Generalized Ulam-Hyers Stability Of System Of Quadratic Functional Equations Originating From Geometrical Equations

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ABSTRACT

In this paper, the authors established the generalized Ulam-Hyers stability of system of quadratic functional equations which originating from Leibniz formula in Euclidean Geometry and median of a triangle in intuitionistic fuzzy Banach space using Hyers direct method.

Keywords—Quadratic functional equations, generalized Ulam – Hyers stability, intuitionistic fuzzy Banach space, Hyers method, Geometry, Leibniz formula in Euclidean Geometry, median of a triangle.

I. INTRODUCTION

The revision of stability problems for functional equations is coupled to a query of Ulam [36,37] concerning the stability of group homomorphisms and confidently responded for Banach spaces by Hyers [20]. It was supplementary generalized and outstanding results was attained by number of authors see ([3,18,26,29,32]). The general solution and generalized Ulam-Hyers-Rassias stability of quadratic functional equation was investigated by Cholewa [15], S. Czerwik [16], Jung [23]. Ravi [30,31].

During the last eight decades, the overhead problems was attempted by numerous authors and its solutions via various forms of functional equations were discussed one can refer [1-2,4-14,17,21-22,24-25,27-28] and references cited there in.

Geometry is a branch of mathematics concerned with properties of space such as the distance, shape, size, and relative position of figures. Geometry is, along with arithmetic, one of the oldest branches of mathematics. A mathematician who works in the field of geometry is called a geometer. Until the 19th century, geometry was almost exclusively devoted to Euclidean geometry, which includes the notions of point, line, plane, distance, angle, surface, and curve, as fundamental concepts.

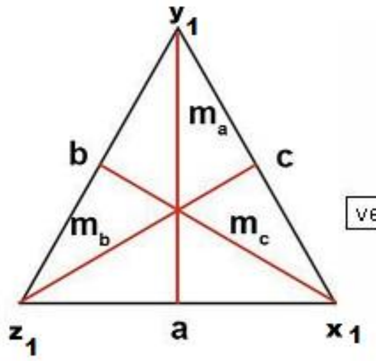
Originally developed to model the physical world, geometry has applications in almost all sciences, and also in art, architecture, and other activities that are related to graphics. Geometry also has applications in areas of mathematics that are apparently unrelated. The geometry that underlies general relativity is a famous application of non-Euclidean geometry.

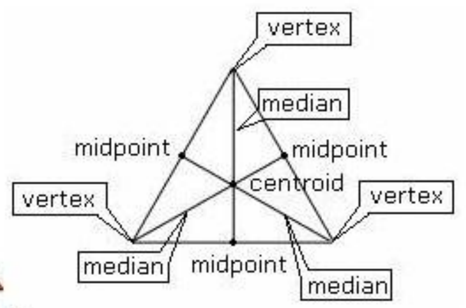
During and late the 19th century several discoveries enlarged dramatically the scope of geometry has been greatly expanded, and the field has been split in many subfields that depend on the underlying methods differential geometry, algebraic geometry, computational geometry, algebraic topology, discrete geometry. Also, the properties of Euclidean spaces that are disregarded projective geometry that consider only alignment of points but not distance and parallelism, affine geometry that omits the concept of angle and distance, finite geometry that omits continuity, and others. This enlargement of the scope of geometry led to a change of meaning of the word space, which originally referred to the three-dimensional space of the physical world and its model provided by Euclidean geometry; presently a geometric space, or simply a space is a mathematical structure on which some geometry is defined. For more detailed see [38].

The main aim of this paper is to convert the geometrical properties into functional equations which satisfies the properties via its solutions.

A MEDIAN OF A TRIANGLE

In geometry, a median of a triangle is a line segment joining a vertex to the midpoint of the opposing side. Every triangle has exactly three medians: one running from each vertex to the opposite side. In the case of isosceles and equilateral triangles, a median bisects any angle at a vertex whose two adjacent sides are equal in length.





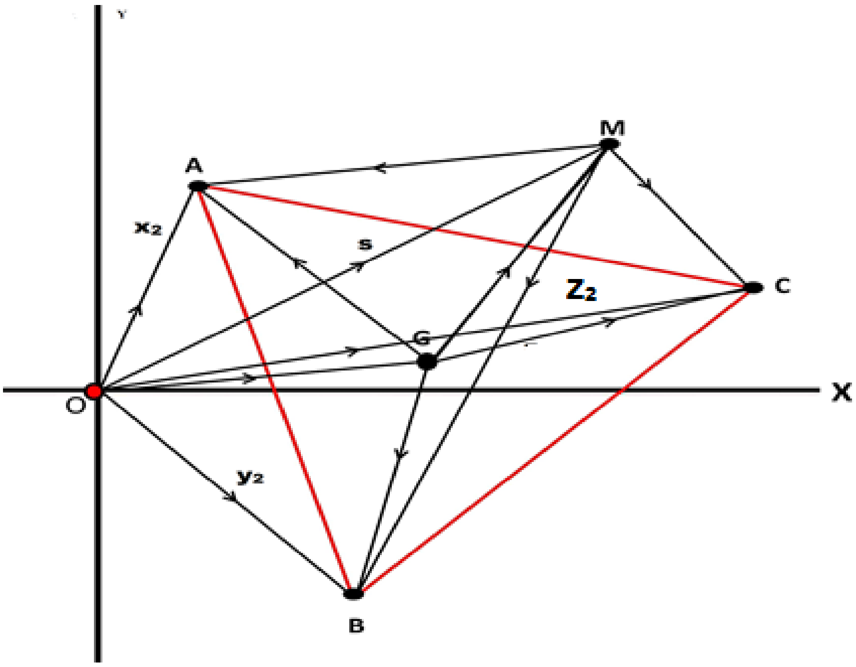
In a above triangle with the sides a, b and c the median drawn to the side c has the length of

$$m_c^2 = \frac{1}{2}\left(a^2 + b^2\right) - \frac{1}{4}c^2. \tag{1.1}$$

B. LEIBNIZ QUADRATIC FORMULA IN EUCLIDEAN GEOMETRY

Let M be an arbitrary point lying on the plane of the triangle ABC and G is the centroid (= Gravity center) of ABC, then

$$\|\underline{MA}|^2 + \|\underline{MB}|^2 + \|\underline{MC}|^2 = 3\|\underline{MG}|^2 + \left(\|\underline{GA}|^2 + \|\underline{GB}|^2 + \|\underline{GC}|^2\right). \tag{1.2}$$



The above equation (1.2), can be transformed into quadratic functional equation of the median from z is given by

$$f_T\left(\frac{x_1 + y_1}{2} - z_1\right) = \frac{1}{2}\left(f_T\left(z_1 - x_1\right) + f_T\left(z_1 - y_1\right)\right) - \frac{1}{4} f_T\left(x_1 - y_1\right). \tag{1.3}$$

Also, equation (1.4), can be transformed into quadratic functional equation of the centroid G is set by

$$f_L(x_2 - s) + f_L(y_2 - s) + f_L(z_2 - s)$$
$$= 3 f_L\left(\frac{x_2 + y_2 + z_2}{3} - s\right) + f_L\left(\frac{2x_2 - y_2 - z_2}{3}\right) + f_L\left(\frac{-x_2 + 2y_2 - z_2}{3}\right) + f_L\left(\frac{-x_2 - y_2 + 2z_2}{3}\right). \tag{1.4}$$

In this paper, the authors established the generalized Ulam-Hyers stability of system of quadratic functional equations (1.3) and (1.4) which originating from Leibniz formula in Euclidean Geometry and median of a triangle in intuitionistic fuzzy Banach space using Hyers direct method.

