k}{v} \cos\left(v+\frac{1}{2}\right)t\right\}\right] \quad (6.3)$$

Writing $I_n(x) = t_n^{(C,\widetilde{2)(E},1)}(f;x) - \tilde{f}(x)$ then

$$|I_n(x)| = |t_n^{(C,\widetilde{2)(E},1)}(f;x) - \tilde{f}(x)|$$

$$\leq \left|\frac{1}{\pi(n+2)(n+1)} \sum_{k=0}^{n} \left[\frac{(n-k+1)}{2^k} \int_0^{\pi} \frac{\phi_x(t)}{\sin^{t}/_{2}} \left\{\sum_{v=0}^{k} \binom{k}{v} \cos\left(v+\frac{1}{2}\right)t\right\}\right]\right| dt \quad (6.4)$$

$$|I_n(x) - I_n(y)| = \left|\frac{1}{\pi(n+2)(n+1)} \sum_{k=0}^{n} \left[\frac{(n-k+1)}{2^k} \int_0^{\pi} \frac{\phi_x(t)-\phi_y(t)}{\sin^{t}/_{2}} \left\{\sum_{v=0}^{k} \binom{k}{v} \cos\left(v+\frac{1}{2}\right)t\right\}\right]\right| dt \quad (6.5)$$

$$= \frac{1}{\pi(n+2)(n+1)} \sum_{k=0}^{n} \left[\frac{(n-k+1)}{2^k} \int_0^{\pi} \frac{|\phi_x(t)-\phi_y(t)|}{\sin^{t}/_{2}} \left\{\sum_{v=0}^{k} \binom{k}{v} \cos\left(v+\frac{1}{2}\right)t\right\}\right] dt$$

$$= \frac{1}{\pi(n+2)(n+1)} \sum_{k=0}^{n} \left[\frac{(n-k+1)}{2^k} \int_0^{\pi} \frac{|\phi(t)|}{\sin^{t}/_{2}} \left\{\sum_{v=0}^{k} \binom{k}{v} \cos\left(v+\frac{1}{2}\right)t\right\}\right] dt$$

$$= \int_0^{\pi} |\phi(t)|\,|M_n(t)|\,dt \qquad \text{using Lemma 1}$$

$$= \left[\int_0^{\pi/_{n+1}} + \int_{\pi/_{n+1}}^{\pi}\right] |\phi(t)|\,|M_n(t)|\,dt$$

$$= I_1 + I_2 \quad (6.6)$$

Now using (5.5) and Lemma3

$$|I_1| = \int_0^{\pi/_{n+1}} |\phi(t)||M_n(t)|\,dt$$

$$= O(1) \int_0^{\pi/(n+1)} t^{-1}\, w(t)dt$$

$$= O\left((n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right). \quad (6.7)$$

Now

$$|I_2| = \int_{\pi/_{n+1}}^{\pi} |\phi(t)||M_n(t)|dt \qquad \text{using (5.5) and Lemma 2}$$

$$= O(1) \int_{\pi/(n+1)}^{\pi} t^{-2}\, w(t)\, dt$$

$$= O\left((n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right). \quad (6.8)$$

Now using (5.4), Lemma 1, we get

$$I_1 = O\left(\frac{1}{n+2}\right) \int_0^{\pi/(n+1)} t^{-1}\, w(|x-y|)dt$$

$$= O\,(w(|x-y|)) \int_0^{\pi/(n+1)} t^{-1}\, dt$$

$$= O\,(\log\,(n+1)\, w(|x-y|)) \quad (6.9)$$

Now using (5.4) and Lemma2

$$I_2 = O\left(\frac{1}{n+2}\right) \int_{\pi/(n+1)}^{\pi} t^{-2}\, w(|x-y|)\, dt$$

$$= O\,(w(|x-y|)). \tag{6.10}$$

We have

$$|I_k| = |I_k|^{1-\beta/\alpha}\,|I_k|^{\beta/\alpha}. \quad \text{when} \quad k = 1, 2 \tag{6.11}$$

By using (6.7) and (6.9) respectively in the first and the second factor on the right of the above identify (6.11) for k = 1 then

$$|I_1| = O\left(\left[(n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha} . [\log(n+1)\, w(|x-y|)]^{\beta/\alpha}\right) \tag{6.12}$$

Again using (6.8) and (6.10) in the first and second factor on the right of the identify (6.11) for k = 2 then

$$|I_2| = O\left(\left[(n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha} . [w(|x-y|)]^{\beta/\alpha}\right) \tag{6.13}$$

Thus from (2.6), (6.12) and (6.13) we get

$$\sup_{x\neq y} \left| \Delta^{w^*} I_n(x, y) \right| = \sup_{\substack{(x,y)\\ x\neq y}} \frac{|I_n(x)-I_n(y)|}{w^*(|x-y|)}$$

$$= O\left\{\frac{w(|x-y|)^{\beta/\alpha}}{w^*(|x-y|)} (\log(n+1))^{\beta/\alpha} \left[(n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}\right\} \tag{6.14}$$

Using the fact that $\tilde{f} \epsilon H_w \Rightarrow \phi_x(t) = O\,(w\,(t))$, then

$$\| I_n \|_c = \sup_{-\pi\leq x\leq\pi} \| t_n^{(C,2)\widetilde{(E,1)}}(f;x) - \tilde{f}(x)\|$$

$$= O\,\left\{(n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right\}. \tag{6.15}$$

Combining the result of (6.14) and (6.15), then

$$\| t_n^{(C,2)\widetilde{(E,1)}}(f;x) - \tilde{f}(x)\|_{w^*} = O\left\{\frac{w(|x-y|)^{\beta/\alpha}}{w^*(|x-y|)} (\log(n+1))^{\beta/\alpha} \left[(n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}\right\} \tag{6.16}$$

Completes the proof of main theorem

7. Corollaries:

The corollaries can be derived from main theorem.

