On Generalized Entropy of Order Statistics and Characterization Results*

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***Abstract*—In this communication, we introduced generalized one-parametric entropy of order statistics and obtain its bounds. The main aim of the paper to characterize the parent distributions based on generalized entropy of order statistics. Further, we extended our proposed measure to truncated random variables. We also characterize exponential distribution, pareto distribution and finite range distribution in respect of proposed residual entropy measure of first order statistics.**

***Index Terms*—Random variable (r.v), Shannon's Entropy, order statistics, hazard rate.**

I. Introduction

Let $\phi(t)$ and $\varphi(t)$ be the distribution function and density function respectively. Suppose $T_1, T_2, ..., T_n$ be independent and identically distributed observations associated with given probability functions. Define order statistics by organizing the preceding r.v's in increasing order, then $T_{1:n} \leq T_{2:n} \leq ... \leq T_{n:n}$ is order statistics of r.v T. The p.d.f of $T_{i:n}$ $(1 \leq i \leq n)$ is

$$\varphi_{i:n}(t) = \frac{\Gamma(n+1)}{\Gamma(n-i+1)\Gamma(i)}[\phi(t)]^{i-1}[1-\phi(t)]^{n-i}\varphi(t)$$

where $\Gamma(a)$ is the gamma function.
The probability integral transform of the r.v. $U = \phi_T(T)$ plays a momenteous role to establish our outcomes. Also, U is uniform over the unit interval. From uniform distribution, the order statistics of a sample $U_1, ..., U_n$ is written as by $M_1 < ... < M_n$ and $M_i, i = 1, 2, ..., n$ has beta distribution with p.d.f

$$g_i(m) = \frac{1}{\beta(i, n-i+1)}m^{i-1}(1-m)^{n-i}, \ \ 0 \leq m \leq 1,$$

where $\beta(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 z_2)}$.
Order statistics is considered by authors and is utilized to solve the queries of reliability analysis, digital image processing, reliability engineering etc. [3] statistical estimation [6], image coding [4] etc. In the problems related to reliability and life testing $T_{i:n}$ refers to the lifetime of a (n-i+1) out of n system [2]. At the phase of intriguing a system analysis of ambiguity in the life time of $(n-i+1)$ out of n system is very significant. To handle such kind of problems, Wong and Chen [16] considered an entropy of order statistics analogous to Shannon [7]. Arghami and Abbasnejad [1] provides generalized entropy of order statistics corresponding to Renyi entropy [14]. In this study, we propose generalized entropy of order statistics using Khinchin's [12] approach. We also propose generalized residual entropy of order statistics to deal with the uncertainty in the lifetime of $(n-i+1)$ out of n systems when it has survived upto time t.

Based on entropies, many authors provided the characterization problems of order statistics. The papers of Wong and Chen [16], Ebrahimi [10], Ebrahimi et. al. [9], Baratpour et. al. [8], Park [13], Gupta et. al. [5], Thapliyal [15], Zarezadeh and Asadi [17], Kayal [11] contains many theroretic results based on order statistics.

Section II introduced the on-parametric entropy of order statistics and compute extreme values of proposed measure for some classic distributions. In Section III, we proposed one parametric generalized residual entropy of order statistics and calculate its extreme values for some distributions. We also present a characterization result based on proposed measures. Section IV concludes the whole paper.

II. Generalized entropy of order statistics

If T is a r.v. with cdf $\phi(t)$ and pdf $\varphi(t)$ then generalized entropy of order α is defined by

$$\Omega_\alpha(T) = \frac{1}{\alpha}\int_{-\infty}^{+\infty}\varphi(t)[1-\varphi^\alpha(t)]dt, \tag{1}$$

$$= \frac{1}{\alpha}\int_0^1[1-\varphi^\alpha(\phi^{-1}(t))]dt. \tag{2}$$

where

$$\lim_{\alpha\to 0}\ \Omega_\alpha(T) = -\int\varphi(t)\log\varphi(t)dt \tag{3}$$

$$= \Omega(T) \tag{4}$$

is the Shannon entropy of T.
We propose generalized entropy of order statistics

$$\Omega_\alpha(X_{i:n}) = \frac{1}{\alpha}\int_{-\infty}^{+\infty}\varphi_{i:n}[1-\varphi_{i:n}^{\alpha}(t)]dt \qquad (5)$$

where

$$\varphi_{i:n}(t) = \frac{\Gamma(n+1)}{\Gamma(n-i+1)\Gamma(i)}[\phi(t)]^{i-1}[1-\phi(t)]^{n-i}\varphi(t)$$

