

# On Convergence of 2 – dimensional $q$ – analogues of Jafari’s Integral Transformation

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## Abstract:

In this paper, we have extended the newly defined  $q$  – analogues of the Jafari’s integral transformation towards its 2 – dimensional integral transformation. The  $q$  – analogues of the Jafari’s integral transformation has simple relationship with other two dimensional  $q$  – integral transformations. As an application we have found the conditions under which 2 – dimensional  $q$  – analogues of Jafari’s integral transformation were convergent.

**Keywords:** Integral Transform,  $q$  – Calculus, Convergence

**AMS (2010) Classification:** 35A22, 05A30, 26A33

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## 1. Introduction

The theory of quantum calculus i.e.  $q$  – calculus [6] which also be defined as calculus without limits now becoming the important topic in the field of Mathematics and Physics mainly dealing with the field of Number theory especially in Cryptography, Combinatory, Mechanics , Theory of Relativity and other sciences quantum theory.

In this paper, we have extend the definition of 1 – dimensional  $q$  – analogues Jafari’s integral transformation towards 2 – dimensional  $q$  – analogues and find out its relationship with other 2 - dimensional Laplace type  $q$  – analogues integral transformations [11,12]. The paper were arranged as follows.

The paper mainly divided into three parts, in the first part the generalized definition of one dimensional  $q$  – analogues of Jafari’s Integral Transformation and some other basic integral transformation definition given, in the second part the generalized definition of 2 – dimensional  $q$  – analogues of Jafari’s Integral Transformation and its relationship with some basic integral transformation were explained. In the last part we have proved the conditions for convergence and uniform convergence of 2 – dimensional  $q$  – analogues of Jafari’s Integral Transformation.

In the following, we present some basic definitions needed in proving the main results.

## 2. Basic Definitions

### Definition 2.1: Jafari's Integral Transformation

If a function  $f(t)$  which is to be integrable and defined for  $t \geq 0$  and  $p(s) \neq 0$  and  $q(s)$  are positive real valued function then its Jafari's integral transformation [5] is given by

$$J\{f(t); s\} = \mathcal{F}(s) = p(s) \int_0^{\infty} f(t) e^{-q(s)t} dt \quad (2.1)$$

Provided that the integral exist for  $q(s)$

### Definition 2.2: q – analogues of Exponential function

The q – analogues of exponential function  $e^t$  is denoted by  $\hat{e}_q(t)$  and  $e_q(t)$  and is given by [6]

$$\hat{e}_q(t) = \prod_{i=1}^{\infty} (1 + (1 - q)q^{i-1}t) = \sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{t^k}{[k]_q!} \quad (2.2.A)$$

$$e_q(t) = \prod_{i=1}^{\infty} (1 - (1 - q)q^{i-1}t)^{-1} = \sum_{k=0}^{\infty} \frac{t^k}{[k]_q!} \quad (2.2.B)$$

### Definition 2.3: q – Derivative

The q – derivative of a function  $f(t)$  is denoted by  $D_q f(t)$  and is given by [6],

$$D_q f(t) = \frac{d_q f}{d_q t} = \frac{f(qt) - f(t)}{(q-1)t} \quad (2.3)$$

### Definition 2.4: Laplace type Integral Transformation

If function  $f(t)$  is continuous piecewise and is of exponential order then its Laplace – type integral transformation [14, 15] is given by:

$$\mathcal{L}_{\varepsilon}\{f(t); s\} = \int_0^{\infty} \varepsilon'(t) e^{-\Phi(s)\varepsilon(t)} f(t) dt \quad (2.4)$$

In the above definition,  $\Phi(s)$  is a function which is invertible such that  $\varepsilon(t) = \int e^{-a(t)} dt$  is exponential function and  $a(t)$  is a function which also invertible.