Corollary7. 1: "If $\beta = 0$ and $\tilde{f} \in Lip(\alpha,\ p)$, $0< \alpha \leq 1$ then

$$\| t_n^{(C,2)\widetilde{(E,1)}}(f;x) - \tilde{f}(x)\|_c = O\left\{\frac{1}{(n+1)^{\alpha}}\right\} \quad \text{for } 0< \alpha < 1.$$

$$= O\left(\frac{\log(n+1)}{(n+1)}\right), \text{ for } \alpha = 1$$

Acknowledgement- We are highly thankful to Dr. U. K. Shrivastava, Head, Department of Mathematics, Govt. E. R. Rao science PG College Bilaspur, Chhattisgarh, India for his encouragement and support to this work.

Conclusion

The summability method (E, 1) includes method of summability like Borel, (E, q), (e, c), F(a, q) and [F, d_n] then by using the result of main theorem we can derive more generalizing result and also the result of Mishra and Khatri [10] can be derived directly.

References

[1] Albayrak, I., Koklu, K. and Bayramov, A., On the Degree of Approximation of Function Belonging to the Lipschitz class by (C, 2) (E, 1) means, Int. Journal of Math. Analysis, 4(49), (2010), 2415 – 2421.

[2] Alexits, G., "Convergence problems of orthogonal series" pergamon press, London 1961.

[3] Chandra, P., On the generalized Fej'er mean in the metric of Hölder Metric, Mathematics nachrichten, 109(1), (1982), 39-45.
[4] Das, G., Ghosh, T. and Ray, B.K. Degree of approximation of function by their Fourier series in the generalized Hölder metric, "Proceeding of the Indian Academy of Science .Mathematical Science,106(2), (1996),139-153.
[5] Hardy, G. H. "Divergent series", first edition, Oxford University Press, 1949.
[6] Khan, H. H., On degree of approximation of function belonging to the class Lip (α, p), Indian J. pure Appl. Math, 5(2), (1974), 132 – 136.
[7] Kushwaha, J. K., On approximation of function by (C, 2)(E,1) product summability method of Fourier series, Ratio Mathematica, 38, (2020), 341-348.
[8] Lal, S. and Kushwaha, J.K., Degree of approximation of Lipschitz function by (C, 1)(E, q)means of its Fourier series, International Math. Forum, 4(43) (2009), 2101-2107.
[9] Lal, S. and Singh, P. N., Degree of approximation of conjugate of Lip (α, p) function by (C, 1)(E, 1) means of conjugate series of a Fourier series, Tamkang Journal of Mathematics, 33(3), (2002).
[10] Mishra, V. N. and Khatri, K., Degree of approximation of function $\tilde{f} \in H_w$ class by the (N_p .E^1) Means in the Hδlder metric, International Journal of Mathematics Mathematical Sciences, http://dx.doi.org/10.1155/2014/837408
[11] Mohapatra, R. N. and Chandra, P., Degree of approximation of function in the Hölder metric, Acta Math. Hung; 41(1983), 67-76.
[12] Nigam, H. K., On (C, 2) (E, 1) product means of Fourier series and its conjugate series, Electronic Journal of Mathematical Analysis and Application, 1(2) (2013), 334 – 344.
[13] Prössdorf's, S., "Zur konvergenz der Fourierreihen Hölder Stetiger Funktionen" Mathematische Nachrichten, 69(1), (1975),7-11.
[14] Rathore, H. L. and Shrivastava, U. K., On the degree of approximation of function belonging to weighted (L_p, ξ(t)) class by (C, 2) (E, q) means of Fourier series, International Journal of Pure and Applied Mathematical Sciences, 5(2), (2012), 79-88.
[15] Rathore, H. L., Shrivastava, U. K. and Mishra, L. N., On Approximation of continuous Function in the Hölder Metric by (C 1) [F, d_n] Means of its Fourier Series, Jnanabha, 51(2), (2021), pp.161-167.
[16] Rathore, H. L., Shrivastava, U. K. and Mishra, V. N., Degree of Approximation of Continuous Function in the Hölder metric by (C 1) F (a, q) means of its Fourier Series. Ganita, 72(2), (2022), 19-30.
[17] Rathore, H. L. and Shrivastava, U. K., and Mishra, V. N., On Approximation of continuous Function in the Hölder Metric by (C, 2)(E, q)Means of its Fourier Series, Material Today Proceeding, https://doi.org/10.1016/j.matpr.2021.11.150, Vol. 57(5), (2022), 2026-2030.
[18] Singh, T. and Mahajan, P. Error Bound of periodic signals in the Hölder metric, International Journal of Mathematics and Mathematical Sciences, doi: 10.1155/2008/495075
[19] Titchmarsh, E.C. "The Theory of function", Oxford University Press, 1939, 402 - 403.
[20] Zygmund, A. "Trigonometric Series", 2nd Rev. Ed., Cambridge University Press, Cambridge, 1968.