Then using probability integral transform we obtain the following representation of generalized entropy of i^{th} order statistic

$$\begin{aligned}\Omega_\alpha(T_{i:n}) &= \frac{1}{\alpha}\Big\{1-\int_0^1 g_i^{\alpha+1}(m)f^{\alpha}(\phi^{-1}(m))dm\Big\} \qquad (6)\\ &= \frac{1}{\alpha}\Big\{1-E_{g_i^{\alpha+1}}[\varphi^{\alpha}(\phi^{-1}(m))]\Big\}. \qquad (7)\end{aligned}$$

where $g_i(m)$ is the density of beta distribution.
Further, we compute the extreme values of one-parametric generalized entropy of order statistics for some classic distributions.

Distribution	P.d.f	$\Omega_\alpha(T_{1:n})$	$\Omega_\alpha(T_{n:n})$
Exponential	$\theta\exp^{-\theta t}$	$\frac{1}{\alpha}\Big\{1-\frac{\theta^\alpha n^{\alpha+1}}{n(\alpha+1)}\Big\}$	$\frac{1}{\alpha}\Big\{1-\theta^\alpha n^{\alpha+1}B(\alpha n+n-\alpha,\alpha+1)\Big\}$
Pareto	$\frac{\theta\beta^\theta}{x^{\theta+1}}, t\geq\beta>0,\theta>0$	$\frac{1}{\alpha}\Big\{1-\frac{\theta^\alpha}{\beta^\alpha}\frac{n^{\alpha+1}}{n(\alpha+1)+\alpha/\theta}\Big\}$	$\frac{1}{\alpha}\Big\{1-\frac{\theta^\alpha n^{\alpha+1}}{\beta^\alpha}B(\alpha n+n-\alpha,\alpha(1+1/\theta)+1)\Big\}$
Finite Range	$\frac{a}{b}\Big(1-\frac{t}{b}\Big)^{a-1}$, $a>1,b>1,0\leq t\leq 1$	$\frac{1}{\alpha}\Big\{1-(\frac{a}{b})^\alpha\frac{n^{\alpha+1}}{\alpha n+n-\alpha+\alpha/a}\Big\}$	$\frac{1}{\alpha}\Big\{1-\frac{a^\alpha}{b^\alpha}n^{\alpha+1}B(\alpha(n-1)+n,\alpha/a+1)\Big\}$

Theorem below gives bound in terms of generalized entropy of data distribution $\Omega_\alpha(T)$ for generalized entropy $\Omega_\alpha(\Omega_{i:n})$.
Theorem 1. If random variable T having generalized entropy $\Omega_\alpha(T)<\infty$ then generalized entropy of order statistics is bounded as shown below
Assume $B_i^{\alpha+1}$ to be the i^{th} term of the binomial probability $[Bin(n-1;pi)]^{\alpha+1}; p_i=\frac{i-1}{n-i}$. Then

$$\Omega_\alpha(T_{i:n}) \geq B_i^{\alpha+1}[\Omega_\alpha(T)-1]+\frac{1}{\alpha}.$$

Proof. For beta distribution, mode is g_i is $p_i=\frac{i-1}{n-i}$. Thus, $g_i^{\alpha+1}(w) \leq B_i^{\alpha+1} = g_i^{\alpha+1}(p_i) = \frac{1}{B(i,n-i+1)^{\alpha+1}}p_i^{(\alpha+1)(i-1)}(1-p_i)^{(\alpha+1)(n-i)}$.

$$\begin{aligned}\Omega_\alpha(T_{i:n}) &= \frac{1}{\alpha}\{1-\int_0^1 g_i^{\alpha+1}(m)f^{\alpha}(\phi^{-1}(m))dm\}\\ &\geq \frac{1}{\alpha}\{1-B_i^{\alpha+1}\int_0^1 f^{\alpha}(\phi^{-1}(m)dm\}\\ &= \frac{1}{\alpha}\{1-B_i^{\alpha+1}\int^{\alpha+1}(t)dt\}\\ &= \frac{1}{\alpha}\Big\{1+B_i^{\alpha+1}\Big[\int\varphi(t)[1-\varphi^{\alpha}(t)]dt-1\Big]\Big\}\end{aligned}$$