II BASIC DEFINITIONS RELATED TO INTUITIONISTIC FUZZY NORMED SPACES

In this section, we provide some basic definitions and notations related to intuitionistic fuzzy normed spaces as in [34,35,36].

Definition 2.1. A binary operation * : $[0,1]\times[0,1]\to[0,1]$ is said to be a continuous t-norm if it satisfies the following conditions :

$(*1)$ * is associative and commutative,

$(*2)$ * is continuous,

$(*3)$ $a*1=a$ for all $a\in[0,1]$,

$(*4)$ $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for each $a,b,c,d\in[0,1]$.

Definition 2.2. A binary operation $\Diamond$: $[0,1]\times[0,1]\to[0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions :

$(\Diamond 1)$ $\Diamond$ is associative and commutative,

$(\Diamond 2)$ $\Diamond$ is continuous,

$(\Diamond 3)$ $a\Diamond 0=a$ for all $a\in[0,1]$,

$(\Diamond 4)$ $a\Diamond b\le c\Diamond d$ whenever $a\le c$ and $b\le d$ for each $a,b,c,d\in[0,1]$.

Definition 2.3. The five-tuple $(X,\mu,\upsilon,*,\Diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm, μ,υ are fuzzy sets on $X\times(0,\infty)$ such that for all $x,y\in X$ and $s,t>0$ satisfying the following conditions :

[IFNS1] $\mu(x,t)+\upsilon(x,t)\le 1,$

[IFNS2] $\mu(x,t)>0,$

[IFNS3] $\mu(x,t)=1$ if and only if $x=0,$

[IFNS4] $\mu(\alpha x,t)=\mu\left(x,\frac{t}{|\alpha|}\right)$ for each $\alpha\ne 0;$

[IFNS5] $\mu(x,t)*\mu(y,s)\ge\mu(x+y,t+s),$

[IFNS6] $\mu(x,.):(0,\infty)\to[0,1]$ is continuous,

[IFNS7] $lim_{t\to\infty}\mu(x,t)=1$ and $lim_{t\to 0}\mu(x,t)=0$

[IFNS8] $\upsilon(x,t)<1,$

[IFNS9] $\upsilon(x,t)=0$ if and only if $x=0,$

[IFNS10] $\upsilon(\alpha x,t)=\upsilon\left(x,\frac{t}{|\alpha|}\right)$ for each $\alpha\ne 0;$

[IFNS11] $\upsilon(x,t)\Diamond\upsilon(y,s)\le\upsilon(x+y,t+s),$

[IFNS12] $\upsilon(x,.):\upsilon(0,\infty)\to[0,1]$ is continuous,

[IFNS13] $lim_{t\to\infty}\upsilon(x,t)=0$ and $lim_{t\to 0}\upsilon(x,t)=1$.

In this case (μ,υ) is called an intuitionistic fuzzy norm

Example 2.3. Let $(X,\|.\|)$ be a normed space, $a*b=ab$ and $a\Diamond b=\min\{a+b,1\}$ for all $a,b\in[0,1]$. For all , $x\in X$, $t>0$, consider $\mu(x,t):=\begin{cases}\frac{t}{t+\|x\|} & if \quad t>0\\ 0 & if \quad t\le 0\end{cases}$ and $\upsilon(x,t):=\begin{cases}\frac{\|x\|}{t+\|x\|} & if \quad t>0\\ 0 & if \quad t\le 0\end{cases}$ then $(X,\mu,\upsilon,*,\Diamond)$ is an IFNS.

Definition 2.5. Let $(X,\mu,\upsilon,*,\Diamond)$ be an IFNS. Then, a sequence $\{x_n\}$ is said to be intuitionistic fuzzy convergent to $L\in X$ if $lim_{n\to\infty}\mu(x_n-L,t)=1$ and $lim_{n\to\infty}\upsilon(x_n-L,t)=0$ for all $t>0$.

Definition 2.6. Let $(X,\mu,\upsilon,*,\Diamond)$ be an IFNS. Then, a sequence $\{x_n\}$ is said to be intuitionistic fuzzy Cauchy sequence if $lim_{n\to\infty}\mu(x_{n+p}-x_n,t)=1$ and $lim_{n\to\infty}\upsilon(x_{n+p}-x_n,t)=0$ for all $t>0$ and $p=1,2,....$

Definition 2.7. Let $(X,\mu,\upsilon,*,\Diamond)$ be an IFNS. Then, $(X,\mu,\upsilon,*,\Diamond)$ is said to be Banach space if every intuitionistic fuzzy convergent in $(X,\mu,\upsilon,*,\Diamond)$.

III INTUITIONISTIC FUZZY STABILITY

From now on (X,μ,υ) be an Intuitionistic Fuzzy normed space (Y,μ',υ') be an Intuitionistic Fuzzy Banach space respectively. In this section, using an idea of Gavruta. We prove the stability of in the spirit of Hyers, Ulam and Rassias. For convenience we use the following abbreviation for a given mapping $f_T : X \to Y$ and $f_L : X \to Y$ by

$$\Delta f_T(x,y,z) = f_T\left(\frac{x_1+y_1}{2}-z_1\right)-\frac{1}{2}\left(f_T(z_1-x_1)+f_T(z_1-y_1)\right)+\frac{1}{4}f_T(x_1-y_1);$$

and

$$\Delta f_{:L}(x,y,z,t) = f_L(x_2-s)+f_L(y_2-s)+f_L(z_2-s) -3f_L\left(\frac{x_2+y_2+z_2}{3}-s\right)-f_L\left(\frac{2x_2-y_2-z_2}{3}\right)-f_L\left(\frac{-x_2+2y_2-z_2}{3}\right)-f_L\left(\frac{-x_2-y_2+2z_2}{3}\right);$$

for all $x_1,y_1,z_1,x_2,y_2,z_2,s \in X$.

