**OPITMALITY CONDITIONS WITH CONSTRAINT QUALIFICATIONS FOR NON LINEAR PROGRAMMING PROBLEMS**

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**Abstract:** The theme of this paper is on optimality conditions for solving nonlinear programming problems (NLPP). We will be discussing some definitions of the cone approach on the Karush Kuhn Tucker optimality conditions and Constraint qualification with the help of well-known Farkas’ lemma. We will be discussing in this paper some of them constraint qualification and observe the weakest of these constraint qualification with respect to the concept of cones and they polars.

**Key Words:** Polar, Cones, The Tangent Cone, Optimality Conditions, Karush-Kuhn-Tucker, Constraint qualifications.

**Introduction:**

 In mathematics, Non Linear Programming (NLP) is the process of solving an optimization problem defined by a system of equalities and inequalities, collectively termed constraints, over a set of unknown real variables, along with an objective function to be maximized or minimized, where some of the constraints or the objective functions are nonlinear. In optimization, the Karush – Kuhn – Tucker (KKT) conditions (also known as Kuhn – Tucker – Conditions) are first order necessary conditions for a solution in nonlinear programming to be optimal, provided that some regularity conditions are satisfied. Allowing inequality constraints, the KKT approach to nonlinear programming generalizes the method of Lagrange multipliers, which allows only equality constraints. The system of equations corresponding to the KKT conditions is usually not solved directly, except in the few special case where a closed – form solution can be derived analytically. In general, many optimization algorithms can be interpreted as methods for numerically solving the KKT system of equations. The KKT conditions were originally named after Harold W. Kuhn, and Albert W. Tucker, who first published the conditions Later scholars discovered that that the necessary conditions for this problem had been stated by William Karush in his master’s thesis and Kjeldsen (2005). Under differentiability and constraint qualifications, the KKT conditions provide necessary conditions for a solution to be optimal. Under convexity, these conditions are also sufficient. If some of the functions are non – differentiable, sub differential versions of KKT conditions are available.

Consider the standard NLP: Min(P): f(𝑥)

 Subject to 𝑔I ( 𝑥) ≤0,∀ i=1,2,…,𝑝

 hi (𝑥) =0,∀ 𝑖=1,2,…,𝑚

Where the functions 𝑓 (𝑥) :Rn⟶R

𝑔i :Rn⟶Rp 𝑎𝑛𝑑

hi :Rn⟶Rm , they are conditionally differentiable

The feasible set of problem (P) will be denoted by Ω i.e.

Ω={ 𝑥∈Rn: 𝑔i 𝑥 ≤0 𝑎𝑛𝑑 hi 𝑥 =0 }

The classical KKT condition at given 𝑥∈Ω, and then there exists Lagrangian multipliers

𝜆∈Rm 𝑎𝑛𝑑 𝜇∈Rp Such that

∇𝑓 (𝑥) + $\sum\_{i=1}^{m}λi ∇h\_{i} (\overline{x}) $+ $\sum\_{j=1}^{p}μ\_{j}$∇$g\_{j}$($\overline{x}$) =0

𝑤h𝑒𝑟𝑒

 𝜇j≥0

𝜇j.𝑔j ($\overline{x})$ =0

 In order to have an optimal solution to the given NLP problem, the KKT necessary condition has to be satisfied. The above optimality criteria has been used to formulate algorithms that solve (P) in the presence of any constraint qualification.

If the problem is unconstraint, then the KKT conditions reduces to ∇𝑓 ($\overline{x}$) =0, which is a necessary optimality condition; however, this will not always be true, see the following counter example:

**COUNTER EXAMPLE:** Consider the problem (P) with h(𝑥) :R2⟶R; 𝑔 (𝑥) :R2⟶R2 𝑓𝑜𝑟 𝑗=1,2 defined by h( 𝑥 )=−𝑥1 , and

 𝑔 (𝑥) = [𝑥2− (1−𝑥1)3, −𝑥2] T

Note that 𝑥∗= (1,0)T , it is minimizers of the problem but the KKT condition do not hold In this paper we discuss the KKT condition supposing the equality between the polar of the tangent cone and the polar of the first order feasible variations. Although this condition is the weakest assumption, it is extremely difficult to be verified. Therefore, other constraint qualifications, which are easier to be verified, will be discussed as: Slater’s Linear independence of Gradient, Mangasarian – Fromovitz’s and quasi – regularity. In general, we call a property of the feasible set a constraint qualification if it guarantees the KKT conditions to hold at a local minimizer. Several mathematicians obtained different constraint qualifications. In this research, we will discuss many of them. A special interest is devoted to show the weakest such qualification.