### 3. Two – dimensional q – analogues of Jafari’s Integral Transformation

In this section, we introduce the extension of q – analogues of Jafari’s integral transformation [13] towards 2 – dimensional q – analogues of Jafari’s integral transformation of along with some properties;

#### Definition:

We consider the definition of 2 – dimensional q – analogues of Jafari’s integral transformation using the definition [13] as;

$$\widehat{J}_q[f(x, t)](u, v) = P(u, v) \int_0^\infty \int_0^\infty e_q[-\varepsilon(u, v, x, t)] f(x, t) d_q x d_q t \quad \text{--} \quad [A]$$

Where,  $\varepsilon(u, v, x, t) = Q(u)x + Q(v)t$  are invertible functions with the property that

$$f(x, t) \in S = \left\{ f(x, t) : \exists k_1, k_2 > 0, |f(x, t)| < M e^{\frac{|x|}{k_1}}, x \in (-1)^j \times [0, \infty), a. e. 't', M > 0 \right\} \text{ and}$$

$$P(u, v) = P(u)P(v)$$

#### 3.1) Relationship with some q – analogues of some integral transformations

##### 3.1.1] 2 – dimensional q – analogues of Laplace transformation

The two dimensional q – analogues of Laplace transformation [13] of a function  $f(x, t)$  can be obtained by taking  $Q(u) = u$  and  $Q(v) = v$ ,  $P(u, v) = 1$  in equation [A] gives;

$$\widehat{L}_q[f(x, t)](u, v) = \int_0^\infty \int_0^\infty e_q[-(ux + vt)] f(x, t) d_q x d_q t$$

##### 3.1.2] 2 – dimensional q – analogues of Elzaki transformation

The two dimensional q – analogues of Elzaki transformation [13] of a function  $f(x, t)$  can be obtained by taking  $Q(u) = \frac{1}{u}$  and  $Q(v) = \frac{1}{v}$ ,  $P(u, v) = uv$  in equation [A] gives;

$$\widehat{T}_q[f(x, t)](u, v) = uv \int_0^\infty \int_0^\infty e_q \left[ -\left( \frac{x}{u} + \frac{t}{v} \right) \right] f(x, t) d_q x d_q t$$

### 3.1.3] 2 – dimensional q – analogues of Sumudu transformation

The two dimensional q – analogues of Sumudu transformation [13] of a function  $f(x, t)$  can be obtained by taking  $Q(u) = \frac{1}{u}$  and  $Q(v) = \frac{1}{v}$ ,  $P(u, v) = \frac{1}{uv}$  in equation [A] gives;

$$\widehat{SuSu}_q[f(x, t)](u, v) = \frac{1}{uv} \int_0^{\infty} \int_0^{\infty} e_q \left[ -\left( \frac{x}{u} + \frac{t}{v} \right) \right] f(x, t) d_q x d_q t$$

### 3.1.4] 2 – dimensional q – analogues of Aboodh transformation

The two dimensional q – analogues of Aboodh transformation of a function  $f(x, t)$  can be obtained by taking  $Q(u) = u$  and  $Q(v) = v$ ,  $P(u, v) = \frac{1}{uv}$  in equation [A] gives;

$$\widehat{AA}_q[f(x, t)](u, v) = \frac{1}{uv} \int_0^{\infty} \int_0^{\infty} e_q[-(ux + vt)] f(x, t) d_q x d_q t$$

### 3.1.5] 2 – dimensional q – analogues of Pourreza transformation

The two dimensional q – analogues of Pourreza transformation of a function  $f(x, t)$  can be obtained by taking  $Q(u) = u^2$  and  $Q(v) = v^2$ ,  $P(u, v) = uv$  in equation [A] gives;

$$\widehat{PP}_q[f(x, t)](u, v) = uv \int_0^{\infty} \int_0^{\infty} e_q[-(xu^2 + tv^2)] f(x, t) d_q x d_q t$$

### 3.1.6] 2 – dimensional q – analogues of Mohand transformation

The two dimensional q – analogues of Mohand transformation of a function  $f(x, t)$  can be obtained by taking  $Q(u) = u$  and  $Q(v) = v$ ,  $P(u, v) = u^2 v^2$  in equation [A] gives;

$$\widehat{MM}_q[f(x, t)](u, v) = u^2 v^2 \int_0^{\infty} \int_0^{\infty} e_q[-(xu + tv)] f(x, t) d_q x d_q t$$