Theorem 3.1 : Let $\beta \in \{-1,1\}$ be fixed and let $\Lambda_T : X^3 \to (0,1];\Lambda_L : X^4 \to (0,1]$ are mappings for some $\alpha > 0$ with $0 < \left(\frac{\alpha}{4}\right)^{\beta} < 1$ satisfying the conditions

$$\begin{cases}\mu'\left(\Lambda_T(2^{\beta}x_1,2^{\beta}y_1,2^{\beta}z_1),t\right) \geq \mu'\left(\alpha^{\beta}\Lambda_T(x_1,y_1,z_1),t\right)\\ \upsilon'\left(\Lambda_T(2^{\beta}x_1,2^{\beta}y_1,2^{\beta}z_1),t\right) \leq \upsilon'\left(\alpha^{\beta}\Lambda_T(x_1,y_1,z_1),t\right)\end{cases};$$
$$\begin{cases}\mu'\left(\Lambda_L(2^{\beta}x_2,2^{\beta}y_2,2^{\beta}z_2.2^{\beta}s),t\right) \geq \mu'\left(\alpha^{\beta}\Lambda_L(x_2,y_2,z_2,s),t\right)\\ \upsilon'\left(\Lambda_L(2^{\beta}x_2,2^{\beta}y_2,2^{\beta}z_2,2^{\beta}s),t\right) \leq \upsilon'\left(\alpha^{\beta}\Lambda_L(x_2,y_2,z_2,s),t\right)\end{cases}; \quad (3.1)$$

$$\begin{cases}\lim_{n\to\infty}\mu'\left(\Lambda_T(2^{n\beta}x_1,2^{n\beta}y_1,2^{n\beta}z_1),4^{n\beta}t\right)=1\\ \lim_{n\to\infty}\upsilon'\left(\Lambda_T(2^{n\beta}x_1,2^{n\beta}y_1,2^{n\beta}z_1),4^{n\beta}t\right)=0\end{cases};$$
$$\begin{cases}\lim_{n\to\infty}\mu'\left(\Lambda_L(2^{n\beta}x_2,2^{n\beta}y_2,2^{n\beta}z_2,2^{n\beta}s),4^{n\beta}t\right)=1\\ \lim_{n\to\infty}\upsilon'\left(\Lambda_L(2^{n\beta}x_2,2^{n\beta}y_2,2^{n\beta}z_2,2^{n\beta}s),4^{n\beta}t\right)=0\end{cases}; \quad (3.2)$$

and $f_T : X \to Y; f_L : X \to Y$ are functions satisfying the inequalities

$$\begin{cases}\mu\left(\Delta f_T(x_1,y_1,z_1),t\right) \geq \mu'\left(\Lambda_T(x_1,y_1,z_1),t\right)\\ \upsilon\left(\Delta f_T(x_1,y_1,z_1),t\right) \leq \upsilon'\left(\Lambda_T(x_1,y_1,z_1),t\right)\end{cases};$$
$$\begin{cases}\mu\left(\Delta f_L(x_2,y_2,z_2,s),t\right) \geq \mu'\left(\Lambda_L(x_2,y_2,z_2,s),t\right)\\ \upsilon\left(\Delta f_L(x_2,y_2,z_2,s),t\right) \leq \upsilon'\left(\Lambda_L(x_2,y_2,z_2,s),t\right)\end{cases}; \quad (3.3)$$

for all $x_1,y_1,z_1,x_2,y_2,z_2,s \in X$ and all $t > 0.$ Then there exists a unique quadratic mappings $Q_T : X \to Y; Q_L : X \to Y$ satisfying (1.3) and (1.4) such that

$$\begin{cases}\mu\left(Q_T(x_1)-f_T(x_1),t\right) \geq \mu'\left(\Lambda_T(x_1,-x_1,0),\frac{|4-\alpha|t}{4}\right)\\ \upsilon\left(Q_T(x_1)-f_T(x_1),t\right) \leq \upsilon'\left(\Lambda_T(x_1,-x_1,0),\frac{|4-\alpha|t}{4}\right)\end{cases};$$
$$\begin{cases}\mu\left(Q_L(x_2)-f_L(x_2),t\right) \geq \mu'\left(\Lambda_L(2x_2,x_2,0,0),|4-\alpha|t\right)\\ \upsilon\left(Q_L(x_2)-f_L(x_2),t\right) \leq \upsilon'\left(\Lambda_L(2x_2,x_2,0,0),|4-\alpha|t\right)\end{cases}; \quad (3.4)$$

for all $x_1,x_2 \in X$ and all $t > 0.$ The mappings $Q_T(x_1);Q_L(x_2)$ are obtained by

$$Q_T(x_1) = (\mu,\upsilon)-\lim_{n\to\infty}\frac{f_T(2^{n\beta}x_1)}{4^{n\beta}};Q_L(x_2) = (\mu,\upsilon)-\lim_{n\to\infty}\frac{f_L(2^{n\beta}x_2)}{4^{n\beta}} \quad (3.5)$$

for all $x_1,x_2 \in X$.

Proof : Assume $\beta=1$. Replacing $(x_1,y_1,z_1);(x_2,y_2,z_2,s)$ by $(x_1,-x_1,0);(2x_2,x_2,0,0)$ in (3.3), using evenness of f and [IFNS4], one can get

$$\begin{cases}\mu\left(\frac{1}{4}f_T(2x_1)-f_T(x_1),t\right)\geq\mu'(\Lambda_T(x_1,-x_1,0),t)\\ \upsilon\left(\frac{1}{4}f_T(2x_1)-f_T(x_1),t\right)\leq\upsilon'(\Lambda_T(x_1,-x_1,0),t)\end{cases};\qquad \begin{cases}\mu\left(\frac{1}{4}f_L(2x_2)-f_L(x_2),\frac{t}{4}\right)\geq\mu'(\Lambda_L(2x_2,x_2,0,0),t).\\ \upsilon\left(\frac{1}{4}f_L(2x_2)-f_L(x_2),\frac{t}{4}\right)\leq\upsilon'(\Lambda_L(2x_2,x_2,0,0),t)\end{cases}; \tag{3.6}$$

for all $x_1,x_2\in X$ and all $t>0$. Replacing (x_1,x_2) by $(2^nx_1,2^nx_2)$ in (3.6), one can obtain

$$\begin{cases}\mu\left(\frac{1}{4}f_T(2^nx_1)-f_T(2^nx_1),t\right)\geq\mu'(\Lambda_T(2^nx_1,-2^nx_1,0),t)\\ \upsilon\left(\frac{1}{4}f_T(2^nx_1)-f_T(2^nx_1),t\right)\leq\upsilon'(\Lambda_T(2^nx_1,-2^nx_1,0),t)\end{cases};\qquad \begin{cases}\mu\left(\frac{1}{4}f_L(2^nx_2)-f_L(2^nx_2),\frac{t}{4}\right)\geq\mu'(\Lambda_L(2.2^nx_2,2^nx_2,0,0),t)\\ \upsilon\left(\frac{1}{4}f_L(2^nx_2)-f_L(2^nx_2),\frac{t}{4}\right)\leq\upsilon'(\Lambda_L(2.2^nx_2,2^nx_2,0,0),t)\end{cases}; \tag{3.7}$$

