BANACH CONTRACTION IN ORTHOGONAL RECTANGULAR b - METRIC SPACES

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Abstract. Using the notion of the orthogonal sets, we introduce the idea of orthogonal contraction in rectangular b-metric space (RbMS). Furthermore, we prove a Banach contraction principle for the purposed contraction.Our results generalize and improve the results of Gordji et al.[27] and many well known results given by some authors in RbMS.

Keywords: Fixed points (FP), orthogonal rectangular b-metric space, orthogonal set, metric space (MS).

1. Introduction

In the past few decades, fixed point theory has been effectively used to investigate a broad range of scientific subjects, bridging pure and practical approaches and even tackling highly relevant computing challenges. In particular, fixed point theory has been created for several applications, such as the study and calculation of integral equation solutions, game theory, physics, engineering, computer science, neural networks, and models in economics and related subjects. Metric FPT depends on the concept of a MS. In mathematical analysis, the most fundamental FP results is the well-known Banach contraction principle (BCP) .One helpful method for demonstrating the existence and uniqueness of solutions for different numerical models is the FP hypothesis.. The task of finding a point $x \in \mathcal{Y}$ such that $\phi(x) = x$ is considered a FP problem. Given a nonempty set $\mathcal{Y}$ and a map ϕ from $\mathcal{Y}$ into itself. The point $x \in \mathcal{Y}$ is referred to as a FP of ϕ.

The literature contains numerous generalizations of the idea of a MS. The concept of RMS given by Branciari [2] , and proved an equivalent of the BCP in such a space. Many FPT for different contractions on rectangular metric space have now been discovered (see [3],[4],[5],[6],[9],[8],[10],[11],[12],[13],[14]).

Bakhtin [15] established b-MS as a MS generalization and demonstrated the analogue of the BCP in b-MS. Many FPT are proved in b-MS. (see [16],[17],[20],[18], [19],[21], [22],[23] and the references therein).

The notion of RbMS, which was not always Hausdorff and generalized the ideas of MS, RMS, and b-MS, was first presented by George et al.[30] . He also demonstrated Kannan's and Banach's FPT for RbMS. The notion of orthogonal sets was recently given by Eshaghi Gordji et al.[28] , who also provided an extension of the BCP. They also provided applications of their findings to guarantee the uniqueness and existence of solutions to differential equations of the first order. His study aims to extend the notion of an orthogonal contraction in the context ofMS, as introduced by Gordji et al. [27]. We presented the notion of an ORbMS and establish some FPT for Banach contractions.

2. Preliminaries

Bakhtin [15] and Czerwikas [19] first proposed a b-MS in the following manner.

Definition 2.1. [19] If $\mathcal{Y} \neq \emptyset$ and $s \geq 1$. Consider $\rho : \mathcal{Y} \times \mathcal{Y} \to [0, \infty)$ fulfill the given conditions $\forall\ \varpi, \varsigma, \varrho \in \mathcal{Y}$:

(1) $\rho(\varpi, \varsigma) = 0$ iff $\varpi = \varsigma$
(2) $\rho(\varpi, \varsigma) = \rho(\varsigma, \varpi)$
(3) $\rho(\varpi, \varsigma) \leq s[\rho(\varpi, \varrho) + \rho(\varrho, \varsigma)]$

Then $(\mathcal{Y}, \rho)$ is called b-MS with coefficient s.

Definition 2.2. [29] If $\mathcal{Y} \neq \emptyset$ and $\rho : \mathcal{Y} \times \mathcal{Y} \ \to [0, \infty)$ fulfills:

(1) $\rho(\varpi, \varsigma) = 0 \iff \varpi = \varsigma\ \forall\ (\varpi, \varsigma \in \mathcal{Y};)$
(2) $\rho(\varpi, \varsigma) = \rho(\varsigma, \varpi)$ for all $\varpi, \varsigma \in \mathcal{Y}$;
(3) $\rho(\varpi, \varsigma) \leq \rho(\varpi, r) + \rho(r, s) + \rho(s, \varsigma)$ for all $\varpi, \varsigma \in \mathcal{Y}$ and all distinct points $r, s \in \mathcal{Y}\ \{\varpi, \varsigma\}$.

