**Hopf Bifurcation and Harvesting Analysis in Prey and Two Predators Model**

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**ABSTRACT**

In this chapter we studied the dynamics of three species model consisting of a single prey and two predators. The model in volved two predators preying on single prey. Two predators are neutral to each other and of generalist type. A delay is induced in the interaction of prey and second predator species and prey species is harvested. The model equations are described by system of delay differential equations. The co-existing state is identified, and local stability analysis is carried out at this point. The hopf bifurcation analysis is carried outat this point. The critical point is identified using numerical simulation. The harvesting effort of prey species is included in the study and identified the efforts which makes the system stable.

**Key wards**: prey, predator Time lags, Hopf Bifurcation, harvesting.

**AMS classification**: 34DXX

**I INTRODUCTION**

Ecological science gained a lot of importance in this era. The population dynamics of species is widely addressed by many authors. The stability analysis was the main aspect in ecological phenomenon. The modelling approach of ecological study was initiated byLokta [1] and Volterra [2]. The modelling of ecological communities isdispensed by using differential equations. the stability aspects using differential equations are dealt by Braun [9] and Simon’s [10]. The Wide range of model in ecological and epidemical is studied by Kapur [3, 4]. The qualitative analysis is the focus in ecological stability and dealt by authors [5-8]. The time delaysare quite common in biological and ecological phenomenon. The delays are of discrete, continuous are distributed type. The time lags approach in population dynamics was dealt by Cushing, J.M [11], Norman [12]. The time delays are stabilizes or destabilizes the system. The tendency of time delays is influential in ecological science. The approaches in delay differential equations are widely elaborated by the authors [13-15]. The approaches are not only in ecology but also in epidemiology. The time delays in epidemiology are also plays a significant role. Karuna [16] and Ranjith [17] studied the instability tendencies (Hopf bifurcation) in HIV and SIR epidemic models.The distributed time lags are also very important in ecological communities. These delays are significantly explaining the role of history of species dynamics. The role of kernel dynamics with different delay kernels in prey-predator and competitor populations are Widley studied by paparao e.tal [19-25].

The prey-predator models are always intersecting, three species models with one prey and two predators were discussed by shiva Reddy [18]. In his study he chooses the model with a prey, predator and super-predator. The dynamics was well established.The delaysare intersectingin prey-predator models. In this chapter we choose a model consisting of a single prey and two predators preying on a same prey. The two predators are having alternative food source. We infuse a discrete time lag (τ) in prey-predator model (logistic) in the interaction ofprey and second predator and also take harvesting of prey species. The stability analysis was carried out at normal study state and hopf bifurcation analysis is also addressed.Finally, the harvesting efforts are also identified in which the system stabilizes for certain harvesting efforts.

**II. FORMATION OF MATHEMATICAL MODE**L.

The mathematical for three species ecological species consists of a single prey and two predators preying on this prey species. Two predators are neutral to each other and have alternative food sources (Generalist predators). A logistic growth model is proposed for all three species. A discrete time lag is induced in the interaction of prey and second species. The harvesting effort on prey species is also taken investigation. The mathematical model is formulated with a system of delay differential equations given below.

(2.1)

With the following notations

x1(t) Prey density ,x2(t) First predator density ,x3(t) second predator density

 : Intrinsic growth rates three species

 : Inter species competition rate of three species

: Mutual interference strengths of three species

Li : Carrying capacities of three species q: Harsing effort, E: effort of harvesting

all the constants are assumed to be positive

**III. EXISTENCE OF EQUILIBRIUM:**

The system (2.1) admits a positive equilibrium point for the co-existence state if the following conditions hold good.

1. (ii)  (iii) 

The co-existing state given by

(3.1)

**IV. LOCAL STABILITY ANALYSIS:**

**Theorem 4.1:** The co-existing statecislocally asymptotically stable

**Proof:** The Jacobean matrix for the system of equations (2.1) is given by

(4.1)

With the characterstic equation is given by

 (4.2)

Where

We need to find the condition for existence of negative real roots

**Case (i) :**For equation (4.2) becomes

In simplest form the above equation can be represented as

Where

By using Routh-Hurwitz criteria, we calculate the following determinants



Clearly 



Hence the co-existing stateis locally asymptotically stable.

**Case (ii) for τ > 0:**

Suppose there is positive  such that the equation (3.1) has pair of purely imaginary root, can be taken as  then satisfies the equation (4.1)

Puttingwe have

On simplication we get

Where

If we assume that then have no positive real roots

Thus if then there is no  such that  is an Eigen value of the characteristic equation of

If  will never be a purely imaginary root of equation, thus the real parts of all Eigen values of

are negative for all summarizing, the above analysis, we have the following theorem

**Theorem 4.2**: Co-existing state is locally asymptotically stable for all ,if following condition hold.

Now if any one of is negative then (2.1) has positive root 

Eliminating  from equation (4.3)&(4.4) we get

(4.5)

**V. HOPFBIFURCATIONS**

**Theorem 5.1:** The sufficient condition for the system (2.1) admits bifurcation along the normal steady state E if

and locally asymptotically stable if

**Proof:** Hopf bifurcation occurs when the real part of become positive when and the steady state become unstable moreover, when passes through the critical value.