for all $x_1,x_2\in X$ and all $t>0$. Using [IFNS4], (3.1) in (3.7), one can arrive

$$\begin{cases}\mu\left(\frac{1}{4^{n+1}}f_T(2^nx_1)-\frac{1}{4^n}f_T(2^nx_1),\frac{t}{4^n}\right)\geq\mu'(\Lambda_T(2^nx_1,-2^nx_1,0),t)\geq\mu'\left(\Lambda_T(x_1,-x_1,0),\frac{t}{\alpha^n}\right)\\ \upsilon\left(\frac{1}{4^{n+1}}f_T(2^nx_1)-\frac{1}{4^n}f_T(2^nx_1),\frac{t}{4^n}\right)\leq\upsilon'(\Lambda_T(2^nx_1,-2^nx_1,0),t)\leq\upsilon'\left(\Lambda_T(x_1,-x_1,0),\frac{t}{\alpha^n}\right)\end{cases};$$
$$\begin{cases}\mu\left(\frac{1}{4^{n+1}}f_L(2^nx_2)-\frac{1}{4^n}f_L(2^nx_2),\frac{1}{4}.\frac{t}{4^n}\right)\geq\mu'(\Lambda_L(2.2^nx_2,2^nx_2,0,0),t)\geq\mu'\left(\Lambda_L(2x_2,x_2,0,0),\frac{t}{\alpha^n}\right)\\ \upsilon\left(\frac{1}{4^{n+1}}f_L(2^nx_2)-\frac{1}{4^n}f_L(2^nx_2),\frac{1}{4}.\frac{t}{4^n}\right)\leq\upsilon'(\Lambda_L(2.2^nx_2,2^nx_2,0,0),t)\leq\upsilon'\left(\Lambda_L(2x_2,x_2,0,0),\frac{t}{\alpha^n}\right)\end{cases}; \tag{3.8}$$

for all $x_1,x_2\in X$ and all $t>0$. Replacing t by α^nt in (3.8), one can have

$$\begin{cases}\mu\left(\frac{1}{4^{n+1}}f_T(2^nx_1)-\frac{1}{4^n}f_T(2^nx_1),\frac{\alpha^nt}{4^n}\right)\geq\mu'(\Lambda_T(x_1,-x_1,0),t)\\ \upsilon\left(\frac{1}{4^{n+1}}f_T(2^nx_1)-\frac{1}{4^n}f_T(2^nx_1),\frac{\alpha^nt}{4^n}\right)\leq\upsilon'(\Lambda_T(x_1,-x_1,0),t)\end{cases};\qquad \begin{cases}\mu\left(\frac{1}{4^{n+1}}f_L(2^nx_2)-\frac{1}{4^n}f_L(2^nx_2),\frac{1}{4}.\frac{\alpha^nt}{4^n}\right)\geq\mu'(\Lambda_L(2x_2,x_2,0,0),t)\\ \upsilon\left(\frac{1}{4^{n+1}}f_L(2^nx_2)-\frac{1}{4^n}f_L(2^nx_2),\frac{1}{4}.\frac{\alpha^nt}{4^n}\right)\leq\upsilon'(\Lambda_L(2x_2,x_2,0,0),t)\end{cases}; \tag{3.9}$$

for all $x_1,x_2\in X$ and all $t>0$. One can easy to see verify that

$$\frac{f_T(2^nx_1)}{4^n}-f_T(x_1)=\sum_{i=0}^{n-1}\frac{f_T(2^{(i+1)}x_1)}{4^{(i+1)}}-\frac{f_T(2^ix_1)}{4^i}$$
$$\frac{f_L(2^nx_2)}{4^n}-f_L(x_2)=\sum_{i=0}^{n-1}\frac{f_L(2^{(i+1)}x_2)}{4^{(i+1)}}-\frac{f_L(2^ix_2)}{4^i} \tag{3.10}$$

for all $x_1,x_2\in X$. From equations (3.9) and (3.10), we have.

$$\begin{cases} \mu\left(\dfrac{f_T\left(2^n x_1\right)}{4^n} - f_T(x_1), \sum_{i=0}^{n-1}\dfrac{\alpha^i t}{4^i}\right) \geq \mu\left(\sum_{i=0}^{n-1}\dfrac{f_T(2^{(i+1)}x_1)}{4^{(i+1)}} - \dfrac{f_T(2^i x_1)}{4^i}, \sum_{i=0}^{n-1}\dfrac{\alpha^i t}{4^i}\right) \\ \qquad\qquad \geq \prod_{i=0}^{n-1}\mu\left(\dfrac{f_T(2^{(i+1)}x_1)}{4^{(i+1)}} - \dfrac{f_T(2^i x_1)}{4^i}, \dfrac{\alpha^i t}{4^i}\right) \geq \mu'\left(\Lambda_T(x_1,-x_1,0),t\right) \\ \upsilon\left(\dfrac{f_T\left(2^n x_1\right)}{4^n} - f_T(x_1), \sum_{i=0}^{n-1}\dfrac{\alpha^i t}{4^i}\right) \leq \upsilon\left(\sum_{i=0}^{n-1}\dfrac{f_T(2^{(i+1)}x_1)}{4^{(i+1)}} - \dfrac{f_T(2^i x_1)}{4^i}, \sum_{i=0}^{n-1}\dfrac{\alpha^i t}{4^i}\right) \\ \qquad\qquad \leq \coprod_{j=1}^{n}\upsilon\left(\dfrac{f_T(2^{(i+1)}x_1)}{4^{(i+1)}} - \dfrac{f_T(2^i x_1)}{4^i}, \dfrac{\alpha^i t}{4^i}\right) \leq \upsilon'\left(\Lambda_T(x_1,-x_1,0),t\right) \end{cases};$$

