**Certain Reduction Formulae for Srivastava–Daoust**

**type Series**

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**ABSTRACT**

 Many of the Physical, Astrophysical, Statistical, and Mathematical problems can be solved by using the allied special functions. Specifically, various multiple hypergeometric series and reduction formulae can be applied to solve such problems. The aim of the present chapter is to derive certain classes of the reduction formulae for the Srivastava–Daoust type double hypergeometric series. To prove these reductions, we use one of the extension results on the Bailey transform developed and studied by Joshi and Vyas in 2005. We also obtain some well–known formulae, e.g. the Kampé de Fériet reduction formula, the Euler transformation formula and Whipple’s quadratic transformation formula, on particularization of some reduction formulae.

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1. **INTRODUCTION**

The Gauss hypergeometric series is



 (1)

This series was studied by the famous German mathematician C.F. Gauss [1].

The part  in “(1)” in terms of Pochhammer’s symbol is denoted by

 (2)

For , the values of is equal to 1. Similar interpretation is for other Pochhammer’s symbol,  and , present in “(1)”.

Thus, “(1)” in terms of Pochhammer’s symbol is given by:

 (3)

The series in “(3)” is convergent if . Eq. “(3)” is convergent if it terminates, i.e., when  or  are negative integer or zero. However, for  it converges if  and diverges if. If, the series “(3)” converges conditionally, see [2, p. 18].

Also, when  or  where and where  in “(3)” then the series “(3)” terminates otherwise it becomes meaningless.

The generalization of “(3)” is as given below:

 (4)

which have arbitrary number of numerator  and denominator  parameters.

Note that  and  are either zero or positive integers and the argument  may take any real or complex value, provided none of the bottom parameters  in “(4)” is zero or negative integer. This generalized hypergeometric function is convergent or divergent with the following restrictions:

1. Converges for all  if ; for all if  and for  with , if

 (5)

1. Diverges for every , , if

The generalization of “(3)” can be given by either increasing numerator and denominator parameters, as we have shown in “(4)” or by increasing the arguments. Such series are called multiple hypergeometric series, e. g. Appell functions in two variables, Kampé de Fériet functions, Horn’s functions, Srivastava’s triple hypergeometric series, Srivastava–Daust series etc. Several authors worked on the reductions and transformations for these multiple hypergeometric series, see ([3], [4], [5], [6], [7], [8], [9], [10]) and references therein. The applications of multiple hypergeometric series in solving a vast number of Physical, Statistical and Mathematical problems can be found in ([11], [12], [2], [13], [14], [15]) and references therein.

 The utility of the reduction formulae for certain classes of double series is discussed in several research papers, see ([16], [17], [18], [19], [20]) and references therein. They have shown that certain reduction formulae for multiple hypergeometric series are applicable in solving astrophysical problems, queuing theory and related stochastic processes, physical and quantum chemical problems, boundary value problems (heat equation) and in the derivation of radial wave functions. The reduction formulae for Srivastava–Daoust hypergeometric functions have been studied and investigated in a number of papers ([21], [22], [23], [24], [25], [26], [27] and references therein).

 In this chapter, we focus on investigating certain Srivastava–Daoust type reduction formulae. Certain reductions are interesting generalizations of some well–known hypergeometric functions e.g. the Euler transformation formula, the Whipple’s quadratic transformation and one of the Kampé de Fériet reductions.

The Kampé de Fériet function with an arbitrary number of numerator and denominator parameters and two arguments is as follows:

 (6)

where the convergence condition is,



 (7)

The Srivastava–Daoust series [2, pp. 26–28], also referred to as the generalized Lauricella function of several variables is as follows.

 (8)

where

 (9)

For further details on notations and convergence conditions for “(8)”, please refer to [7] and [9, pp. 157–158]. Eq. “(8)” reduces to “(6)” when .

Note that, in many of the papers concerning reductions or transformations of Srivastava–Daoust double hypergeometric series, the parameters  and ’s appearing in “(8)” to “(9)” are given some particular constant values. For example, see [24]. With the help of one of the extension results on the Bailey transform (“(10)” and “(11)”), we can express these parameters in terms of  that can be assigned any arbitrary integer values. The results with arbitrary values of these parameters are not available in the literature till date. Moreover, it is always possible to derive general reduction formulae involving arbitrary bounded sequence of complex numbers in place , provided that the involved series are convergent. Further, the obvious and straightforward generalizations of the results of this paper to reductions or transformations of  fold series to *m*–fold series can always be developed after getting the idea of applying Saalschütz summation theorem used in this chapter.