Then $(\mathcal{Y}, \rho)$ is called a RMS.

We define a RbMS as follows:

Definition 2.3. [29] If $\mathcal{Y} \neq \emptyset$ and $\rho : \mathcal{Y} \times \mathcal{Y} \ \to [0, \infty)$ satisfies:

(1) $\rho(\varpi, \varsigma) = 0 \iff \varpi = \varsigma\ \forall\ \varpi, \varsigma \in \mathcal{Y}$;
(2) $\rho(\varpi, \varsigma) = \rho(\varsigma, \varpi)$ for all $\varpi, \varsigma \in \mathcal{Y}$;
(3) $\exists\ s \geq 1$ s.t. $\rho(\varpi, \varsigma) \leq s[\rho(\varpi, p) + \rho(p, q) + \rho(q, \varsigma)]$ for all $\varpi, \varsigma \in \mathcal{Y}$ and all distinct points $p, q \in \mathcal{Y} \diagdown \{\varpi, \varsigma\}$.

Then $(\mathcal{Y}, \rho)$ is called a RbMS.

Gordji et al. [28] presented the notion of the orthogonal set as follows:

Definition 2.4. [27] Consider a set $\mathcal{Y} \neq \phi$ and a binary relation $\perp \subseteq \mathcal{Y} \times \mathcal{Y}$. Then $(\mathcal{Y}, \perp)$ referred as an orthogonal set if following criterion is satisfied $\forall\ \varsigma \in \mathcal{Y}\ \exists\ \varpi_0$ such that $(\varsigma \perp \varpi_0)$ or $(\varpi_0 \perp \varsigma)$, where ϖ_0 is orthogonal element.

Definition 2.5. [27] Consider a set $\mathcal{Y} \neq \phi$ and a binary relation $\perp \subseteq \mathcal{Y} \times \mathcal{Y}$. Any two elements from $\mathcal{Y}$ are orthogonally connected if $\varpi, \varsigma \in \mathcal{Y}$ such that $\varpi \perp \varsigma$.

Definition 2.6. [27] Consider $\mathcal{Y} \neq \phi$ and $(\mathcal{Y}, \perp)$ is O-set then,

(i) a sequence $\{\varpi_m\}$ is known as an orthogonal sequence. if, $\varpi_m \perp \varpi_{m+1}$ or $\varpi_{m+1} \perp \varpi_m, \forall\ m \in \mathbb{N}$;
(ii) similarly, a sequence $\{\varpi_m\}$ is known as Cauchy orthogonal sequence if,

$$\varpi_m \perp \varpi_{m+1} \text{ or } \varpi_{m+1} \perp \varpi_m, \forall\ m \in \mathbb{N};$$

Definition 2.7. [27] Consider that $(\mathcal{Y}, \perp, \rho)$ is an O - MS. Then

(i) $(\mathcal{Y}, \perp, \rho)$ is complete O-MS if every Cauchy O-sequence is converges in $\mathcal{Y}$
(ii) And completeness of metric space imply O-completeness but inverse isn't really true.

Definition 2.8. [27]Consider $(\mathcal{Y}, \perp, \rho)$ be an O-MS. Then

(i) a mapping $T : \mathcal{Y} \to \mathcal{Y}$ is known as O-continuous if for each O- sequence $\{\varpi_m\}_{m \in \mathbb{N}} \to \varpi \Rightarrow T(\varpi_m) \to T(\varpi)$ as $m \to \infty$.
(ii) O-continuity is relatively weak than classical continuity in classical metric spaces.

Definition 2.9. [27] Consider $\mathcal{Y} \neq \phi$ and a pair $(\mathcal{Y}, \perp)$ be an O-set. Any mapping $T : \mathcal{Y} \to \mathcal{Y}$ is weakly $\perp$-preserving if $T(\varpi) \perp T(\varsigma)$ or $T(\varsigma) \perp T(\varpi)$ whenever $\varpi \perp \varsigma$ and $\perp$-preserving if $T(\varpi) \perp T(\varsigma)$ whenever $\varpi \perp \varsigma$

3. Main Results

Theorem 3.1. Consider $(\mathcal{Y}, \rho, \perp)$ be a O-complete RbMS with coefficient $s \geq 1$ and suppose that $T : \mathcal{Y} \to \mathcal{Y}$ be $\perp$-continuous and $\perp$-preserving satisfying :

$$\rho(T\varpi, T\varsigma) \leq \alpha\rho(\varpi, \varsigma) \tag{3.1}$$

for all $\varpi, \varsigma \in \mathcal{Y}$, where α are nonnegative constants with $\alpha < 1$. Then T has a unique FP.