$$\begin{cases} \mu\left(\dfrac{f_L\left(2^n x_2\right)}{4^n} - f_L(x_2), \dfrac{1}{4}\sum_{i=0}^{n-1}\dfrac{\alpha^i t}{4^i}\right) \geq \mu\left(\sum_{i=0}^{n-1}\dfrac{f_L(2^{(i+1)}x_2)}{4^{(i+1)}} - \dfrac{f_L(2^i x_2)}{4^i}, \dfrac{1}{4}\sum_{i=0}^{n-1}\dfrac{\alpha^i t}{4^i}\right) \\ \qquad\qquad \geq \prod_{i=0}^{n-1}\mu\left(\dfrac{f_T(2^{(i+1)}x_1)}{4^{(i+1)}} - \dfrac{f_T(2^i x_1)}{4^i}, \dfrac{1}{4}\dfrac{\alpha^i t}{4^i}\right) \geq \mu'\left(\Lambda_L(2x_2,x_2,0,0),t\right) \\ \upsilon\left(\dfrac{f_L\left(2^n x_2\right)}{4^n} - f_L(x_2), \dfrac{1}{4}\sum_{i=0}^{n-1}\dfrac{\alpha^i t}{4^i}\right) \leq \upsilon\left(\sum_{i=0}^{n-1}\dfrac{f_L(2^{(i+1)}x_2)}{4^{(i+1)}} - \dfrac{f_L(2^i x_2)}{4^i}, \dfrac{1}{4}\sum_{i=0}^{n-1}\dfrac{\alpha^i t}{4^i}\right) \\ \qquad\qquad \leq \coprod_{j=1}^{n}\upsilon\left(\dfrac{f_T(2^{(i+1)}x_1)}{4^{(i+1)}} - \dfrac{f_T(2^i x_1)}{4^i}, \dfrac{1}{4}\dfrac{\alpha^i t}{4^i}\right) \leq \upsilon'\left(\Lambda_L(2x_2,x_2,0,0),t\right) \end{cases} \quad (3.11)$$

for all $x_1, x_2 \in X$ and $t > 0$ where $\prod_{j=1}^{n} a_j = a_1 * a_2 * ......* a_n, \coprod_{j=1}^{n} a_j = a_1 \lozenge a_2 \lozenge .....\lozenge a_n$

for all $x_1, x_2 \in X$ and all $t > 0$. Replacing x by $\sigma^m x$ in (3.11) and using (3.1), [IFNS4], one can obtain

$$\begin{cases} \mu\left(\dfrac{f_T\left(2^{n+m}x_1\right)}{4^{n+m}} - \dfrac{f_T(2^m x_1)}{4^m}, \sum_{i=0}^{n-1}\dfrac{\alpha^i t}{4^{i+m}}\right) \geq \mu'\left(\Lambda_T(x_1,-x_1,0), \dfrac{t}{\alpha^m}\right) \\ \upsilon\left(\dfrac{f_T\left(2^{n+m}x_1\right)}{4^{n+m}} - \dfrac{f_T(2^m x_1)}{4^m}, \sum_{i=0}^{n-1}\dfrac{\alpha^i t}{4^{i+m}}\right) \leq \upsilon'\left(\Lambda_T(x_1,-x_1,0), \dfrac{t}{\alpha^m}\right) \end{cases};$$

$$\begin{cases} \mu\left(\dfrac{f_L\left(2^{n+m}x_2\right)}{4^{n+m}} - \dfrac{f_L(2^m x_2)}{4^m}, \dfrac{1}{4}\sum_{i=0}^{n-1}\dfrac{\alpha^i t}{4^{i+m}}\right) \geq \mu'\left(\Lambda_L(2x_2,x_2,0,0), \dfrac{t}{\alpha^m}\right) \\ \upsilon\left(\dfrac{f_L\left(2^{n+m}x_2\right)}{4^{n+m}} - \dfrac{f_L(2^m x_2)}{4^m}, \dfrac{1}{4}\sum_{i=0}^{n-1}\dfrac{\alpha^i t}{4^{i+m}}\right) \leq \upsilon'\left(\Lambda_L(2x_2,x_2,0,0), \dfrac{t}{\alpha^m}\right) \end{cases}; \quad (3.12)$$

for all $x_1, x_2 \in X$ and all $t > 0$ and all $m, n \geq 0$. Replacing t by $\alpha^m t$ in (3.12), one can get

$$\begin{cases} \mu\left(\dfrac{f_T\left(2^{n+m}x_1\right)}{4^{n+m}} - \dfrac{f_T(2^m x_1)}{4^m}, \sum_{i=0}^{n-1}\dfrac{\alpha^{i+m} t}{4^{i+m}}\right) \geq \mu'\left(\Lambda_T(x_1,-x_1,0),t\right) \\ \upsilon\left(\dfrac{f_T\left(2^{n+m}x_1\right)}{4^{n+m}} - \dfrac{f_T(2^m x_1)}{4^m}, \sum_{i=0}^{n-1}\dfrac{\alpha^{i+m} t}{4^{i+m}}\right) \leq \upsilon'\left(\Lambda_T(x_1,-x_1,0),t\right) \end{cases};$$

$$\begin{cases} \mu\left(\dfrac{f_L\left(2^{n+m}x_2\right)}{4^{n+m}} - \dfrac{f_L(2^m x_2)}{4^m}, \dfrac{1}{4}\sum_{i=0}^{n-1}\dfrac{\alpha^{i+m} t}{4^{i+m}}\right) \geq \mu'\left(\Lambda_L(2x_2,x_2,0,0),t\right) \\ \upsilon\left(\dfrac{f_L\left(2^{n+m}x_2\right)}{4^{n+m}} - \dfrac{f_L(2^m x_2)}{4^m}, \dfrac{1}{4}\sum_{i=0}^{n-1}\dfrac{\alpha^{i+m} t}{4^{i+m}}\right) \leq \upsilon'\left(\Lambda_L(2x_2,x_2,0,0),t\right) \end{cases}; \quad (3.13)$$

for all $x_1, x_2 \in X$ and all $t > 0$ and all $m, n \geq 0$. Using [IFNS4] in (3.13), one can obtain

$$\begin{cases} \mu\left(\frac{f_T\left(2^{n+m}x_1\right)}{4^{n+m}} - \frac{f_T(2^m x_1)}{4^m}, t \right) \geq \mu'\left(\Lambda_T(x_1,-x_1,0), \frac{t}{\sum_{i=0}^{n-1} \frac{\alpha^{i+m}}{4^{i+m}}} \right) \\ \upsilon\left(\frac{f_T\left(2^{n+m}x_1\right)}{4^{n+m}} - \frac{f_T(2^m x_1)}{4^m}, t \right) \leq \upsilon'\left(\Lambda_T(x_1,-x_1,0), \frac{t}{\sum_{i=0}^{n-1} \frac{\alpha^{i+m}}{4^{i+m}}} \right) \end{cases};$$

$$\begin{cases} \mu\left(\frac{f_L\left(2^{n+m}x_2\right)}{4^{n+m}} - \frac{f_L(2^m x_2)}{4^m}, t \right) \geq \mu'\left(\Lambda_L(2x_2,x_2,0,0), \frac{t}{\frac{1}{4}\sum_{i=0}^{n-1} \frac{\alpha^{i+m}}{4^{i+m}}} \right) \\ \upsilon\left(\frac{f_L\left(2^{n+m}x_2\right)}{4^{n+m}} - \frac{f_L(2^m x_2)}{4^m}, t \right) \leq \upsilon'\left(\Lambda_L(2x_2,x_2,0,0), \frac{t}{\frac{1}{4}\sum_{i=0}^{n-1} \frac{\alpha^{i+m}}{4^{i+m}}} \right) \end{cases}; \tag{3.14}$$

for all $x_1, x_2 \in X$ and all $t > 0$ and all $m,n \geq 0$.