So

$$\Omega_\alpha(T_{i:n}) \geq B_i^{\alpha+1}[\Omega_\alpha(T)-1]+\frac{1}{\alpha}.$$

Next, we have done some theorems on entropy of order statistics on the subject of ordering properties. Definitions below with T and K be r.v's with distribution function ϕ_T and ϕ_k, density functions φ_t and φ_k and survival functions $\bar{\phi}_T(t)=1-\phi_T(t)$ and $\bar{\phi}_t(t)=1-\phi_T(t)$.
Definition 1. If $\bar{\phi}_t(v) \leq \bar{\phi}_K(v)$, $\forall v$, then r.v T is stochastically less than or equal to K, written as T $\overset{st}{\leq}$ K.
Definition 2. If $\varphi_T(t)\leq\varphi_K(k)$ is non increasing in t, then r.v T is less than or equal to K in likelihood ratio ordering, written as T $\overset{lr}{\leq}$ K.
Definition 3. If $\Omega_\alpha(T)\leq\Omega_\alpha(K)$, then r.v T is less than or equal to K in entropy ordering, written as $T\overset{GE}{\leq}K$.
Theorem 2. Let T and K be two non-negative r.v's. If $T\overset{disp}{\leq}K$ then $K\overset{GE}{\leq}T$.
Proof. We have
$\Omega_\alpha(K)-\Omega_\alpha(T)=$

$$\frac{1}{\alpha}\Big\{\int g(t)[1-g^{\alpha}(t)]dt-\int\varphi(x)[1-\varphi^{\alpha}(t)]dt\Big\}$$

$$\begin{aligned}&= \frac{1}{\alpha}\int_0^1[\varphi^{\alpha+1}(\phi^{-1}(m))-g^{\alpha+1}(G^{-1}(m))]dm\\ &\leq 0.\end{aligned}$$

Example 1 shows the utilization of Theorem 2.
Example 1. Suppose $\varphi(t)=e^{-2t}$ and $g(t)=e^t$. Then $T\overset{disp}{\leq}K$. Therefore, by using Theorem 2, we have $T\overset{disp}{\leq}K$.
Theorem 3. Let T denotes a r.v and $K_i, i=1,...,n$, be its order statistics. If $\varphi_T(\phi_T^{-1}(t))$ is non-decreasing in t, then $\Omega_\alpha(K_i)$ is increasing in i.
Proof. We have
$\Omega_\alpha(K_{i+1})-\Omega_\alpha(K_i)=$

$$\frac{1}{\alpha}\Big\{E_{g_{i+1}^{\alpha+1}}(\varphi^{\alpha}(\phi^{-1}(m_{i+1})))-E_{g_i^{\alpha+1}}(\varphi^{\alpha}(\phi^{-1}(m_i)))\Big\}.$$

Here, we have stochastically ordered order statistics, so $M_i\overset{st}{\leq}M_{i+1}$ implies that for any non-decreasing function ψ,

$$E_{g_i^{\alpha+1)}}[\psi(M_i)] < E_{g_{i+1}^{\alpha+1}}[\psi(M_{i+1})],$$

Thus, $\Omega_\alpha(K_{i+1})-\Omega_\alpha(T_i)\geq 0$, and the result follows.
Theorem 4. Let T and K be two r.v's, at least one of which is 'DFR'. Then $T\overset{hr}{\leq}K \implies T\overset{GE}{\leq}K$.
Proof. Since $T\overset{hr}{\leq}K \implies T\overset{disp}{\leq}K$, the result follows.

Theorem 5. Let T and K be two continuous r.v's having cdf $\phi(t)$ and $G(k)$ and pdf's $\varphi(t)$ and $g(k)$ respectively. Suppose

$$\beta_1 = \Big\{0 < m < 1 | g(G^{-1}(m)) \leq \varphi(\phi^{-1}(m)\Big\},$$
$$\beta_2 = \Big\{0 < m < 1 | g(G^{-1}(m) \geq f(F^{-1}(m))\Big\},$$

and $T \overset{GE}{\leq} K$. Let $h(m) = u^{i-1}(1-u)^{n-i}$. If $B_1 = \phi, B_2 = \phi$ or $\inf_{B_1} h(m) \geq \inf_{B_1} h(m)$, then $T_{i:n} \overset{GE}{\leq} K_{i:n}$.