Proof. Let $\varpi_0 \in \mathcal{Y}$ be an orthogonal element in $\mathcal{Y}$,then by definition

$$(\forall\varsigma \in \mathcal{Y}, \varsigma \perp \varpi_0) \ or \ (\forall\varsigma \in \mathcal{Y}, \varpi_0 \perp \varsigma).$$

It follow that $(\varpi_0 \perp T(\varpi_0))$ or $(T(\varpi_0) \perp \varpi_0)$. Let

$$\varpi_1 = T(\varpi_0), \quad \varpi_2 = T(\varpi_1) = T^2(\varpi_0), \quad \varpi_{v+1} = T(\varpi_v) = T^{v+1}(\varpi_0), \quad \forall\ v \in \mathbb{N}$$

Since T is $\perp$ preserving, $\{\varpi_v\}$ is an O-sequence.
Setting $\rho_v = \rho(\varpi_v, \varpi_{v+1})$. From (1), it follows that

$$\rho(\varpi_v, \varpi_{v+1}) = \rho(T\varpi_{v-1}, T\varpi_v) \leq \alpha\rho(\varpi_{v-1}, \varpi_v)$$

i.e.

$$\rho(\varpi_v, \varpi_{v+1}) \leq \alpha\rho(\varpi_{v-1}, \varpi_v)$$

$$\rho_v \leq \alpha\rho_{v-1}.$$

By going through this process again, we get

$$\rho_v \leq \alpha^v \rho_0 \tag{3.2}$$

Suppose that ϖ_0 is not a periodic point of T. If $\varpi_0 = \varpi_v$, then for any $v \geq 2$,

$$\rho(\varpi_0, T\varpi_0) = \rho(\varpi_v, T\varpi_v)$$

$$\rho(\varpi_0, \varpi_1) = \rho(\varpi_v, \varpi_{v+1})$$

$$\rho_0 = \rho_v$$

$$\rho_0 \leq \alpha^v \rho_0$$

a contradiction. Therefore, $\rho_0 = 0$ i.e., $\varpi_0 = \varpi_1$.
$\Rightarrow \varpi_0$ is a FP of T. Assume that $\varpi_v \neq \varpi_u\ \forall$ distinct $u, v \in \mathbb{N}$. Again put $\rho(\varpi_v, \varpi_{v+2}) = \rho_v^*$.
From (3.1) for any $v \in \mathbb{N}$, we get

$$\rho(\varpi_v, \varpi_{v+2}) = \rho(T\varpi_{v-1}, T\varpi_{v+1}) \leq \alpha\rho(\varpi_{v-1}, \varpi_{v+1})$$

$$\rho_v^* \leq \rho_{v-1}^*$$

By going through this process again, we get

$$\rho(\varpi_v, \varpi_{v+2}) \leq \alpha^v \rho_0^* \tag{3.3}$$

For the sequence ϖ_v we consider $\rho(\varpi_v, \varpi_{v+w})$ in two cases. If w is odd say $2u+1$ then using (3.2) we obtain