Since $0 < \alpha < 4$ and $\sum_{i=0}^{n} \left(\frac{\alpha}{4}\right)^i < \infty$, the Cauchy Criterion for convergence in IFNS, it shows that

$$\left\{ \frac{f_T\left(2^n x_1\right)}{4^n} \right\}; \left\{ \frac{f_L\left(2^n x_2\right)}{4^n} \right\}$$

are Cauchy sequences in (Y,μ',υ') and it is complete, this sequences converges to some point $Q_T(x_1) \in Y; Q_L(x_2) \in Y$. So, one can we define the mapping $Q_T : X \to Y; Q_L : X \to Y$ by

$$Q_T(x_1) = (\mu,\upsilon) - \lim_{n\to\infty} \frac{f_T(2^n x_1)}{4^n};$$

$$Q_L(x_2) = (\mu,\upsilon) - \lim_{n\to\infty} \frac{f_L(2^n x_2)}{4^n};$$

for all $x_1, x_2 \in X$. Letting $m = 0$ in (3.14), one can get

$$\begin{cases} \mu\left(\frac{f_T\left(2^n x_1\right)}{4^n} - f_T(x_1), t \right) \geq \mu'\left(\Lambda_T(x_1,-x_1,0), \frac{t}{\sum_{i=0}^{n-1} \frac{\alpha^i}{4^i}} \right) \\ \upsilon\left(\frac{f_T\left(2^n x_1\right)}{4^n} - f_T(x_1), t \right) \leq \upsilon'\left(\Lambda_T(x_1,-x_1,0), \frac{t}{\sum_{i=0}^{n-1} \frac{\alpha^i}{4^i}} \right) \end{cases};$$

$$\begin{cases} \mu\left(\frac{f_L\left(2^n x_2\right)}{4^n} - f_L(x_2), t \right) \geq \mu'\left(\Lambda_L(2x_2,x_2,0,0), \frac{t}{\frac{1}{4}\sum_{i=0}^{n-1} \frac{\alpha^i}{4^i}} \right) \\ \upsilon\left(\frac{f_L\left(2^n x_2\right)}{4^n} - f_L(x_2), t \right) \leq \upsilon'\left(\Lambda_L(2x_2,x_2,0,0), \frac{t}{\frac{1}{4}\sum_{i=0}^{n-1} \frac{\alpha^i}{4^i}} \right) \end{cases}; \tag{3.15}$$

for all $x_1, x_2 \in X$ and all $t > 0$. Letting $n \to \infty$ in (3.15) and using [IFNS6], one can arrive.

$$\begin{cases}\mu\left(Q_T(x_1)-f_T(x_1),t\right)\geq \mu'\left(\Lambda_T(x_1,-x_1,0),\dfrac{(4-\alpha)t}{4}\right)\\ \upsilon\left(Q_T(x_1)-f_T(x_1),t\right)\leq \upsilon'\left(\Lambda_T(x_1,-x_1,0),\dfrac{(4-\alpha)t}{4}\right)\end{cases};$$
$$\begin{cases}\mu\left(Q_L(x_2)-f_L(x_2),t\right)\geq \mu'\left(\Lambda_L(2x_2,x_2,0,0),(4-\alpha)t\right)\\ \upsilon\left(Q_L(x_2)-f_L(x_2),t\right)\leq \upsilon'\left(\Lambda_L(2x_2,x_2,0,0),(4-\alpha)t\right)\end{cases}; \qquad (3.16)$$

for all $x_1,x_2\in X$ and $t>0$.

Now, we need to prove Q_T satisfies (1.3) and Q_L satisfies (1.4), replacing $(x_1,y_1,z_1);(x_2,y_2,z_2,s)$ by $(2^n x_1,2^n y_1,2^n z_1);(2^n x_2,2^n y_2,2^n z_2,2^n s)$ in (3.3) respectively, one can obtain

$$\begin{cases}\mu\left(\dfrac{1}{4^n}\left\{f_T\left(\dfrac{2^n x_1+2^n y_1}{2}-2^n z_1\right)-\dfrac{1}{2}\left(f_T\left(2^n z_1-2^n x_1\right)+f_T\left(2^n z_1-2^n y_1\right)\right)+\dfrac{1}{4}f_T\left(2^n x_1-2^n y_1\right)\right\},t\right)\\ \qquad\qquad \geq \mu'\left(\Lambda_T\left(2^n x_1,2^n y_1,2^n z_1\right),4^n t\right)\\ \upsilon\left(\dfrac{1}{4^n}\left\{f_T\left(\dfrac{2^n x_1+2^n y_1}{2}-z_1\right)-\dfrac{1}{2}\left(f_T\left(2^n z_1-2^n x_1\right)+f_T\left(2^n z_1-2^n y_1\right)\right)+\dfrac{1}{4}f_T\left(2^n x_1-2^n y_1\right)\right\},t\right)\\ \qquad\qquad \leq \upsilon'\left(\Lambda_T\left(2^n x_1,2^n y_1,2^n z_1\right),4^n t\right)\end{cases}; \qquad (3.18)$$

$$\begin{cases}\mu\left(\dfrac{1}{4^n}\left\{f_L(2^n x_2-2^n s)+f_L(2^n y_2-2^n s)+f_L(2^n z_2-2^n s)-3f_L\left(\dfrac{2^n x_2+2^n y_2+2^n z_2}{3}-2^n s\right)\right.\right.\\ \qquad \left.\left.-f_L\left(\dfrac{2.2^n x_2-2^n y_2-2^n z_2}{3}\right)-f_L\left(\dfrac{-2^n x_2+2.2^n y_2-2^n z_2}{3}\right)-f_L\left(\dfrac{-2^n x_2-2^n y_2+2.2^n z_2}{3}\right)\right\},t\right)\\ \qquad\qquad \geq \mu'\left(\Lambda_L\left(2^n x_2,2^n y_2,2^n z_2,2^n s\right),4^n t\right)\\ \upsilon\left(\dfrac{1}{4^n}\left\{f_L(2^n x_2-2^n s)+f_L(2^n y_2-2^n s)+f_L(2^n z_2-2^n s)-3f_L\left(\dfrac{2^n x_2+2^n y_2+2^n z_2}{3}-2^n s\right)\right.\right.\\ \qquad \left.\left.-f_L\left(\dfrac{2.2^n x_2-2^n y_2-2^n z_2}{3}\right)-f_L\left(\dfrac{-2^n x_2+2.2^n y_2-2^n z_2}{3}\right)-f_L\left(\dfrac{-2^n x_2-2^n y_2+2.2^n z_2}{3}\right)\right\},t\right)\\ \qquad\qquad \leq \upsilon'\left(\Lambda_L\left(2^n x_2,2^n y_2,2^n z_2,2^n s\right),4^n t\right)\end{cases} \qquad (3.19)$$