Proof. If $\beta_1 = \phi$ or $\beta_2 = \phi$, the result is obvious.

So, let $\beta_1 \neq \phi$ and $B_2 \neq \phi$. Since $T \overset{GE}{\leq} K$, we have

$$\int_0^1 [\varphi^\alpha(\psi^{-1}(m)) - g^\alpha(G^{-1}(m))]dw \geq 0 \tag{8}$$

From (7), we have
$\Omega_\alpha(T_{i:n}) = \frac{1}{\alpha}\int_{-\infty}^{+\infty} \varphi_{i:n}(t)[1 - \varphi^\alpha_{i;n}(t)]$.
Let $m = \psi(t)$, then $\Omega_\alpha(T_{i:n}) = \frac{1}{\alpha}\Big\{1 - \frac{1}{B(i,n-i+1)^{\alpha+1}} \int_0^1 m^{(\alpha+1)(i-1)}(1-m)^{(\alpha+1)(n-i)} f(F^{-1}(m)dm\Big\}$.
Using the transformation $m = G(k)$, we have

$$\Omega_\alpha(T_{i:n}) = \frac{1}{\alpha}\Big\{1 - \frac{1}{B(i,n-i+1)^{\alpha+1}}\int_0^1 m^{(\alpha+1)(i-1)} (1-m)^{(\alpha+1)(n-i)} g(G^{-1}(m))dm\Big\}.$$

It follows that

$$\Omega_\alpha(T_{i:n}) - \Omega_\alpha(K_{i:n}) = -\frac{1}{B(i,n-i+1)^{\alpha+1}}\int_0^1 h^{\alpha+1}(m) \Big[\varphi^{\alpha+1}(\phi^{-1}(m)) - g^{\alpha+1}(G^{-1}(m))\Big]dm = -\frac{1}{B(i,n-i+1)^{\alpha+1}}\ D.$$

Now,

$$\begin{aligned}
D &= \int_0^1 h^{\alpha+1}(m)[\varphi^{\alpha+1}(\phi^{-1}(m)) - g^{\alpha+1}(G^{-1}(m))]dm \\
&= \int_{\beta_1} h^{\alpha+1}(m)[\varphi^{\alpha+1}(\phi^{-1}(m)) - g^{\alpha+1}(G^{-1}(m))]dm \\
&\quad + \int_{\beta_1} h^{\alpha+1}(m)[\varphi^{\alpha+1}(\phi^{-1}(m)) - g^{\alpha+1}(G^{-1}(m))]dm \\
&= \int_{\beta_1} h^{\alpha+1}(m)[\varphi^{\alpha+1}(\phi^{-1}(m)) - g^{\alpha+1}(G^{-1}(m))]dm \\
&\quad + \int_{\beta_1} h^{\alpha+1}(m)[\varphi^{\alpha+1}(\phi^{-1}(m)) - g^{\alpha+1}(G^{-1}(m))]dm \\
&\geq (\inf_{\beta_1} h(m))^{\alpha+1}\int_{\beta_1}[\varphi^{\alpha+1}(\phi^{-1}(m)) - g^{\alpha+1}(G^{-1}(m))]dm \\
&\quad + (\sup_{\beta_2})^{\alpha+1}\int_{\beta_2}[\varphi^{\alpha+1}(\phi^{-1}(m)) - g^{\alpha+1}(G^{-1}(m))]dm \\
&\geq v(\sup_{\beta_2} h(m))^{\alpha+1}\int_{\beta_1}[\varphi^{\alpha+1}(\phi^{-1}(m)) - g^{\alpha+1}(G^{-1}(m))]dm \\
&\quad + (\sup_{\beta_2})^{\alpha+1}\int_{\beta_2}[\varphi^{\alpha+1}(\phi^{-1}(m)) - g^{\alpha+1}(G^{-1}(m))]dm \\
&= (\sup_{\beta_2} h(m))^{\alpha+1}\int_0^1[\varphi^{\alpha+1}(\phi^{-1}(m)) - g^{\alpha+1}(G^{-1}(m))]dm \\
&\geq 0,
\end{aligned}$$

where 2nd inequality is captured by using the condition $\inf_{\beta_1} h(m) \geq \sup_{\beta_2} h(m)$. The third inequality is obtained by using (8). Thus, $T_{i:n} \overset{GE}{\leq} K_{i:n}$.

