##### Banach Fixed Contraction Mapping Theorem in Vector $S$-metric Spaces

Pooja Yadav [[1]](#footnote-1), Mamta Kamra[[2]](#footnote-2)

[1,2] Department of Mathematics, Indira Gandhi University, Meerpur, Rewari, Haryana-122502, India

 **Abstract**

We demonstrate the Banach contraction mapping theorem on vector $S$-metric space. We also give an example to explain our results.

**Keywords:** Vector metric space, Vector lattice, Vector $S$-metric space.

## 1 Introduction

 Banach Contraction Principle(BCP) was demonsted firstly by S. Banach [2] in 1922. It has a vital role in fixed point(FP) theory and became very famous due to iterations. Many researchers are establishing new results in various generalizations of metric spaces. $S$-metric space is one of the generalizations in metric spaces. In 2012, $S$-metric space was defined by Sedghi et al.[7]. We start with some definitions and results for vector $S$-metric spaces(VSMS).

**Definition 1.**[4] On a set $∁$, a relation $⪯$ is a partial order if it follows the conditions stated below:

(1) $Δ\_{1}⪯Δ\_{1}$ (reflexive)

(2) $Δ\_{1}⪯Δ\_{2}$ and $Δ\_{2}⪯Δ\_{1}$ implies $Δ\_{1}=Δ\_{2}$ $ (anti-symmetry)$

(3) $Δ\_{1}⪯Δ\_{2}$ and $Δ\_{2}⪯Δ\_{3}$ implies $Δ\_{1}⪯Δ\_{3}$ $ (transitivity)$

 $∀$ $Δ\_{1},Δ\_{2},Δ\_{3}\in ∁$ .

The set $∁$ with partial order $⪯$ is known as partially ordered set (poset).

A partially ordered set $(∁,⪯)$ is called linearly ordered if for $Δ\_{1},Δ\_{2}\in ∁$, we have either $Δ\_{1}⪯Δ\_{2}$ or $Δ\_{2}⪯Δ\_{1}$.

**Definition 2.**[4] Let $∁$ be linear space which is real and $(∁,⪯)$ be a poset . Then the poset $(∁,⪯)$ is said to be an ordered linear space if it follows the properties mentioned below:

(1) $℘\_{1}⪯℘\_{2}⟹℘\_{1}+℘\_{3}⪯℘\_{2}+℘\_{3}$

(2) $℘\_{1}⪯℘\_{2}⟹ω℘\_{1}⪯ω℘\_{2}$

$∀℘\_{1},℘\_{2},℘\_{3}\in ∁$ and $ω>0$.

**Definition 3.**[4] A poset is called lattice if each set with two elements has an infimum and a supremum.

**Definition 4.**[4] An ordered linear space where the ordering is lattice is called vector lattice(VL).

**Definition 5.**[4] A VL $K$ is called Archimedean if $inf\{\frac{1}{m}Ω\}=0$ for every $Ω\in K^{+}$ where

 $K^{+}=\{Ω\in K:Ω⪰0\}.$

**Definition 6.**[3] Let $K$ be VL and $R$ be a nonvoid set. A function $d:R×R\rightarrow K$ is called vector metric on $R$ if it follows the conditions stated below:

(1)$d(Ω\_{1},Ω\_{2})=0$ iff $Ω\_{1}=Ω\_{2}$

(2)$d(Ω\_{1},Ω\_{2})⪯d(Ω\_{1},Ω\_{3})+d(Ω\_{3},Ω\_{2})$ $∀Ω\_{1},Ω\_{2},Ω\_{3}\in R$

The triple $(R,d,K)$ is called vector metric space.

Now, vector valued $S$-metric space is defined as follows:

**Definition 8.** [10} Let $K$ be VL and $R$ be a nonvoid set. A function $S:R×R×R\rightarrow K$ is called vector $S$-metric on $R$ that satisfies the conditions mentioned below:

(1) $S(℘\_{1},℘\_{2},℘\_{3})⪰0$,

(2) $S(℘\_{1},℘\_{2},℘\_{3})=0$ iff $℘\_{1}=℘\_{2}=℘\_{3}$,

(3) $S(℘\_{1},℘\_{2},℘\_{3})⪯S(℘\_{1},℘\_{2},α)+S(℘\_{2},℘\_{2},α)$+$ S(℘\_{3},℘\_{3},α), $

 $ $

for all $℘\_{1},℘\_{2},℘\_{3},α\in R$.