for all $x_1,y_1,z_1,x_2,y_2,z_2,s\in X$ and all $t>0$. Now

$$\mu\left(Q_T\left(\frac{x_1+y_1}{2}-z_1\right)-\frac{1}{2}\left(Q_T\left(z_1-x_1\right)+Q_T\left(z_1-y_1\right)\right)+\frac{1}{4}Q_T\left(x_1-y_1\right)\right)$$
$$\geq \mu\left(Q_T\left(\frac{x_1+y_1}{2}-z_1\right)-\frac{1}{4^n}f_T\left(\frac{2^n x_1+2^n y_1}{2}-z_1\right),t\right)*$$
$$\mu\left(-\frac{1}{2}\left(Q_T\left(z_1-x_1\right)+Q_T\left(z_1-y_1\right)\right)+\frac{1}{4^n}\frac{1}{2}\left(f_T\left(2^n z_1-2^n x_1\right)+f_T\left(2^n z_1-2^n y_1\right)\right),t\right)*$$
$$\mu\left(\frac{1}{4}Q_T\left(x_1-y_1\right)-\frac{1}{4^n}\frac{1}{4}f_T\left(2^n x_1-2^n y_1\right),t\right)*$$
$$\mu\left(\frac{1}{4^n}\left\{f_T\left(\frac{2^n x_1+2^n y_1}{2}-2^n z_1\right)-\frac{1}{2}\left(f_T\left(2^n z_1-2^n x_1\right)+f_T\left(2^n z_1-2^n y_1\right)\right)+\frac{1}{4}f_T\left(2^n x_1-2^n y_1\right)\right\},t\right)$$

$$\upsilon\left(Q_T\left(\frac{x_1+y_1}{2}-z_1\right)-\frac{1}{2}\left(Q_T\left(z_1-x_1\right)+Q_T\left(z_1-y_1\right)\right)+\frac{1}{4}Q_T\left(x_1-y_1\right)\right)$$

$$\le\upsilon\left(Q_T\left(\frac{x_1+y_1}{2}-z_1\right)-\frac{1}{4^n}f_T\left(\frac{2^n x_1+2^n y_1}{2}-z_1\right),t\right)*$$

$$\upsilon\left(-\frac{1}{2}\left(Q_T\left(z_1-x_1\right)+Q_T\left(z_1-y_1\right)\right)+\frac{1}{4^n}\frac{1}{2}\left(f_T\left(2^n z_1-2^n x_1\right)+f_T\left(2^n z_1-2^n y_1\right)\right),t\right)*$$

$$\upsilon\left(\frac{1}{4}Q_T\left(x_1-y_1\right)-\frac{1}{4^n}\frac{1}{4}f_T\left(2^n x_1-2^n y_1\right),t\right)*$$

$$\upsilon\left(\frac{1}{4^n}\left\{f_T\left(\frac{2^n x_1+2^n y_1}{2}-2^n z_1\right)-\frac{1}{2}\left(f_T\left(2^n z_1-2^n x_1\right)+f_T\left(2^n z_1-2^n y_1\right)\right)+\frac{1}{4}f_T\left(2^n x_1-2^n y_1\right)\right\},t\right)$$

Letting $n\to\infty$ in (3.18) and using (IFN7), (IFN13), (IFN3), (IFN9), we can see that Q_T satisfies (1.3). Similarly, we can prove the another result.

To prove the uniqueness of $Q_T:X\to Y; Q_L:X\to Y$, assume there exists a mapping $Q_T':X\to Y$ satisfies (1.3), $Q_L':X\to Y$ satisfies (1.4) and (3.4). Hence,

$$\begin{cases}\mu(Q_T(x_1)-Q_T'(x_1),t)\ge\mu\left(\frac{Q_T(2^n x_1)}{4^n}-\frac{f_T(2^n x_1)}{4^n},\frac{t}{2}\right)*\mu\left(\frac{f_T(2^n x_1)}{4^n}-\frac{Q_T'(2^n x_1)}{4^n},\frac{t}{2}\right)\\ \qquad\ge\mu'\left(\Lambda_T(x_1,-x_1 0),\frac{(4-\alpha)t}{8.4^n}\right)*\mu'\left(\Lambda_T(x_1,-x_1 0),\frac{(4-\alpha)t}{8.4^n}\right)\\ \upsilon(Q_T(x_1)-Q_T'(x_1),t)\le\upsilon\left(\frac{Q_T(2^n x_1)}{4^n}-\frac{f_T(2^n x_1)}{4^n},\frac{t}{2}\right)*\upsilon\left(\frac{f_T(2^n x_1)}{4^n}-\frac{Q_T'(2^n x_1)}{4^n},\frac{t}{2}\right)\\ \qquad\le\upsilon'\left(\Lambda_T(x_1,-x_1 0),\frac{(4-\alpha)t}{8.4^n}\right)*\upsilon'\left(\Lambda_T(x_1,-x_1 0),\frac{(4-\alpha)t}{8.4^n}\right)\end{cases}$$

$$\begin{cases}\mu(Q_L(x_2)-Q_L'(x_2),t)\ge\mu\left(\frac{Q_L(2^n x_2)}{4^n}-\frac{f_L(2^n x_2)}{4^n},\frac{t}{2}\right)*\mu\left(\frac{f_L(2^n x_2)}{4^n}-\frac{Q_L'(2^n x_2)}{4^n},\frac{t}{2}\right)\\ \qquad\ge\mu'\left(\Lambda_L(2x_2,x_2,0,0),\frac{(4-\alpha)t}{2.4^n}\right)*\mu'\left(\Lambda_L(2x_2,x_2,0,0),\frac{(4-\alpha)t}{2.4^n}\right)\\ \upsilon(Q_L(x_2)-Q_L'(x_2),t)\le\upsilon\left(\frac{Q_L(2^n x_2)}{4^n}-\frac{f_L(2^n x_2)}{4^n},\frac{t}{2}\right)*\upsilon\left(\frac{f_L(2^n x_2)}{4^n}-\frac{Q_L'(2^n x_2)}{4^n},\frac{t}{2}\right)\\ \qquad\le\upsilon'\left(\Lambda_L(2x_2,x_2,0,0),\frac{(4-\alpha)t}{2.4^n}\right)*\upsilon'\left(\Lambda_L(2x_2,x_2,0,0),\frac{(4-\alpha)t}{2.4^n}\right)\end{cases}\tag{3.20}$$