A. Characterization Results

We present some characterization results using Stone-Weierstrass Theorem.

Lemma 1. If $\int_0^1 t^n \eta(t)dt = 0$ for $n \geq 0$, where η:continuous function on $[0,1]$ then $\eta(t) = 0\ \forall\ t \in [0,1]$.

Theorem 6. Suppose T and K are two non-negative r.v's possessing an absolutely continuous distribution function $\phi(t)$ and $G(k)$ and pdf's $\varphi(t)$ and $g(k)$, respectively, whose supports L_1 and L_2 have a common lower bound b, then for a fixed i $(1 \leq i \leq n)$,

$$T \overset{D}{=} K \Leftrightarrow \Omega_\alpha(T_{i:n}) = \Omega_\alpha(K_{i:n}), \forall n \geq m.$$

Proof. The necessity is trivial, hence we have to prove only sufficiency part. Assume $\Omega_\alpha(T_{i:n}) = \Omega_\alpha(K_{i:n}), \forall n \geq m$.

$$\int_{L_1} \frac{n!}{(i-1)!(n-i)!}\phi^{i-1}(t)\bar{\phi}^{n-i}(t)\varphi(t)\Big\{1 - \Big(\frac{n!}{(i-1)!(n-i)!}\Big)^\alpha$$

$$\times \phi^{\alpha(i-1)}(t)\bar{\phi}^{\alpha(n-i)}(t)\varphi^\alpha(t)\Big\}dt = \int_{L_2}\frac{n!}{(i-1)!(n-i)!}G^{i-1}(k)$$

$$\times \bar{G}^{n-i}(k)g(k)\Big\{1 - \Big(\frac{n!}{(i-1)!(n-i)!}\Big)^\alpha G^{\alpha(i-1)}(k)\bar{G}^{\alpha(n-i)}(k)g^\alpha(k)\Big\}dk$$

Using $m = \bar{\phi}(t)$ and $m = \bar{G}(k)$ and taking $n - i = r$, it follows that

$$\int_0^1 (1-m)^{\alpha(i-1)+i-1}\big[\varphi^\alpha(\phi^{-1}(1-m)) - g^\alpha(G^{-1}(1-m))\big]m^{r(\alpha+1)}dm = 0,$$

for all $r \geq 0$. By Lemma 1, we can conclude that

$$\varphi^\alpha(\phi^{-1}(1-m)) = g^\alpha(G^{-1}(1-m)), \forall m \in (0,1).$$

By taking $1 - m = u$, we have

$$\varphi^\alpha(\phi^{-1}(u)) = g^\alpha(G^{-1}(u)),\ \forall u \in (0,1).$$

Thus $(\phi^{-1})'(u) = (G^{-1})'(u) \forall u \in (0,1)$.
Hence $\phi^{-1}(u) = G^{-1}(u) + q \forall u \in (0,1)$, where q is constant.
Also, $\lim_{u\to 0}\phi^{-1}(u) = \lim_{u \to 0} G^{-1}(u) = b$, we have $\phi^{-1}(u) = G^{-1}(u)$ for all $u \in (0,1)$. Hence the result.

Remark 1. Under the assumptions of Theorem 6, for i = 1, we have

$$Y \overset{D}{=} Z \Leftrightarrow H_\alpha(Y_{1:n}) = H_\alpha(Z_{1:n}), \forall n \geq 1.$$

III. Generalized residual entropy of order statistics

We propose residual information energy for the i^{th} order statistic $T_{i:n}$ as

$\Omega_\alpha(\varphi_{i:n},t)$

$$= \frac{1}{\alpha}\int_t^\infty \frac{\varphi_{i:n}(t)}{\bar{\phi}_{i:n}(t)}\Big[1-\frac{\varphi^\alpha_{i:n}}{\bar{\phi}^\alpha_{i:n}}\Big]dt \quad (9)$$

$$= \frac{1}{\alpha}\Big\{1-[\bar{B}(i,n-i+1)]^{\alpha+1}\int_0^1 m^{(i-1)(\alpha+1)} (1-m)^{(n-i)(\alpha+1)}\varphi^\alpha(\phi^{-1}(m))dm\Big\} \quad (10)$$

Now we calculate 1^{st} and n^{th} order statistic of generalized residual entropy.

