η-Ricci Solitons on Sasakian Manifolds Admitting General Connection

By

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Abstract

In this paper paper we study η-Ricci soliton on Sasakian manifold admitting general connection and obtain some conditions for the soliton to be shrinking, steady and expanding.

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1 Introduction

Hamilton [8] in 1982, introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Ricci flow has becomes an important tool for the study of Riemannian manifold. The Ricci flow is an evolution equation for metrices on a Riemannian manifold defined as

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solution of the Ricci flow. A solution of the Ricci flow is called Ricci Soliton. It is a natural generalization of Einstein metric such that,

$$(L_V g)(X, Y) + 2S(X, Y) + 2\alpha g(X, Y) = 0. \tag{1.1}$$

Where $L_V g$ denotes the lie derivative of the Riemannian metric g along the vector field V, α is a constant, S is Ricci tensor and X and Y are arbitrary vector fields on $\chi(M)$. A Ricci soliton is known as *shrinking*, *steady* or *expanding* according as α is negative, zero or positive respectively. A Ricci soliton with $V = 0$, is reduced to Einstein equation. In the last two decades, many geometers explore the geometry of Ricci solitons in the different settings of manifolds. Ricci soliton has been studied in contact geometry by many authors such as Sharma [16], Tripathi [17], Bagewadi et al. [2, 1, 11], Bejan and Crasmareanu [3], Chandra et al. [6], and many others. It becomes more popular when Gregory Perelman applied Ricci solitons to solve the long standing Poincare conjecture which was posed in 1904 .

On the other hand, η-Ricci soliton is a generalization of Ricci soliton and was introduced by Cho and Kimura [7] in 2009. An η-Ricci soliton is a tuple (g, V, α, β) satisfying

$$(L_V g)(X, Y) + 2S(X, Y) + 2\alpha g(X, Y) + 2\beta\eta(X)\eta(Y) = 0. \tag{1.2}$$

Where β is a constant and other notations are same as for Ricci soliton. In particular when $\beta = 0$, η-Ricci soliton becomes Ricci soliton. Blaga obtained several results concerning η-Ricci soliton on para-Kenmotsu manifold [5] and on Lorentzian para-Sasakian manifolds [4]. Further, Sardar and De [15], Pahan [13], Haseeb and Prasad [9], Hui and Chakraborty [10] and other authors also explores η-Ricci soliton on different structure.

Recently, Biswas and Baishya [18] introduced a new connection, called general connection in the setting of Sasakian geometry as

$$\bar{\nabla}_X Y = \nabla_X Y + \lambda[(\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi] + \mu\eta(X)\phi Y \tag{1.3}$$

for all $X, Y \in \chi(M)$. Where λ and μ are real constants.

***Remark* 1.** *The beauty of general connection $\bar{\nabla}$ lies in the fact that it reduces to:*

1. *Quater symmetric metric connection for (λ, μ)=(0,-1).*
2. *Schouten-van Kampen connection for (λ, μ)=(1,0).*
3. *Tanaka-Webster connection for (λ, μ)=(1,-1).*
4. *Zamkovoy connection for (λ, μ)=(1,1).*

The present paper organized as follow. After introduction, we revisit Sasakian manifold and collect some known results of Sasakian manifold admitting general connection in second section. In third section we study η-Ricci soliton on Sasakian manifold admitting general connection and obtained some results.

2 Preliminaries

An almost contact structure on a smooth manifold M of dimension $n(= 2m + 1)$ is a triplet (ϕ, ξ, η), where ϕ is a (1,1)-tensor field, ξ is a vector field, and η is a 1-form on M satisfying

$$\phi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1. \tag{2.1}$$

Equation (2.1) implies that

$$\phi(\xi) = 0, \qquad \eta(\phi X) = 0, \qquad rank(\phi) = 2n. \tag{2.2}$$

A smooth manifold M endowed with an almost contact structure is called an almost contact manifold. A Riemannian metric g on M is said to be compatible with an almost contact structure (ϕ, ξ, η), if

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie algebra of all vector fields on M. An almost contact manifold endowed with a compatible Riemannian metric is said to be an almost contact metric manifold and is denoted by $M(\phi, \xi, \eta, g)$. The fundamental 2-form Φ on $M(\phi, \xi, \eta, g)$ is defined by $\Phi(X, Y) = g(X, \phi Y) = -\Phi(Y, X)$ for all $X, Y \in \chi(M)$. An almost contact metric manifold is said to be Sasakian manifold if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.4}$$

where ∇ denotes covariant differentiation with respect to the Riemannian connection of g. From the above equation, we deduce that for a Sasakian structure

$$\nabla_X \xi = -\phi X. \tag{2.5}$$

Further, on a Sasakian manifold the following relation holds:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.6}$$

$$R(\xi, X)Y = -R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.7}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{2.8}$$

$$Q\xi = (n-1)\xi, \tag{2.9}$$

$$(\nabla_X \eta)Y = g(X, \phi Y), \tag{2.10}$$

for all vector fields X,Y and Z on M. Where R, S and Q are Riemann Curvature tensor, Ricci tensor and Ricci operator respectively. We now recall some definitions for later use.