**Notation:** Given 𝑥∈Ω, and the set ($\overline{x}$ ) it denotes the set of inequality active constraint indices i.e.

A($\overline{x}$ ) ={i: 𝑔i ($\overline{x}$) =0,1≤i≤𝑝}

**Definition – 1:** A sub set 𝑆 𝑜𝑓 Rn, it is called a **convex set** if two points 𝑥,∈𝑋 𝑎𝑛𝑑 𝜆∈[0,1], such that (1−𝜆) 𝑥+𝜆𝑦∈𝑆

**Definition – 2:** A subset 𝐶 𝑜𝑓 Rn, it is a **cone** when 𝑡𝑑∈𝐶, 𝑎𝑙𝑙 𝑡≥0 𝑎𝑛𝑑 𝑑∈𝐶

**Definition – 3:** Given a set 𝑆⊂Rn, the **polar of** 𝑺, it is given by 𝑃 (𝑆) = {𝑝∈Rn /𝑝T 𝑥≤0,𝑜𝑟 𝑎𝑙𝑙 𝑥∈𝑆}

**Farkas’ Lemma:** Let S⊂Rn, it is a closed convex cone, and then 𝑃 (𝑃 (S)) =S

**Definition – 4:** Let 𝑆 be a nonempty set in Rn, and let 𝑥∈Ω. The **cone of feasible direction** of 𝑆 𝑎𝑡 𝑥 is denoted by D, and given by D = {$\vec{d}$∈Rn : $\vec{d}$≠ 0 , $\overline{x}$+𝜆$\vec{d }$∈𝑆,fo𝑟 𝑎𝑙𝑙 𝜆∈ (0,𝛿 )𝑓𝑜𝑟 𝑠𝑜𝑚𝑒 𝛿>0}

Each nonzero vector $\vec{d}$∈D, is called a cone of **feasible direction**.

**Definition – 5:** Given a function 𝑓:Rn⟶R, **the cone of descent directions** at 𝑥 it is denoted by 𝐹, and given by 𝐹 ($\overline{x}$) =$\{\vec{d}$ ∈Rn : 𝑓 ($\overline{x}$+𝜆$\vec{d }$) < f($\overline{x}$)𝑎𝑙𝑙 𝜆∈ (0,𝛿 )𝑓𝑜𝑟 𝑠𝑜𝑚𝑒 𝛿>0}

Each direction $\vec{d}$∈𝐹, it is called a **descent direction** of 𝑓 𝑎𝑡 $\overline{x}$

**Lemma :** Let 𝑓:Rn⟶R, it is a differentiable function at a point 𝑥∈Rn, and then

a. ∇𝑓 ($\overline{x}$)T𝑑≤0 ,𝑓𝑜𝑟 𝑎𝑙𝑙 𝑑∈ 𝐹($\overline{x}$ )

b. If $\vec{d}$∈Rn satisfies ∇𝑓 (𝑥)T$\vec{d}$<0 , 𝑡h𝑒𝑛 $\vec{d}$∈ 𝐹 ($\overline{x}$) , we get the set and denoted by

F0( $\overline{x}$) = $\vec{\{d}$∈Rn :∇𝑓 (𝑥)T$d<$0}

 **Definition – 6:** The **set of first order feasible variations** at a point $\overline{x}$∈Ω, it is the set

D($\overline{x}$ ) = {𝑑∈Rn: ∇h𝑖 ($\overline{x}$)T𝑑=0,∀ 𝑖=1,…,𝑚 𝑎𝑛𝑑 ∇𝑔i $\overline{(x}$ )T 𝑑≤0,∀ i∈𝐴($\overline{x}$ )} 𝑤h𝑒𝑟𝑒 𝐴 ($\overline{x})$ , is the active set

 Note that D ( $\overline{x}$ ) , it is a nonempty closed, convex cone; called linear approximation of the feasible set.

 Again, given $\overline{x}$∈Ω, define the cone

𝐺 ($\overline{x}$) = { $\sum\_{i=1}^{m}λi ∇h\_{i} (\overline{x}) $+ $\sum\_{j\in A(\overline{x}) }^{}μj$∇$g\_{j}$($\overline{x}$)/ 𝜇j≥ 0 $∀ j\in A(\overline{x} )\} $

 **The Tangent Cone:**

**Definition – 7:** A vector 𝑑∈Rn, it is called tangent direction to Ω⊂Rn 𝑜𝑚 $\overline{x}$∈Ω, when either 𝑑=0, or there exists a sequence of feasible points 𝑥k ⊂Ω 𝑠𝑢𝑐h 𝑡h𝑎𝑡 𝑥k⟶$\overline{x}$, and also

 $x^{k}$−𝑥/$‖x^{k}-x‖$ ⟶𝑑 /$‖d‖$

Clearly, the set T($\overline{x}$ ) of the tangent directions to Ω 𝑓𝑟𝑜𝑚$\overline{ x}$ , it is a cone. This set is said to be a tangent cone.