Distribution	P.d.f	$\Omega_\alpha(T_{1:n})$	$\Omega_\alpha(T_{n:n})$
Exponential	$\theta\exp^{-\theta t}$	$\frac{1}{\alpha}\Big\{1-\frac{e^{-\theta tn(\alpha+1)}\theta^\alpha}{n^{\alpha+1}n(\alpha+1)}\Big\}$	$\frac{1}{\alpha}\Big\{1-\Big(\frac{1}{n}\frac{[1-e^{-\theta t}]^n}{n}\Big)^{\alpha+1} B(\alpha n+n-\alpha,\alpha+1)\Big\}$
Pareto	$\frac{\theta\beta^\theta}{t^{\theta+1}}, x\geq\beta>0,\theta>0$	$\frac{1}{\alpha}\Big\{1-\frac{\beta^{\theta n(\alpha+1)}}{nt^{\theta n(\alpha+1)}}\frac{\theta^\alpha}{\beta^\alpha(\alpha n+n+\alpha/\theta)}\Big\}$	$\frac{1}{\alpha}\Big\{1-\Big(\frac{\bar{\phi}^n(t)}{n}\Big)^{\alpha+1}\frac{\theta^\alpha}{\beta^\alpha}\beta(\alpha n+n-\alpha, \alpha(1+1/\theta+1))\Big\}$
Finite Range	$\frac{a}{b}\Big(1-\frac{t}{b}\Big)^{a-1}$, $a>1,b>1,0\leq x\leq 1$	$\frac{1}{\alpha}\Big\{1-\Big(\frac{a^\alpha}{b^\alpha\alpha(n-1+1/a)+n}\Big)\Big(\frac{\bar{\phi}^n(t)}{n}\Big)^{\alpha+1}\Big\}$	$\frac{1}{\alpha}\Big\{1-\frac{a^\alpha}{nb^\alpha}[1-\bar{\phi}^n]^{\alpha+1}\beta(\alpha n+n-\alpha, \alpha/a+1)\Big\}$

A. Characterization Result

Next, we have mentioned two results related to existence and uniqueness solutions of initial value problem (IVP) encountered in the study of certain differential equation. These results will help us to characterize the proposed measure in terms of distribution function.

Theorem 8. Suppose that f satisfy Lipschitz condition in D where $D\subset R^2$ is domain set and f be a cont. function. Then $y=\psi(t)$ is having the IVP $y'=\varphi(x,y)$, $\psi(t_0)=y_0, t\in I$, is unique.

Lemma 2. Assume f to be continuous in $D\subset R^2$, $\frac{\partial f}{\partial y}$ exists and is continuous in D, where D is convex region. Then φ satisfies the Lipschitz condition in D.

Theorem 9. Suppose T be a continuous r.v. having distribution function $\phi(.)$ and $\Omega_\alpha(\varphi_{i:n};t)<\infty,\ \ \forall t\geq 0$, be the generalized residual entropy function. Then, $\Omega_\alpha(\varphi_{i:n};t)$ characterizes the distribution function.

Proof. Let us assume two functions $\phi_{i:n}$ and $G_{i:n}$ such that

$$\Omega_\alpha(f_{i:n};t)=\Omega_\alpha(g_{i:n};t),\ \ \forall t\geq 0.$$

We know that

$$\Omega_\alpha(\varphi_{i:n};t) = \frac{1}{\alpha}\int_t^\infty\frac{\varphi_{i:n}(x)}{\bar{F}_{i:n}(t)}\Big[1-\frac{\varphi^\alpha_{i:n}(x)}{\bar{\phi}^\alpha_{i:n}(t)}\Big]dx.$$

Differentiating w.r.t. t, we have

$$\Omega'_\alpha(\varphi_{i:n},t) = -(\alpha+1)\beta(i,n-i+1)\lambda_{\phi_{i:n}}(t)\Big[\frac{1}{\alpha} -\Omega_\alpha(\varphi_{i:n},t)\Big]+\frac{\lambda^{\alpha+1}_{\phi_{i:n}(t)}}{\alpha}.$$

Differentiating w.r.t. t again, we have

$$\lambda'_{\phi_{i:n}}(t) = \frac{\alpha(\Omega''_\alpha(\varphi_{i:n},t)+\lambda_{\phi_{i:n}}(t)\Omega'_\alpha(\varphi_{i:n},t))}{(\alpha+1)[\lambda^\alpha_{\phi_{i:n}}(t)-\beta(i,n-i+1)(1-\alpha\Omega_\alpha(\varphi_{i:n},t))]}.$$