***Definition* 2.1.** *A Sasakian manifold is said to be Quasi-conformal like flat with respect to general connection if*

$$\bar{\omega}(X, Y)Z = 0.$$

Where $\bar{\omega}$ is Quasi-conformal like curvature tensor with respect to general connection, and given by

$$\begin{aligned}\bar{\omega}(X, Y)Z &= \bar{R}(X, Y)Z + a[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y] \\ &\quad -\frac{c\bar{r}}{n}\left(\frac{1}{n-1} + a + b\right)[g(Y, Z)X - g(X, Z)Y] \\ &\quad +b[g(Y, Z)\bar{Q}X - g(X, Z)\bar{Q}Y]\end{aligned} \tag{2.11}$$

for all $X, Y, Z \in \chi(M)$.

***Remark* 2.** *The Quasi-conformal like curvature tensor has the following flavours:*

1. *Riemann curvature tensor $\bar{R}$ for $(a, b, c) = (0, 0, 0)$.*
2. *Conformal curvature tensor $\bar{C}$ for $(a, b, c) = (\frac{-1}{n-2}, \frac{-1}{n-2}, 1)$.*
3. *Conharmonic curvature tensor $\bar{L}$ for $(a, b, c) = (\frac{-1}{n-2}, \frac{-1}{n-2}, 0)$.*
4. *Concircular curvature tensor $\bar{E}$ for $(a, b, c) = (0, 0, 1)$.*
5. *Projective curvature tensor $\bar{P}$ for $(a, b, c) = (\frac{-1}{n-1}, 0, 0)$.*
6. *m-projective curvature tensor $\bar{H}$ for $(a, b, c) = (\frac{-1}{2n-2}, \frac{-1}{2n-2}, 0)$.*
7. *$\bar{W}_1$ curvature tensor for $(a, b, c) = (\frac{1}{n-1}, 0, 0)$.*
8. *$\bar{W}_2$ curvature tensor for $(a, b, c) = (0, \frac{-1}{n-1}, 0)$.*
9. *$\bar{W}_4$ curvature tensor for $(a, b, c) = (0, 0, \frac{n}{r})$.*

We now collect some results on Sasakian manifold admitting general connection [18]. Let M be a Sasakian manifold, then on M we have the following results with respect to general connection $\bar{\nabla}$.

$$\bar{\nabla}_X Y = \nabla_X Y + \lambda[g(X, \phi Y)\xi + \eta(Y)\phi X] + \mu\eta(X)\phi Y. \tag{2.12}$$

$$\bar{\nabla}_X \xi = (\lambda - 1)\phi X \qquad \bar{\nabla}_\xi X = (\mu - 1)\phi X \tag{2.13}$$

$$\bar{\nabla}_X \eta(Y) = \bar{\nabla}_X g(Y, \xi) = \eta(\nabla_X Y) + \lambda g(X, \phi Y) + (\lambda - 1)g(Y, \phi X) \tag{2.14}$$

$$\bar{\nabla}_X(\phi Y) = \nabla_X(\phi Y) - \lambda g(\phi X, \phi Y)\xi - \mu\eta(X)Y + \mu\eta(X)\eta(Y)\xi. \tag{2.15}$$

$$\begin{aligned}\bar{\nabla}_X g(Y, \phi Z) &= g(\nabla_X Y, \phi Z) + \mu\eta(X)g(\phi Y, \phi Z) + g(Y, \nabla_X(\phi Z)) \\ &\quad -\mu\eta(X)g(Y, Z) + \mu\eta(X)\eta(Y)\eta(Z).\end{aligned} \tag{2.16}$$

The Riemann curvature tensor $\bar{R}$, with respect to general connection $\bar{\nabla}$ is given by:

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z + (\lambda^2 - 2\lambda)[g(Z,\phi X)\phi Y + g(Y,\phi Z)\phi X] \\ &\quad -2\mu g(Y,\phi X)\phi Z + (\lambda - \lambda\mu + \mu)[g(X,Z)\eta(Y)\xi \\ &\quad -\eta(X)g(Y,Z)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \end{aligned} \tag{2.17}$$

The Ricci tensor $\bar{S}$, with respect to general connection $\bar{\nabla}$ is given by:

$$\begin{aligned}\bar{S}(X,Y) &= S(X,Y) - (\lambda^2 - \lambda - \mu - \lambda\mu)g(X,Y) \\ &\quad +(\lambda^2 + (n-2)\lambda\mu - n(\lambda+\mu))\eta(X)\eta(Y) \end{aligned} \tag{2.18}$$