$$\begin{aligned}
\rho(\varpi_v, \varpi_{v+2u+1}) &\leq s[\rho(\varpi_v, \varpi_{v+1}) + \rho(\varpi_{v+1}, \varpi_{v+2}) + \rho(\varpi_{v+2}, \varpi_{v+2u+1})] \\
&\leq s[\rho_v + \rho_{v+1}] + s^2[\rho(\varpi_{v+2}, \varpi_{v+3}) + \rho(\varpi_{v+3}, \varpi_{v+4}) + \rho(\varpi_{v+4}, \varpi_{v+2u+1})] \\
&\leq s[\rho_v + \rho_{v+1}] + s^2[\rho_{v+2} + \rho_{v+3}] + s^3[\rho_{v+4} + \rho_{v+5}] + ... + s^u \rho_{v+2u} \\
&\leq s[\alpha^v \rho_0 + \alpha^{v+1}\rho_0] + s^2[\alpha^{v+2}\rho_0 + \alpha^{v+3}\rho_0] + s^3[\alpha^{v+4}\rho_0 + \alpha^{v+5}\rho_0] + ... + s^u \alpha^{v+2u}\rho_0 \\
&\leq s\alpha^v[1 + s\alpha^2 + s^2\alpha^4 + ...]\rho_0 + s\alpha^{v+1}[1 + s\alpha^2 + s^2\alpha^4 + ...]\rho_0 \\
&\leq \frac{1+\alpha}{1-s\alpha^2} s\alpha^v \rho_0 \ (s\alpha^2 < 1)
\end{aligned}$$

Therefore,

$$\rho(\varpi_v, \varpi_{v+2u+1}) \leq \frac{1+\alpha}{1-s\alpha^2} s\alpha^v \rho_0 \tag{3.4}$$

If w is even say 2u then using (3.2)and (3.3) we obtain

$$\begin{aligned}
\rho(\varpi_v, \varpi_{v+2u}) &\leq s[\rho(\varpi_v, \varpi_{v+1}) + \rho(\varpi_{v+1}, \varpi_{v+2}) + \rho(\varpi_{v+2}\varpi_{v+2u})] \\
&\leq s[\rho_v + d_{v+1}] + s^2[\rho(\varpi_{v+2}, \varpi_{v+3}) + \rho(\varpi_{v+3}, \varpi_{v+4}) + \rho(\varpi_{v+4}, \varpi_{v+2u})] \\
&\leq s[\rho_v + \rho_{v+1}] + s^2[\rho_{v+2}, +\rho_{v+3}] + s^3[\rho_{v+4}, +\rho_{v+5}] \\
&+ ... + s^{u-1}[\rho_{2u-4} + \rho_{2u-3}] + s^{u-1}\rho(\varpi_{v+2u-2}, \varpi_{v+2u}) \\
&\leq s[\alpha^v\rho_0 + \alpha^{v+1}\rho_0] + s^2[\alpha^{v+2}\rho_0 + \alpha^{v+3}\rho_0] + s^3[\alpha^{v+4}\rho_0 + \alpha^{v+5}\rho_0] \\
&+ ... + s^{u-1}[\alpha^{2u-4}\rho_0 + \alpha^{2u-3}\rho_0] + s^{u-1}\alpha^{v+2u-2}\rho_0 \\
&\leq s\alpha^v[1 + s\alpha^2 + s^2\alpha^4 + ...]\rho_0 + s\alpha^{v+1}[1 + s\alpha^2 + s^2\alpha^4 + ...]\rho_0 \\
&+ s^{u-1}\alpha^{v+2u-2}\rho_0^*,
\end{aligned}$$

i.e.

$$\begin{aligned}
\rho(\varpi_v, \varpi_{v+2u}) &\leq \frac{1+\alpha}{1-s\alpha^2} s\alpha^v\rho_0 + s^{u-1}\alpha^{v+2u-2}\rho_0^* \\
&< \frac{1+\alpha}{1-s\alpha^2} s\alpha^v\rho_0 + (s\alpha)^{2u}\alpha^{v-2}\rho_0^* \\
&\leq \frac{1+\alpha}{1-s\alpha^2} s\alpha^v\rho_0 + \alpha^{v-2}\rho_0^*
\end{aligned}$$

Therefore,

$$\rho(\varpi_v, \varpi_{v+2u}) \leq \frac{1+\alpha}{1-s\alpha^2} s\alpha^v\rho_0 + \alpha^{v-2}\rho_0^* \tag{3.5}$$

from (3.4) and (3.5) that

$$\lim_{v\to\infty} \rho(\varpi_v, \varpi_{v+w}) = 0 \ \forall \ w > 0 \tag{3.6}$$