Suppose that

$$\Omega_\alpha(\varphi_{i:n},t)=\Omega_\alpha(g_{i:n},t)=v(t),\ \ 1\leq i\leq n,\ t\geq 0,$$

where $\phi_{i:n}$ and $G_{i:n}$ are two distributions of T. Then, $\forall t\geq 0$, $\lambda'_{\phi_{i:n}}(t)=\psi(t,\lambda_{\phi_{i:n}}(t))$, $\lambda'_{G_{i:n}}(t)=\psi(t,\lambda_{G_{i:n}}(t))$ where

$$\psi(t,y) = \frac{\alpha(v''(t)+yv'(t))}{(\alpha+1)[y^\alpha-\beta(i,n-i+1)(1-\alpha v(t))]}.$$

Using above theorem and lemma, we get the desired result.

$\lambda_{\phi_{i:n}}(t)=\lambda_{G_{i:n}}(t)$. Hence the result is proved.

Theorem 10. Suppose T be a non-negative r.v. with d.f. $\phi(.)$ and $\Omega_\alpha(\varphi_{i:n},\varphi;t)<\infty, t\geq 0$ denotes the dynamic residual generalized inaccuracy of the first order statistics based on a random sample of size n. Also, $\lambda_\phi(t)$ be hazard rate function of X and

$$\Omega_\alpha(\varphi_{1:n},\varphi;t) = \frac{1}{\alpha}\Big[1-\frac{n^{\alpha+1}c}{\alpha+1}\lambda^\alpha_\phi(t)\Big],\quad \alpha>0,$$

where c is a constant. So, T has

(i) an exp dist iff $c=1$,

(ii) a Pareto dist iff $c>1$,

(iii) a finite range dist iff $c<1$.

Proof. Suppose

$$\Omega_\alpha(\varphi_{1:n},\varphi;t) = \frac{1}{\alpha}\Big[1-\frac{n^{\alpha+1}c}{\alpha+1}\lambda^\alpha_\phi(t)\Big],\quad \alpha>0,$$

Differentiating w.r.t. 't', we have

$$\frac{d}{dt}[\Omega_\alpha(\varphi_{1:n},\varphi;t)] = -(\alpha+1)n^2\lambda_\phi(t)\Big[\frac{1}{\alpha}-\Omega_\alpha(\varphi_{1:n},t)\Big] +\frac{n^{\alpha+1}}{\alpha}\lambda^{\alpha+1}_\phi(t), \quad (11)$$

where $\lambda_\phi(t)$ and $\lambda_{\phi_{1:n}}(t)$ be the hazard rates of T and $T_{1:n}$. Since $\lambda_{\phi_{1:n}}(t)=n\lambda_\phi(t)$. Using $I_\alpha(\varphi_{1:n},\varphi;t)=\frac{1}{\alpha}\Big[1-\frac{n}{c}\lambda^\alpha_\phi(t)\Big]$ and substituting $\lambda_{\phi_{1:n}}(t)$, (12) becomes

$$\lambda'_F(t) = \frac{n^2}{\alpha}\frac{c-1}{c}(\alpha+1)\lambda^2_\phi(t).$$

It's solution is given by

$$\lambda_{\varphi}(t) = \frac{1}{kt+b}, \tag{12}$$

where $k = \frac{n^2}{\alpha}\frac{c-1}{c}(\alpha+1)$ and $b = \lambda_{\phi}(0)$.
(i) For $c = 1 \implies k = 0$ and using (12), $\lambda_{\varphi}(t)$ turned to be HR of exp dist.
(ii)For $c > 1 \implies k > 0$ and using (12), $\lambda_{\varphi}(t)$ turned to be HR of Pareto dist.
(iii) For $c < 1 \implies k < 0$ and using (12), $\lambda_{\varphi}(t)$ turned to be HR of finite range dist.
Using characterization of exponential, Pareto and finite range distributions in terms of hazard rate function, the only if part of above theorem follows.

IV. Conclusion

The one-parametric generalized entropy measures plays an important role as a measure of uncertainty and complexity in diverse fields viz. electronics, physics and engineering to illustrate several disorganized systems. We considered the generalized entropy measure of order statistics and generalized residual entropies in terms of order statistics utilizing probability integral transformation. Characterization results are also done for the proposed measures. Finally we examined few properties of the proposed measures for exp dist.

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