One of the two extension results on the Bailey transform [28] due to Joshi and Vyas [29] is stated as follows:

If

 and

(10)



then, subject to convergence conditions

 (11)

where  and  are any functions of only and  and  are any arbitrary integers. We use two transforms given in “(10)” to derive the reduction formulas stated in Section 2.

This chapter has three sections. The additional reduction formulae and their derivations are given in Section 2. The particular cases of certain reduction formulae are discussed in Section 3.

**II. REDUCTION FORMULAE FOR SRIVASTAVA–DAOUST TYPE FUNCTIONS AND THEIR DERIVATIONS**

In this section, first we state the reduction formulae for Srivastava–Daoust type functions and then demonstrate the proof of reduction formulae one by one.

 (12)

 (13)

 (14)

 (15)

 (16)

 (17)

 (18)

 (19)

 (20)

 (21)

 (22)

 (23)

 (24)

 (25)

1. **Derivations of the results (12) to (25)**

To obtain the reduction formulae “(12)” to “(25)” listed in Section II, we set different expressions for  and in “(10)”, which yields  and, closed form for  when the Saalschütz summation theorem [28, p. 243, “(III.2)”] is applied. The final results are obtained with the help of “(11)”. Note that,  and  are positive integers, while and  are arbitrary integers.

1. Choosing

 and in “(10)” and using “(11)”, we get the reduction “(12)”.

1. Selecting  and in “(10)” and using “(11)”, we obtain the reduction “(13)”.
2. Letting

 and in “(10)” and using “(11)”, we get the result “(14)”.

1. Selecting  and in “(10)” and using “(11)”, we obtain the reduction “(15)”.
2. Selecting  and in “(10)” and using “(11)”, we obtain the result “(16)”.
3. Choosing  and in “(10)” and using “(11)”, we obtain the reduction “(17)”.
4. Letting and in “(10)” and using “(11)”, we obtain the result “(19)”.
5. Selecting  and in “(10)” and using “(11)”, we obtain the reduction “(20)”.
6. Letting  and in “(10)” and using “(11)”, we obtain the result “(21)”.

1. Taking  and in “(10)” and using “(11)”, we obtain the result “(22)”.
2. Selecting  and in “(10)” and using “(11)”, we obtain the reduction “(23)”.
3. Taking  and in “(10)” and using “(11)”, we obtain the reduction “(24)”.
4. Letting  and in “(10)” and using “(11)”, we obtain the result “(25)”.
5. Choosing  in “(10)” and using “(11)”, we obtain the reduction “(26)”.
6. **PARTICULAR CASES OF INVESTIGATED RESULTS**

The reductions stated in previous section i.e. “(12)” to “(25)” have arbitrary variables  and . By assigning different integer values to these variables  and , we obtain the well–known results, e.g., the Kampé de Fériet reduction formula, the Euler transformation formula and the Whipple’s quadratic transformation formula as recorded in [2, p. 28, “(34)”] and [30, p. 60] and [31, p. 633, “(E.4.3)”], respectively, are obtained.

1. Choosing  in “(12)”, a Kampé de Fériet reduction formula [2, p. 28, “(34)”] follows.
2. Selecting  in “(13)”, the well–known Whipple's quadratic transformation [31, p. 633, “(E.4.3)”] follows.
3. The “(12)” is a generalization of both the Euler transformation [30, p. 60] and the Kampé de Fériet reduction given by [2, p. 28, “(34)”] which follows respectively, when  and .
4. Eq. “(15)” is a generalization of both Whipple’s quadratic transformation [31, p. 633, “(E.4.3)”] and Horn’s  reduction:

 (26)

which follows respectively, when  and .

1. When  in “(16)”, the Whipple’s quadratic transformation given by [31, p. 633, “(E.4.3)”] follows.
2. The “(20)” generalizes and unifies the Horn's  reduction as mentioned in part “(iv)” of this section and the Kampé de Fériet reduction mentioned in [2, p. 28, “(34)”], which follows when  and , respectively.
3. The “(21)” generalizes the Kampé de Fériet reduction mentioned in [2, p. 28, “(34)”], which follows when  or .
4. Choosing in “(23)”, the Whipple’s quadratic transformation given [31, p. 633, “(E.4.3)”] follows.
5. When  in “(25)”, we obtain a Srivastava–Daoust transformation due to [21, p. 21]. Note that, the Gauss theorem can not be applied in the right side of “(3.1)”, see [21, p. 24] since convergence conditions of Gauss theorem are violated.

Further, for , “(25)” also generalizes the Bailey's cubic transformation given by [32, p. 190] for .

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