To check this result, differentiating (4.2) W .r t., we get

=

Put in the above equation

Real part of

=

On substituting

=

=

By using this conditionN1>0, N2>0, N3>0 we have

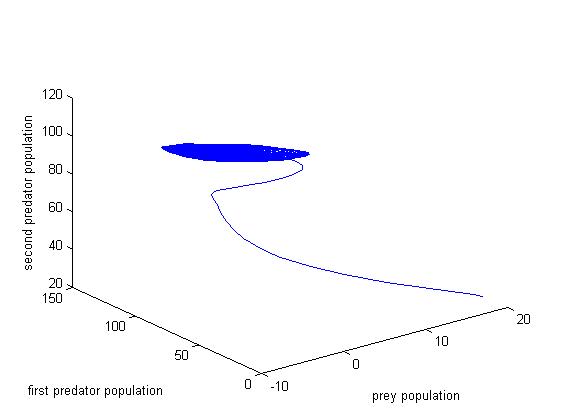
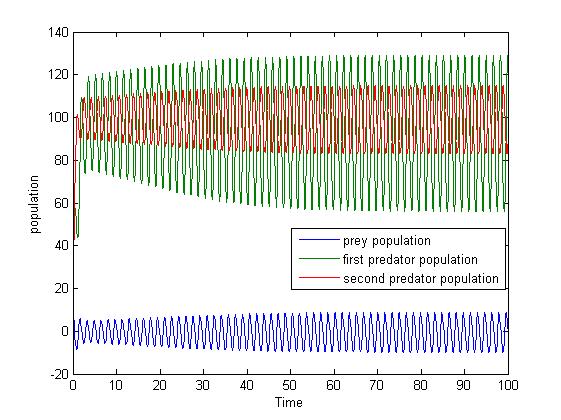
Therefore, the Hopf bifurcation occurs at

**VI. NUMERICAL EXAMPLE:**

Numerical simulation is carried out in support of bifurcation analysis along co-existing state for the system of equations (2.1). Four set of parametric values are chosen for investigation and identified the bifurcation parameter (τ0) shownbelow.

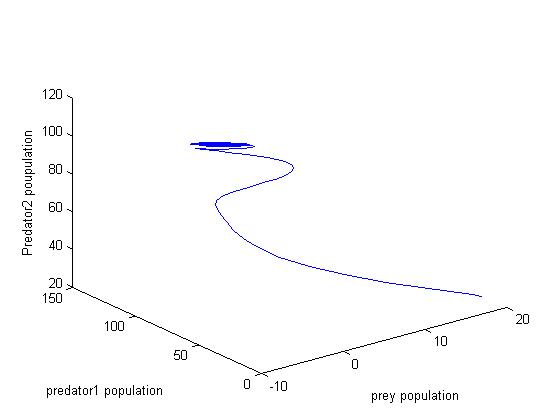
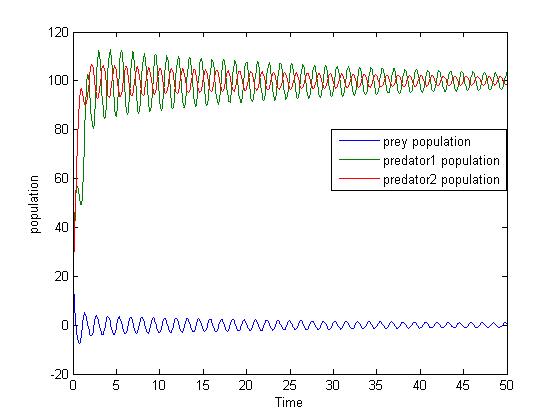
In each graph, figure (a) represents Time series responsesand (b) represents Phase portraits

**Example 6.1:** Let us parametric values



**Fig 6.1.1(A) Fig 6.1.1(B)**

The system possesses unbounded periodic solutions hence the system (2.1) becomes unstable for the critical parameter τ = 0.7.



**Fig 6.1.2(A) Fig 6.1.2(B)**

The system becomes stable for τ = 0.65 for the system of equations (2.1).

The critical parameter ‘τ ‘is identified using numerical simulation where the stable system becomes unstable when the critical parameter ‘τ ‘value greater than 0.65.

Now the harvesting parameter dynamics is also discussed with various catch ability coefficient (q) values and effort (E) is shown below.