The Ricci operator $\bar{Q}$, and scalar curvature $\bar{r}$ with respect to general connection $\bar{\nabla}$ is given by:

$$\bar{Q}X = QX - (\lambda^2 - \lambda - \mu - \lambda\mu)X + (\lambda^2 + (n-2)\lambda\mu - n(\lambda+\mu))\eta(X)\xi \tag{2.19}$$

$$\bar{r} = r - (\lambda^2 - \lambda - \mu - \lambda\mu)n + (\lambda^2 + (n-2)\lambda\mu - n(\lambda+\mu)) \tag{2.20}$$

3 η-Ricci soliton on Sasakian manifold admitting general connection

Let M be a Sasakian manifold admitting η-Ricci soliton with respect to general connection, then by (1.2), we have

$$(\bar{L}_\xi g)(X,Y) + 2\bar{S}(X,Y) + 2\alpha g(X,Y) + 2\beta\eta(X)\eta(Y) = 0. \tag{3.1}$$

Expressing the Lie derivative along ξ with respect to general connection we obtain

$$\begin{aligned}(\bar{L}_\xi g)(X,Y) &= \bar{L}_\xi g(X,Y) - g(\bar{L}_\xi X, Y) - g(X, \bar{L}_\xi Y) \\ &= \bar{L}_\xi g(X,Y) - g([\xi,X],Y) - g(X,[\xi,Y]) \\ &= \bar{\nabla}_\xi g(X,Y) - g(\nabla_\xi X - \nabla_X \xi, Y) - g(X, \nabla_\xi Y - \nabla_Y \xi) \end{aligned} \tag{3.2}$$

By (1.3), above equation becomes

$$(\bar{L}_\xi g)(X,Y) = 2g(\bar{\nabla}_\xi X, Y) + 2g(X, \bar{\nabla}_\xi Y) + g(\bar{\nabla}_X \xi, Y) + g(X, \bar{\nabla}_Y \xi, X) \tag{3.3}$$

which reduces to, by the help of (2.13), the following form

$$(\bar{L}_\xi g)(X,Y) = 0. \tag{3.4}$$

So (3.1) becomes

$$\bar{S}(X,Y) = -\alpha g(X,Y) - \beta\eta(X)\eta(Y) \tag{3.5}$$

Setting $X = Y = \xi$, and using (2.18) we get

$$\alpha + \beta = (n-1)(\lambda - \lambda\mu + \mu - 1). \tag{3.6}$$

Summarizing the above, we therefore state the following:

***Theorem* 3.1.** *Let (g,ξ,α,β) is an η-Ricci soliton on a Sasakian manifold M, then the η-Ricci soliton is-*

1. *Shrinking, steady or expanding according as* $2(1-n) < \beta$, $2(1-n) = \beta$ *or* $2(1-n) > \beta$ *respectively, for quater-symmetric metric connection.*
2. *Shrinking, steady or expanding according as* $\beta > 0$, $\beta = 0$ *or* $\beta < 0$ *respectively, for Schouten-van Kampen connection, Tanaka-Webster connection and Zamkovoy connection.*

Now, consider that the Sasakian manifold is Quasi-conformal like flat with respect to general connection, then by definition (2.1) we have

$$\begin{aligned} \bar{R}(X,Y)Z &= -a[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y] \\ &+ \frac{c\bar{r}}{n}\left(\frac{1}{n-1} + a + b\right)[g(Y,Z)X - g(X,Z)Y] \\ &- b[g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] \end{aligned} \tag{3.7}$$

Taking inner product with W, we obtain

$$\begin{aligned} \bar{R}(X,Y,Z,W) &= -a[\bar{S}(Y,Z)g(X,W) - \bar{S}(X,Z)g(Y,W)] \\ &+ \frac{c\bar{r}}{n}\left(\frac{1}{n-1} + a + b\right)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &- b[g(Y,Z)\bar{S}(X,W) - g(X,Z)\bar{S}(Y,W)] \end{aligned} \tag{3.8}$$

let $\{e_1, e_2, e_3, ...e_n\}$ be a orthonormal basis for M, putting $Y = Z = e_i$ in (3.8) and taking summation over i, we get

$$\bar{S}(X,W) = \left[\frac{c\bar{r}}{n}\left(\frac{1}{n-1} + a + b\right)\left(\frac{n-1}{1+bn-a-b}\right) - \left(\frac{a\bar{r}}{1+bn-a-b}\right)\right] g(X,W) \tag{3.9}$$

using (3.5) in (3.9), we get

$$\begin{aligned} -\alpha g(X,W) - \beta\eta(X)\eta(W) &= \left[\frac{c\bar{r}}{n}\left(\frac{1}{n-1} + a + b\right)\left(\frac{n-1}{1+bn-a-b}\right)\right] g(X,W) \\ &- \left[\left(\frac{a\bar{r}}{1+bn-a-b}\right)\right] g(X,W) \end{aligned} \tag{3.10}$$