The triplet $(R,S,K)$ is called vector $S$-metric space(VSMS).

**Example 1**  *Let* $R$ *be a nonvoid set and* $K$ *be a VL. A function* $S:R×R×R\rightarrow K$ *is defined by*

 $S(℘\_{1},℘\_{2},℘\_{3})=|(℘\_{1},℘\_{3})|+|(℘\_{2},℘\_{3})| ∀℘\_{1},℘\_{2},℘\_{3}\in R$

then the triplet $(R,S,K)$ is VSMS.

**Definition 9.** A sequence $〈ℏ\_{n}〉$ in VSMS $(R,S,K)$ is called $K$-convergent to some $ℏ\in K$ if there is a sequence $〈μ\_{n}〉$ in $K$ satisfying $μ\_{n}\downright 0$ and $S(ℏ\_{n},ℏ\_{n},ℏ)\leq μ\_{n}$ and denote it by $μ\_{n}→ℏ$.

**Definition 10.** A sequence $〈ℏ\_{n}〉$ in VSMS $(R,S,K)$ is known as $K$-Cauchy sequence if there is a sequence $〈μ\_{n}〉$ in $K$ satisfying $μ\_{n}\downright 0$ and $S(ℏ\_{n},ℏ\_{n},ℏ\_{n+q})\leq μ\_{n}$ holds for all $q$ and $n$.

**Definition 11.** If each $K$-Cauchy sequence in $R$ is $K$-converges to a limit in $R$ then VSMS $(R,S,K)$ is called $K$-complete .

**Lemma**[8] For VSMS $(R,S,K)$,

$$ S(ℏ,ℏ,μ)=S(μ,μ,ℏ) ∀μ,ℏ\in R.$$

## 2 Main Results

**Theorem 1** *Let* $(R,S,K)$ *be a VSMS which is K-complete and* $K$ *be Archimedean. Suppose the transformation* $f:R\rightarrow R$ *satisfies*

 $S(fΩ,fΩ,fℏ)⪯qS(Ω,Ω,℘) ∀Ω,℘\in R$

where $q\in [0,1)$. Then $f$ has FP in $R$ which is unique and for any $℘\_{0}\in R$, iterative sequence $〈℘\_{m}〉$ defined by $℘\_{m}=f℘\_{m-1}$, for all $m\in N$, $K$-converges to FP of $f$.

**Proof** Let $℘\_{0}\in R$ and $〈℘\_{m}〉$ defined by $℘\_{m}=f℘\_{m-1}$ for $m\in N$.Then we have

 $S(℘\_{m},℘\_{m},℘\_{m+1})=S(f℘\_{m-1},f℘\_{m-1},f℘\_{m})⪯qS(℘\_{m-1},℘\_{m-1},℘\_{m})⪯$

 $…⪯q^{m}S(℘\_{0},℘\_{0},℘\_{1})$

 Thus for $m,p\in N$

 $S(℘\_{m},℘\_{m},℘\_{m+p})⪯2S(℘\_{m},℘\_{m},℘\_{m+1})+2S(℘\_{m+1},℘\_{m+1},℘\_{m+2})+$

 $ …+S(℘\_{m+p-1},℘\_{m+p-1},℘\_{m+p})$

 $ ⪯2S(℘\_{m},℘\_{m},℘\_{m+1}) 2S(℘\_{m+1},℘\_{m+1},℘\_{m+2})+$

 $ …+ 2S(℘\_{m+p-1},℘\_{m+p-1},℘\_{m+p})$

 $⪯2(q^{m}+q^{m+1}+…+q^{m+p-1}) S(℘\_{0},℘\_{0},℘\_{1})$

 $⪯2q^{m+p-1}(1+q+q^{2}+…) S(℘\_{0},℘\_{0},℘\_{1})$

 $⪯2\frac{q^{m+p-1}}{1-q}S(℘\_{0},℘\_{0},℘\_{1}).$

Thus $〈℘\_{m}〉$ is a $K$-Cauchy sequence because $K$ be Archimedean. Then by $K$-completeness of $R$, there exist $℘\in R$ such that $℘\_{m}→℘$. So there exist $〈b\_{m}〉$ in $K$ such that $b\_{m}\downright 0$ and $S(℘\_{m},℘\_{m},℘)⪯b\_{m}$. Since

 $S(f℘,f℘,℘)⪯2S(f℘\_{m},f℘\_{m},f℘)+ S(f℘\_{m},f℘\_{m},℘)$

 $⪯2qS(℘\_{m},℘\_{m},℘)+S(℘\_{m+1},℘\_{m+1},℘)$

 $ ⪯2qb\_{m}+b\_{m+1}$

 $⪯2(q+1)b\_{m},$

 then $S(f℘,f℘,℘)=0$, i.e. $f℘=℘$.