### 3.1.7] 2 – dimensional q – analogues of Sawi transformation

The two dimensional q – analogues of Sawi transformation of a function  $f(x, t)$  can be obtained by taking  $Q(u) = \frac{1}{u}$  and  $Q(v) = \frac{1}{v}$ ,  $P(u, v) = \frac{1}{u^2 v^2}$  in equation [A] gives;

$$\widehat{\mathcal{S}}_q[f(x, t)](u, v) = \frac{1}{u^2 v^2} \int_0^\infty \int_0^\infty e_q \left[ -\left(\frac{x}{u} + \frac{t}{v}\right) \right] f(x, t) d_q x d_q t$$

In the similar manner, by substitution of various values of  $Q(u)$ ,  $Q(v)$  and  $P(u, v)$  one can obtain the relationship with q – analogues of Natural Transformation, and q – analogues of G Transformation of order  $\alpha$ .

## 4. Convergence of Two – dimensional q – analogues of Jafari’s Integral Transformation

### Theorem 4.1:

If  $f(x, t)$  is continuous on  $[0, \infty) \times [0, \infty)$  and integral converges at  $Q(u_0)$  and  $Q(v_0)$ . Then the two – dimensional q – analogues of Jafari’s Integral transform of  $f(x, t)$  converges on for  $Q(u) > Q(u_0)$  and  $Q(v) > Q(v_0)$  where  $\varepsilon(u, v, x, t) \geq 0$  in the positive quadrant.

To prove the proof we will use the following lemmas.

**Lemma:** If  $\widehat{\mathcal{J}}_q[f(x, t)](u_0) = \int_0^t P(u_0, v) f(x, t) e_q[-Q(u_0)\varepsilon(x, t)] d_q x$  is bounded on  $[0, \infty)$  then the two – dimensional q – analogues of Jafari’s Integral transform w.r.t u converges for  $Q(u) > Q(u_0)$  and  $\varepsilon(x, t) = x \geq 0$  in the positive quadrant such that  $\varepsilon(x, t) = x$  bounded in first variable.

**Proof:** Consider the set

$S_1 = \left\{ (x, t): g(x, t) = P(u_0, v) \int_0^t f(x, t) e_q[-Q(u_0)\varepsilon(x, t)] d_q x < \infty \right\}$  for  $0 < t < \infty$ . Then by property of  $S_1$  we have,

$g(x, 0) = 0$  and  $\lim_{t \rightarrow \infty} g(x, t)$  will exist and bounded this is because integral is bounded on  $[0, \infty)$

Then by fundamental theorem of calculus, we get

$$g_t(x, t) = P(u_0, v) f(x, t) e_q[-Q(u_0)\varepsilon(x, t)]$$

Where  $P(u_0, v) \neq 0$

Now, we Choose  $\delta_1$  and  $R_1$  with  $0 < \delta_1 < R_1$ , Then the integral

$$\begin{aligned} I &= \int_{\delta_1}^{R_1} P(u_0, v) f(x, t) e_q[-Q(u)\varepsilon(x, t)] d_q x \\ &= \int_{\delta_1}^{R_1} g_t(x, t) e_q[-[Q(u) - Q(u_0)]\varepsilon(x, t)] d_q x \quad \text{with } P(u_0, v) \neq 0 \end{aligned}$$

Applying integration by parts then the integral turns out to be

$$I = \left[ e_q[-[Q(u) - Q(u_o)]\varepsilon(x, t)]g(x, t) \right]_{\delta_1}^{R_1} + \int_{\delta_1}^{R_1} [Q(u) - Q(u_o)]e_q[-[Q(u) - Q(u_o)]\varepsilon(x, t)]g_t(x, t) dx$$

Now let,  $\delta_1 \rightarrow 0$

$$\Rightarrow I = \left[ e_q[-[Q(u) - Q(u_o)]\varepsilon(x, R_1)]g(x, R_1) + \int_0^{R_1} [Q(u) - Q(u_o)]e_q[-[Q(u) - Q(u_o)]\varepsilon(x, R_1)]g(x, t) d_q x \right]$$

Now let  $R_1 \rightarrow \infty$  then  $e_q[-[Q(u) - Q(u_o)]\varepsilon(x, R_1)] \rightarrow 0$  as  $Q(u) > Q(u_o)$  and  $\varepsilon(x, t) \geq 0$  in the positive quadrant and bounded in first variable. Which exist as the integral is bounded  $Q(u) > Q(u_o)$ .