Putting $X = W = \xi$, in (3.10), we get

$$-(\alpha+\beta) = \left[\frac{c\bar{r}}{n}\left(\frac{1}{n-1} + a + b\right)\left(\frac{n-1}{1+bn-a-b}\right) - \left(\frac{a\bar{r}}{1+bn-a-b}\right)\right] \tag{3.11}$$

Which reduces to, by the help of (2.20), in the following form

$$\begin{aligned} -(\alpha+\beta) &= \Big[\frac{c[r - (\lambda^2 - \lambda - \mu - \lambda\mu)n + (\lambda^2 + (n-2)\lambda\mu - n(\lambda+\mu))]}{n} \\ &\left(\frac{1}{n-1} + a + b\right)\left(\frac{n-1}{1+bn-a-b}\right)\Big] \\ &- \left[\frac{a[r - (\lambda^2 - \lambda - \mu - \lambda\mu)n + (\lambda^2 + (n-2)\lambda\mu - n(\lambda+\mu))]}{1+bn-a-b}\right] \end{aligned} \tag{3.12}$$

Therefore we can state the following theorems:

***Theorem* 3.2.** *If (g, ξ, λ, μ) is an η-Ricci soliton on the Quasi-conformal like flat Sasakian manifold with respect to general connection $\bar{\nabla}$, then α and β are related by the equation (3.12).*

***Theorem* 3.3.** *If (g, ξ, λ, μ) is an η-Ricci soliton on concircularly flat or projectively flat or m-projectively flat Sasakian manifold, then the η-Ricci soliton is:*

1. *shrinking, steady or expanding according as $\beta + \frac{r}{n}$ is positive, zero or negative, with respect to quater symmetric metric connection.*
2. *shrinking, steady or expanding according as $\beta + \frac{r+1-n}{n}$ is positive, zero or negative, with respect to Schouten-van Kampen connection.*
3. *shrinking, steady or expanding according as $\beta + \frac{r-3n+3}{n}$ is positive, zero or negative, with respect to Tanaka-Webster connection.*
4. *shrinking, steady or expanding according as $\beta + \frac{r+n-1}{n}$ is positive, zero or negative, with respect to Zamkovoy connection.*

Proof. For $(a, b, c) = (0, 0, 1)$, or $(\frac{-1}{n-1}, 0, 0)$, or $(\frac{-1}{2n-2}, \frac{-1}{2n-2}, 0)$, we get, from (3.12),

$$-(\alpha + \beta) = \frac{r - (\lambda^2 - \lambda - \mu - \lambda\mu)n + \lambda^2 + (n-2)\lambda\mu - n(\lambda + \mu)}{n}. \qquad (3.13)$$

Hence, the theorem follows by choosing suitable values of λ and μ. □

***Theorem* 3.4.** *If (g, ξ, λ, μ) is an η-Ricci soliton on a Sasakian manifold which is flat with respect to $\bar{W}_1$-curvature tensor, then the η-Ricci soliton is:*

1. *shrinking, steady or expanding according as $\frac{r}{n-2} - \beta$ is negative, zero or positive, with respect to quater symmetric metric connection.*
2. *shrinking, steady or expanding according as $\frac{r+1-n}{n-2} - \beta$ is negative, zero or positive, with respect to Schouten-van Kampen connection.*
3. *shrinking, steady or expanding according as $\frac{r-3n+3}{n-2} - \beta$ is negative, zero or positive, with respect to Tanaka-Webster connection.*
4. *shrinking, steady or expanding according as $\frac{r+n-1}{n-2} - \beta$ is negative, zero or positive, with respect to Zamkovoy connection.*

Proof. The proof is similar as of Theorem (3.3). □

***Theorem* 3.5.** *If (g, ξ, λ, μ) is an η-Ricci soliton on a Sasakian manifold which is flat with respect to $\bar{W}_4$-curvature tensor, then the η-Ricci soliton is:*

1. *shrinking, steady or expanding according as $\beta + r^2$ is positive, zero or negative, with respect to quater symmetric metric connection.*
2. *shrinking, steady or expanding according as $\beta + r^2 + r - nr$ is positive, zero or negative, with respect to Schouten-van Kampen connection.*
3. *shrinking, steady or expanding according as $\beta + r^2 - 3nr + 3r$ is positive, zero or negative, with respect to Tanaka-Webster connection.*
4. *shrinking, steady or expanding according as $\beta + r^2 + rn - r$ is positive, zero or negative, with respect to Zamkovoy connection.*

Proof. The proof is similar as of Theorem (3.3). □

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