We can also verify the following theorem as above.

**Theorem 2** *Let* $(R,S,K)$ *be a VSMS which is complete and* $K$ *be Archimedean. Suppose the transformation* $f:R\rightarrow R$ *satisfies*

 $S(fΩ,fΩ,f℘)⪯\{a\_{1}S(Ω,Ω,fΩ)+a\_{2}S(℘,℘,f℘) +a\_{3}S(Ω,Ω,f℘)+ a\_{4}S(℘,℘,fΩ) +a\_{5}S(Ω,Ω,℘)\} $

 for all $Ω,℘\in R$, where $a\_{1},a\_{2},a\_{3},a\_{4}$ and $a\_{5}$ are positive and $a\_{1}+a\_{2}+a\_{3}+a\_{4}+a\_{5}<1$. Then $f$ has FP in $R$ and for any $℘\_{0}\in R$, iterative sequence $〈℘\_{m}〉$ defined by $y\_{m}=f℘\_{m-1}$, $m\in N$, $K$-converges to FP of $f$.

**Example 2** *Let* $K=R\_{+}^{2}$ *with coordinatewise ordering and let*

 $R=\{(0,℘)\in R^{2}:0⪯℘⪯1\}∪ (℘,0)\in R^{2}:0⪯℘⪯1\}.$

The mapping $S:R×R×R\rightarrow K$ is defined by

 $S((Ω,0),(Ω,0),(℘,0))=(\frac{4}{3}|Ω-℘|,|Ω-℘|)$

 $S((0,Ω),(0,Ω),(0,℘))=(|Ω-℘|,\frac{2}{3}|Ω-℘|)$

 $S((Ω,0),(Ω,0),(0,℘))=(\frac{4}{3}Ω+℘,Ω+\frac{2}{3}℘)$

 Then $R$ is VSMS which is complete.

**References**

[1] Aliprantis C. D., Border K. C., *Infinite Dimensional Analysis*, Verlag, Berlin, 1999.

[2] Banach S., Sur les operations dans les ensembles abstraits el leur application aux equations integrals, Fund. Math., 3 (1992), 133–181.

[3] Cevik C., Altun I., *Vector metric spaces and some properties*, Topal. Met. Nonlin. Anal; 34(2), 375-382, 2009.

[4] Kamra M., Kumar S., Sarita K., *Some fixed point theorems for self mappings on vector b-metric spaces*, Global Journal of Pure and Applied Mathematics 14(11), 1489-1507, 2018.

[5] Kim J. K., Sedghi S., Gholidahneh A., Razaee M. M., *Fixed points theorems in S-metric spaces*, East Asian Math. J. 32(5), 677 - 684, 2016 .

[6] Prudhvi K., *Fixed point theorems in S- metric spaces*, Universal Journal of Computational Mathematics 3(2), 19-21, 2015.

[7] Sedghi S., Shobe N., Aliouche A., *A generalization of fixed point theorem in S-metric spaces*, Mat. Vesnik 64, 258-266, 2012.

[8] Shahraki M., Sedghi S., Aleomraninejad S. M. A., Mitrovic Z. D., *Some fixed point results on S-metric spaces*, Acta Univ. Sapientiae, Mathematica, 12(2), 347-357, 2020.

[9] Yadav P, Kamra M, Rajpal. Common fixed point theorems for self mappings on a complete vector S-metric space. Spec Ugdym. 2022;43(1):6439-6450.

[10] Yao J., Yang L., *Common fixed point results in S-metric spaces*, Journal of Advances in Applied Mathematics,4(2), 2019.

1. poojayadav.math.rs@igu.ac.in [↑](#footnote-ref-1)
2. mkhaneja15@gmail.com [↑](#footnote-ref-2)