In the similar manner we can prove that if the integral

$I_1 = \int_0^\infty P(u, v_0)f(x, t)e_q[-Q(v)\varepsilon(x, t)]d_q t$  Converges at  $Q(v_0)$  then the integral converges for  $Q(v) > Q(v_0), \varepsilon(x, t) = t \geq 0$  and  $0 < x < \infty$ . Hence the theorem hold.

#### Theorem 4.2:

If  $f(x, t)$  is continuous and bounded on  $[0, \infty) \times [0, \infty)$  and integral converges at  $Q(u_o)$  and  $Q(v_o)$ . Then the 2 – dimensional q – analogues of Jafari's Integral transform of  $f(x, t)$  converges uniformly on  $[u, \infty) \times [v, \infty)$  if  $Q(u) > Q(u_o)$  and  $Q(v) > Q(v_o)$  where  $\varepsilon(x, t) \geq 0$  in the positive quadrant.

To prove the proof we will use the following lemmas.

**Lemma:** If  $\widehat{J}_q[f(x, t)]; (u_o) = \int_0^t P(u_o, v)f(x, t)e_q[-Q(u_o)\varepsilon(x, t)]d_q x$  is bounded on  $[0, \infty)$  then the 2 – dimensional q – analogues of Jafari's Integral transform converges uniformly on  $[u, \infty)$ . If  $Q(u) > Q(u_o)$  and  $\varepsilon(x, t) \geq 0$  in the positive quadrant and bounded in first variable.

$S_1 = \left\{ (x, t): g(x, t) = P(u_o, v) \int_0^t f(x, t)e_q[-Q(u_o)\varepsilon(x, t)]d_q x < \infty \right\}$  for  $0 < t < \infty$ . Then by property of  $S_1$  we have,

$g(x, 0) = 0$  and  $\lim_{t \rightarrow \infty} g(x, t)$  will exist and bounded this is because integral is bounded on  $[0, \infty)$

So by fundamental theorem of calculus, we get

$$g_t(x, t) = P(u_o, v)f(x, t)e_q[-Q(u_o)\varepsilon(x, t)] , \text{ where } P(u_o, v) \neq 0 \text{ ----- I}$$

Now, we choose  $\delta_1$  and  $\delta$  such that  $0 < \delta < \delta_1$ , then the integral

$$\begin{aligned} I &= \int_{\delta}^{\delta_1} P(u_o, v) f(x, t) e_q[-Q(u)\varepsilon(x, t)] d_q x \\ &= \int_{\delta_1}^{\delta_1} e_q[-[Q(u) - Q(u_o)]\varepsilon(x, t)] g_t(x, t) d_q x \quad \text{with } P(u_o, v) \neq 0 \end{aligned}$$

Applying integration by parts then the integral turns out to be

$$\begin{aligned} I &= \left[ e_q[-[Q(u) - Q(u_o)]\varepsilon(x, t)] g(x, t) \right]_{\delta_1}^{R_1} \\ &\quad + \int_{\delta_1}^{R_1} [Q(u) - Q(u_o)] e_q[-[Q(u) - Q(u_o)]\varepsilon(x, t)] g(x, t) d_q x \\ &= [e_q[-[Q(u) - Q(u_o)]\varepsilon(R_1, t)] g(R_1, t) - e_q[-[Q(u) - Q(u_o)]\varepsilon(\delta_1, t)] g(\delta_1, t) \\ &\quad + \int_{\delta_1}^{R_1} [Q(u) - Q(u_o)] e_q[-[Q(u) - Q(u_o)]\varepsilon(x, t)] g(x, t) d_q x \end{aligned}$$

By property of bounded ness  $\exists M > 0$  such that

$$|g(x, t)| \leq M \text{ it gives us;}$$

### Conclusion:

The paper gives the conditions about convergence and uniform convergence of the 2 – D q – analogues of Jafari’s Integral Transformation along with its relationship with some other q – Integral transformation.

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