**Optimality Conditions and Constraint Qualifications:**

In this topic we prove the KKT theorem assuming the weakest qualification condition and discuss other ones easier to be verified, supposes that the objective function increases along tangent direction.

**Constraint Qualifications:**

The Kuhn – Tucker conditions are only if some regularity conditions are satisfied. These conditions are called the constraint qualification which imposes a certain restriction on constraint functions of a Nonlinear Programming problem for the specific purpose of ruling out certain irregularities on the boundary of the feasible set that would available KKT conditions should be the optimal solution occurs there.

**Quasi – regularity constraint Qualification:** We say that the quasi – regularity constraint qualification is satisfied at $\overline{x}$ 𝑤h𝑒𝑛 𝑇 ( $\overline{x)}$=𝐷( $\overline{x}$)Where 𝑇 ($\overline{x})$𝑎𝑛𝑑 𝐷 ( ,$\overline{x})$ they are the tangent cone and the set of first order feasible variations cone at $\overline{x}.$

**Note:** These conditions are not equivalent

 for example Consider the functions h (𝑥) :R2⟶R; 𝑔 (𝑥) :R2⟶R 𝑓𝑜𝑟 𝑗=1,2 defined by h(𝑥) =x1x2 ,and 𝑔 (𝑥) : =−x1 - x2 , and the feasible point$ \overline{x}$ = (0,0)T

it is easy to see that

𝑇 ($\overline{x})$ = {(𝑑1,d2) ∈R2 𝑑1≥0,𝑑2≥0 𝑎𝑛𝑑 𝑑1𝑑2=0} , and also

𝐷 ($\overline{x}$) = {(𝑑1,d2 )∈R2  𝑑1≥0,𝑑2≥0 },

and also 𝑃 (𝐷 $\overline{(x)}$) =𝑃 (𝑇 $(\overline{x})$ ) = {𝑑1,𝑑2 ∈R2 : 𝑑1≤0,𝑑2≤0}

 **Slater Constraint Qualification:** Regarding the problem (P), we say that the Slater constraint qualification holds if h is linear and 𝑔 is convex and then there exists 𝑥 ∈Ω, such that h$(\overline{x}$ ) =0 𝑎𝑛𝑑 𝑔 ($\overline{x})$ <0 The Slater constraint is, in fact, a constraint qualification.

**Note: T**he Slater conditions hold and T ($\overline{x}$ ) =𝐷 ($\overline{x)}$,𝑜𝑟 𝑎𝑙𝑙 𝑥 ∈Ω

 **Linear Independence Constraint Qualification – LICQ:** This is the most known constraint qualification and states that the equality constraints gradients ∇h𝑖 ($\overline{x)}$𝑓𝑜𝑟 𝑖=1,…, 𝑚, and the active inequality constraint gradients ∇𝑔i ($\overline{x)}$ 𝑓𝑜𝑟 i∈𝐴($\overline{x})$, they are linearly independent. Although easy to check, this condition is a very strong assumption .

**For example**, consider

Min 𝑓 (𝑥) = (𝑥1−3)2+ (x2−2)2

 Subject to 𝑔I ( x) =2𝑥1+𝑥2−6≤0

 𝑔2 (𝑥) =𝑥1+2𝑥2−6≤0

 In this problem we have ∇f(x)=[ 2 (𝑥1−3) ,2( 𝑥2−2)] ; ∇𝑔1( 𝑥) = [2,1] ,∇𝑔2 (𝑥) = [1,2]

 ⟹The gradients of 𝑔, they are linearly independent.

We consider the following cases:

Case – I: For 𝐴 ($\overline{x}$ ) =∅

From KKT conditions we get both 𝜆1=0 𝑎𝑛𝑑 𝜆2=0 𝑡h𝑒𝑛

∇f($x)$ = [2 (𝑥1−3) ,2 (𝑥2−2)] =0

⟹𝑥1=3 𝑎𝑛𝑑 𝑥2=2

But 2𝑥1+𝑥2−6=2≰0,.𝑒.𝑥, it is not feasible.it cannot be a local minimum.