for all $x_1,x_2\in X$ and all $t>0$. Since,

$$\lim_{n\to\infty}\frac{(4-\alpha)t}{8.4^n}=\infty;\ \lim_{n\to\infty}\frac{(4-\alpha)t}{2.4^n}=\infty$$

for all $t>0$ and by [IFNS5] and [IFNS13], one can arrive

$$\begin{cases}\lim_{n\to\infty}\mu'\left(\Lambda_T(x_1,-x_1 0),\frac{(4-\alpha)t}{8.4^n}\right)=1\\ \lim_{n\to\infty}\upsilon'\left(\Lambda_T(x_1,-x_1 0),\frac{(4-\alpha)t}{8.4^n}\right)=0\end{cases};$$

$$\begin{cases}\mu'\left(\lim_{n\to\infty}\Lambda_L(2x_2,x_2,0,0),\frac{(4-\alpha)t}{2.4^n}\right)=1\\ \upsilon'\left(\lim_{n\to\infty}\Lambda_L(2x_2,x_2,0,0),\frac{(4-\alpha)t}{2.4^n}\right)=0\end{cases};$$

for all $x_1,x_2\in X$ and all $t>0$. Therefore

$$\begin{cases}\mu\left(Q_T(x_1)-Q_T'(x_1),t\right)=1\\ \upsilon\left(Q_T(x_1)-Q_T'(x_1),t\right)=0\end{cases};$$

$$\begin{cases}\mu\left(Q_L(x_2)-Q_L'(x_2),t\right)=1\\ \upsilon\left(Q_L(x_2)-Q_L'(x_2),t\right)=0\end{cases};$$

for all $x_1,x_2\in X$ and all $t>0$.Hence,

$$Q_T(x_1)=Q_T'(x_1);\ Q_L(x_2)=Q_L'(x_2),\ \forall\ x_1,x_2\in X.$$

Therefore $Q_T(x_1);\ Q_L(x_2)$ are unique. So, the proof holds for $\beta=1$.

Now, (x_1,x_2) by $\left(\frac{x_1}{2},\frac{x_2}{2}\right)$ in (3.6) and using [IFNS4], one can get

$$\begin{cases}\mu\left(f_T(x_1)-4f_T\left(\frac{x_1}{2}\right),4t\right)\ge\mu'\left(\Lambda_T\left(\frac{x_1}{2},-\frac{x_1}{2},0\right),t\right)\\ \upsilon\left(f_T(x_1)-4f_T\left(\frac{x_1}{2}\right),4t\right)\le\upsilon'\left(\Lambda_T\left(\frac{x_1}{2},-\frac{x_1}{2},0\right),t\right)\end{cases};$$

$$\begin{cases}\mu\left(f_L(x_2)-4f_L\left(\frac{x_2}{2}\right),t\right)\ge\mu'\left(\Lambda_L\left(x_2,\frac{x_2}{2},0,0\right),t\right).\\ \upsilon\left(f_L(x_2)-4f_L\left(\frac{x_2}{2}\right),t\right)\le\upsilon'\left(\Lambda_L\left(x_2,\frac{x_2}{2},0,0\right),t\right)\end{cases};\qquad(3.21)$$

for all $x_1,x_2\in X$ and all $t>0$. The rest of the proof is analogous to that of prior case. So, the proof holds for $\beta=-1$. This concludes the proof of the theorem.

Corollary 3.2. Assume $f_T:X\to Y;\ f_L:X\to Y$ are functions satisfying the inequalities

$$\begin{cases}\mu\left(\Delta f_T(x_1,y_1,z_1),t\right)\ge\begin{cases}\mu'\left(G,t\right)\\ \mu'\left(G(|x_1|^H+|y_1|^H+|z_1|^H),t\right)\end{cases}\\ \upsilon\left(\Delta f_T(x_1,y_1,z_1),t\right)\le\begin{cases}\upsilon'\left(G,t\right)\\ \upsilon'\left(G(|x_1|^H+|y_1|^H+|z_1|^H),t\right)\end{cases}\end{cases};$$

$$\begin{cases}\mu\left(\Delta f_L(x_2,y_2,z_2,s),t\right)\ge\begin{cases}\mu'\left(G,t\right)\\ \mu'\left(G(|x_2|^H+|y_2|^H+|z_2|^H+|s|^H),t\right)\end{cases}\\ \upsilon\left(\Delta f_L(x_2,y_2,z_2,s),t\right)\le\begin{cases}\upsilon'\left(G,t\right)\\ \upsilon'\left(G(|x_2|^H+|y_2|^H+|z_2|^H+|s|^H),t\right)\end{cases}\end{cases};\qquad(3.22)$$

for all $x_1,y_1,z_1,x_2,y_2,z_2,s\in X$ and all $t>0.$ Then there exists a unique quadratic mappings $Q_T:X\to Y;\ Q_L:X\to Y$ satisfying (1.7) such that

$$\begin{cases}\mu\left(Q_T(x_1)-f_T(x_1),t\right)\ge\begin{cases}\mu'\left(G,\frac{|3|\,t}{4}t\right)\\ \mu'\left(G(3|x_1|^H,\frac{|4-2^H|t}{4}t\right);H\ne1\end{cases}\\ \upsilon\left(Q_T(x_1)-f_T(x_1),t\right)\le\begin{cases}\upsilon'\left(G,\frac{|3|t}{4}t\right)\\ \upsilon'\left(G(3|x_1|^H,\frac{|4-2^H|t}{4}t\right);H\ne1\end{cases}\end{cases};$$

$$\begin{cases} \mu\left(Q_L(x_2) - f_L(x_2), t\right) \geq \begin{cases} \mu'\left(G, |3|t\right) \\ \mu'\left(G((2^H+1)\,|\,x_2\,|^H, \left|4-2^H\right|t\right); H \neq 1 \end{cases} \\ \upsilon\left(Q_L(x_2) - f_L(x_2), t\right) \leq \begin{cases} \upsilon'\left(G, |3|t\right) \\ \upsilon'\left(G((2^H+1)\,|\,x_2\,|^H, \left|4-2^H\right|t\right); H \neq 1 \end{cases} \end{cases}; \qquad (3.23)$$

for all $x_1, x_2 \in X$ and all $t > 0$.

VI CONCLUSIONS

In this paper, we establish the geometrical mathematical properties convert into quadratic functional equations and analyzed the generalized Ulam-Hyers stability of the functional equations in intuitionistic fuzzy Banach space using Hyers direct method.

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