Thus ϖ_v is a Cauchy sequence in $\mathcal{Y}$. By completeness of $(\mathcal{Y}, \rho)$

$$\lim_{v\to\infty} \varpi_v = {\varpi_v}^* \tag{3.7}$$

We shall show that ${\varpi_v}^*$ is a fixed point of T. Again, for any $v \in \mathbb{N}$ we have

$$\begin{aligned}
\rho({\varpi_v}^*, T{\varpi_v}^*) &\leq s[\rho({\varpi_v}^*, \varpi_v) + \rho(\varpi_v, \varpi_{v+1}) + \rho(\varpi_{v+1}, T{\varpi_v}^*)] \\
&= s[\rho({\varpi_v}^*, \varpi_v) + \rho_v + \rho(T\varpi_v, T{\varpi_v}^*)] \\
&\leq s[\rho({\varpi_v}^*, \varpi_v) + \rho_v + \alpha\rho(\varpi_v, {\varpi_v}^*)].
\end{aligned} \tag{3.8}$$

Using (3.6) and (3.7) it follows from above inequality that $\rho(\varpi_v{}^*, T\varpi_v{}^*) = 0$ i.e., $T\varpi_v{}^* = \varpi_v{}^*$. Thus $\varpi_v{}^*$ is a fixed point of T. For uniqueness, let $\varsigma^* \in \mathcal{Y}$ be another FP of T. So we obtain $T^v\varpi^* = \varpi^*$ and $T^v\varsigma^* = \varsigma^*$ $\forall$ $v \in \mathbb{N}$. By the definition of orthogonality, $\exists$ $\varpi_0 \in \mathcal{Y}$ so that

$$[\varpi_0 \perp \varpi^* and\ \varpi_0 \perp \varsigma^*]$$

or

$$[\varpi^* \perp \varpi_0\ and\ \varsigma^* \perp \varpi_0]$$

Since T is $\perp$ - preserving, we have

$$[T^v\varpi_0 \perp T^v\varpi^*\ and\ T^v\varpi_0 \perp T^v\varsigma^*]$$

or

$$[T^v\varpi^* \perp T^v\varpi_0\ and\ T^v\varsigma^* \perp T^v\varpi_0]$$

$\forall$ $v \in \mathbb{N}$.Then we obtain

$$\begin{aligned} \rho(\varpi^*, \varsigma^*) = \rho(T^v\varpi^*, T^v\varsigma^*) &\leq \alpha\rho(\varpi^*, \varsigma^*) \\ &< \rho(\varpi^*, \varsigma^*) \end{aligned}$$

a contradiction. Therefore, $\rho(\varpi^*, \varsigma^*) = 0, i.e., \varpi^* = \varsigma^*$ .Thus FP is unique.

Example 3.1. Consider $\mathcal{Y} = [0, \infty)]$ and orthogonal relation $\perp$ defined on $\mathcal{Y}$ by $\varpi \perp \varsigma \iff \varpi\varsigma \leq \varpi$, i.e., $\varpi = 0$ *or* $\varsigma \leq 1$. Let $\rho : \mathcal{Y} \times \mathcal{Y} \to [0, \infty)$ be defined by $\rho(\varpi, \varsigma) = |\varpi - \varsigma|^2$, then ρ is a RbMS with $s = 2$. It is easy to see that $(\mathcal{Y}, \perp, \rho)$ is O-complete ORbMS.
Define a mapping $T : \mathcal{Y} \to \mathcal{Y}$ by

$$T\varpi = \begin{cases} \frac{\varpi}{3}, 0 \leq \varpi \leq 3 \\ 0, 3 < \varpi \leq 12 \end{cases}$$

It is easy to check T is an OP and OC selfmap on $\mathcal{Y}$ and $|T\varpi - T\varsigma|^2 \leq \frac{1}{9}|\varpi - \varsigma|^2$ $\forall$ $\varpi, \varsigma \in \mathcal{Y}$.
So T satisfy all the condition of theorem 3.1, then T has a unique FP.

4. CONCLUSION

In the past decade, there has been a lot of research focused on the study of fixed points of mappings that satisfy orthogonal sets.Many mathematicians were able to produce more results in this direction as a result.The notion of a novel generalized orthogonal contractive condition in rectangular b-metric spaces is presented in this study. From our primary results, we may also obtain certain fixed point results for mappings meeting an orthogonal contractive condition in metric spaces. The primary conclusions of Gordji et al.[27] are improved and generalized by these results.

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