 Case – II: For $A(\overline{x}$) = 1

From KKT conditions we get 𝜆2=0 𝑡h𝑒𝑛

∇𝑓 ( x$)$+𝜆1∇𝑔1(x$)$ = [2 (𝑥1−3) ,2 (𝑥2−2)] +𝜆1 [2,1] =0

⟹ x 1=3 −𝜆1 𝑎𝑛𝑑 𝑥2=2 − $\frac{λ1}{2}$ and the assumption

𝑔1( 𝑥) =0, it gives theses 𝑥1 𝑎𝑛𝑑 𝑥2, as follows

2(3 −𝜆1 ) + (2 − $\frac{λ1}{2}$ ) – 6 = 0 ⟹𝜆1=$\frac{4}{5}$

⟹𝜆1 𝑎𝑛𝑑 𝜆2 Satisfies KKT condition and then we get

𝑥1=3 −4/5=11/5 𝑎𝑛𝑑 𝑥2=2 −1/2 (4/5) =8/5

Finally 𝑔2( 𝑥 )=11/5+2 (8/5) −6=−3/5≤0, and hence KKT conditions are satisfied.

 Case – III: For 𝐴 ( $\overline{x})$= 2

From KKT conditions we get 𝜆2=0 𝑡h𝑒𝑛 ∇𝑓 (𝑥) +𝜆2∇𝑔2 (𝑥 )= [2 (𝑥1−3) ,2 (𝑥2−2 )]+𝜆2 [1,2 ]=0 ⟹𝑥1=3 −𝜆2/2 𝑎𝑛𝑑 𝑥2=2 −𝜆2 , and the assumptions 𝑔2 (𝑥) =0, it gives 𝑥1 𝑎𝑛𝑑 𝑥2, as follows

(3 −𝜆2/2 )+2 (2 −𝜆2 )−6=0⟹𝜆2=2/5

⟹𝜆1 𝑎𝑛𝑑 𝜆2 Satisfies KKT condition and then we get 𝑥1=3 −1/5=14/5 𝑎𝑛𝑑 𝑥2=2 – 2/5 =8/5

But 𝑔1( 𝑥) =2 (14/5 )+8/5−6=1/5>0,

 this is violated the condition of 𝑔1( 𝑥) ≤0

Case – IV: For 𝐴 ($\overline{x})$ = 1,2

Finally 𝑔1( 𝑥) =2𝑥1+𝑥2−6 𝑎𝑛𝑑 𝑔2( 𝑥) =𝑥1+2𝑥2−6 , gives that 𝑥1=2=𝑥2

 But KKT condition ∇( 𝑥 )+𝜆1∇𝑔1 (𝑥) +𝜆2∇𝑔2( 𝑥) =0

 ⟹[ 2 (𝑥1−3) ,2 (𝑥2−2)] +𝜆1 [2 ,1] +𝜆2 [1,2 ]=0 ,

gives with theses 𝑥1 𝑎𝑛𝑑 𝑥2, as follows

 −2+2𝜆1+𝜆2=0 𝑎𝑛𝑑 𝜆1+2𝜆2=0

⟹𝜆1=4/3 𝑎𝑛𝑑 𝜆2=−2/3, and since 𝜆1>0 𝑎𝑛𝑑 𝜆2<0.

 It does not satisfies KKT condition .

**Note:** Many problems satisfy KKT conditions without LICQ,

for example with 𝑥\*=0

Consider Min 𝑓 (x )=𝑥2

Subject to 𝑔1( 𝑥) =(𝑥1)2+𝑥2≤0

𝑔2( 𝑥) =−𝑥2≤0

Clearly it satisfies KKT conditions without LICQ.

Ma**ngasarian – Fromovitz’s Constraint Qualification – MFCQ:** Another well known condition which ensures KKT is due to Mangasarian – Fromovitz’s. We say that MFCQ holds at $\overline{ x} ,$when the equality constraint gradients are linearly independent and there exists a vector 𝑑∈Rn such that

 ∇h𝑖 ($\overline{x}$)T 𝑑=0 𝑓𝑜𝑟 𝑖=1,…𝑛 𝑎𝑛𝑑 ∇𝑔i($\overline{x}$ )Td≤0 𝑓𝑜𝑟 𝑎𝑙𝑙 i∈𝐴 ($\overline{x}$ ).

The best known necessary optimality criterion for a mathematical programming problem is the KKT optimality conditions however, the above MFCQ condition is in a sense more general. In order for the KKT conditions to hold, one must impose a constraint qualification on the constraints of the problem. On the other hand, no such qualification need be imposed on the constraints in order that the MFCQ to hold. Moreover, the MFCQ itself can be used to drive a form of the constraint qualification for KKT conditions.

**Conclusion:** In this paper we observe that, KKT optimality condition for NLP has been proved by lagrange Multipliers conditions. Such conditions are called constraint qualifications which have been investigated extensively. Some of them were discussed as: Slater, Linear Independence Gradients (LICQ), Mangasarian – Fromovitz’s (MFCQ), and Quasi – Regularity conditions.

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