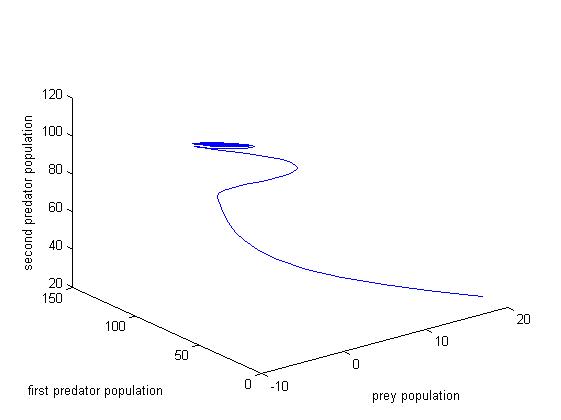
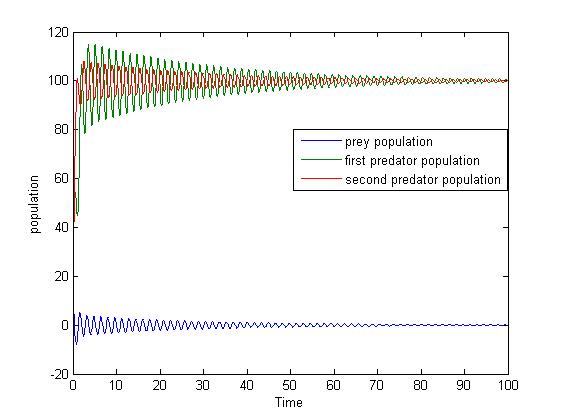
**Case (i)** :When harvesting effort was induced on prey species the dynamics of q = 1 and E= 5

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| --- | --- |
|  |  |

**Fig 6.1.3(A) Fig 6.1.3(B)**

The system admits unbounded periodic solutions with τ = 0.7 and q =1, E=5 makes the system (2.1) unstable.

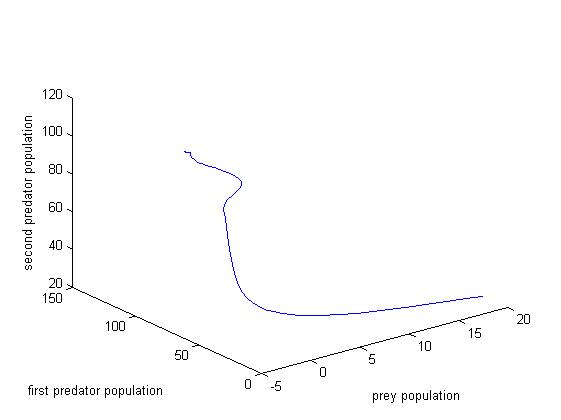
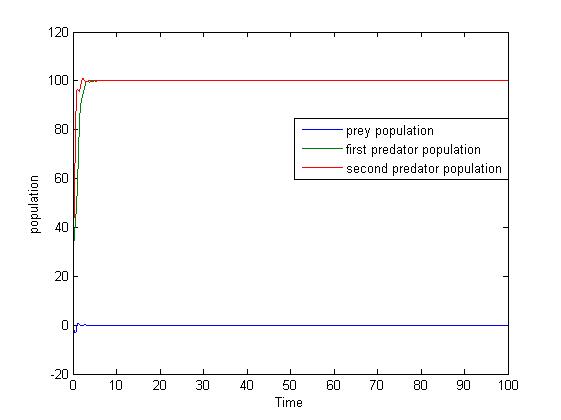
**Case (ii)** : When harvesting effort was induced on prey species the dynamics of q = 0.1 and E= 5



**Fig 6.1.4(A) Fig6.1.4(B)**

The system of equations (2.1) possesses bounded variations and makes the system asymptotically stable for τ = 0.7 and q =0.1, E=5.

**Case(iii) :**When harvesting effort was induced on prey species the dynamics of q = 1 and E= 10



**Fig 6.1.5(A) Fig 6.1.5(B)**

The system is stable for τ = 0.7 and q=1, E= 10.

**VII. DISCUSSION AND CONCLUSION**

The proposed model is asymptotically stable and exhibit Hopf bifurcation nature for certain critical valueτ = 0.65.The sufficient condition for Hopf bifurcation is derived and the critical value (τ0) is identified forone different set of parameters. The critical values for system (2.1) with four set examples are shown in the above graphs [6.1.1 -6.1.2]. The harvesting efforts are also identified in which the system becomes stable is identified using numerical simulation.

The system still admits unstable nature for τ = 0.7 and q=1,E= 5 and becomes asymptotically stable for τ = 0.7 and becomes stable for τ = 0.7 and q=1,E= 10.

The system undergoes hopf bifurcation along the co-existing state and harvesting efforts are significant in stability analysis. The harvesting efforts are identified in which the unstable system becomes stable. Hence harvesting efforts are also significant in stability analysis.

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