Error estimation of function via $(C, \beta, \gamma)(E, 1)$ means of its Fourier-Laguerre series

Smita Sonker[1*] and Neeraj Devi[2]

[2,1*]Department of Mathematics, National Institute of Technology, Kurukshetra, India.

*Corresponding author(s). E-mail(s): smitafma@nitkkr.ac.in; Contributing authors: neerajkundu@gmail.com;

Abstract

In the present work, we proposed the error estimation of functon belonging to $\boldsymbol{L[0,\infty)}$- class by $\boldsymbol{(C,\beta,\gamma)(E,1)}$ means using its Fourier-Laguerre series at point y=0. Our findings generalise earlier results by Krasniqi who studied function approximation by $\boldsymbol{(C,1)(E,q)}$ means and Sonker who assessed the degree of approximation by $\boldsymbol{(C,2)(E,q)}$ means for $\boldsymbol{q=1}$. We also introduced the approximation theorem using product summability along-with some graphical interpretations.

Keywords: Fourier-Laguerre approximation, $\boldsymbol{(C,\beta,\gamma)}$ mean, $\boldsymbol{(E,1)}$ mean, Error estimation

1 Introduction

A wide field of mathematics is represented by the idea of summability when it comes to the study of Analysis and Functional Analysis. It is widely used in operator theory (to approximate the functions of positive linear operators), approximation theory, the theory of orthogonal series, numerical analysis (to assess the rate of convergence), and other domains. The idea of approximation theory originated with Weierstrass's famous theorem which asserts that a polynomial can approximate a continuous function in a certain interval. Summability theory also extends to the theorems of sequence spaces and fuzzy numbers [4]. Single summability approaches [1, 3, 7] are less effective than product summability methods [14, 15]. This fact inspired a variety of mathematicians engaged in summability and approximation theory research. Krasniqi [17] who worked on the error estimation of function by $(C,1)(E,q)$ product summability technique of Fourier-Laguerre series released a paper in 2013 employing the idea of product summability methods. As a result of his use of the product of two summabilities namely $(C,1)$ and (E,q), Fourier-Laguerre series can be approximated more accurately than using individual means. This paper was further developed in 2014 by Sonker [13] at $y=0$ employing $(C,2)(E,q)$ means. In 2015, Mittal and Singh [11] employed the $(T.E_q)$ summable approach in response to the aforementioned results to determine the error in the approximation of the same series function. The error estimation of Fourier-Laguerre series was later determined in 2016 by Khatri and Mishra [8] by superimposing Harmonic Means on Euler Means and using the $(H,1)(E,1)$ product summability technique in this way under the appropriate set of conditions. Sharma [9] investigated the (T,C_δ) technique of the Fourier-Laguerre series in 2021 which is an extension of past research [12] in the area. This inspired us to estimate a function's error at the frontier point of $y=0$ using the $(C,\beta,\gamma)(E,1)$ composite summable technique of its Fourier-Laguerre series. Our findings are compared to the results given by Krasniqi[17] and Sonker [13] in order to show the efficiency of proposed summation method. We also introduced the error estimation theorem using product summability along-with some graphical interpretations.

A function $g(y) \in L(0,\infty)$ is expanded by Fourier-Laguerre method as

$$g(y) \equiv \sum_{m=0}^{\infty} c_m {L_m}^{\delta}(y), \tag{1}$$

where

$$c_m = \frac{1}{\Gamma(\delta+1)\binom{m+\delta}{m}} \int_0^{\infty} e^{-u} u^{\delta} f(u) {L_m}^{\delta}(u) du, \tag{2}$$

and ${L_m}^{\delta}(y)$ denotes the mth Laguerre polynomial of order $\delta \geq -1$, defined by generating function

$$\sum_{m=0}^{\infty} {L_m}^{\delta}(y) w^m = (1-w)^{-\delta-1} e^{(\frac{-yw}{1-w})},$$

and the integral (2) exists. Also,

$$\psi(u) = \frac{1}{\Gamma(\beta+1)} e^{-u} u^{\delta} [f(u) - f(0)]. \tag{3}$$

Let the sequence $\{s_m(g;y)\}$ be the mth partial sum of the Fourier-Laguerre series (1) given by

$$s_m(g;y) = \sum_{h=0}^{m} c_h {L_h}^{\delta}(y),$$

is also known as Fourier?Laguerre polynomial of degree (or order)$\geq m$. We denote $C_m^{\beta,\gamma}$ or (C,β,γ) the m^{th} Cesào mean of order (β,γ) with $\beta+\gamma > -1$ of the sequence

$\{s_m(g;y)\}$ i.e.

$$C_m^{\beta,\gamma} = \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} s_h,$$

where

$$A_m^{\beta+\gamma} = O(m^{\beta+\gamma}), \beta+\gamma > -1 \text{ and } A_0^{\beta+\gamma} = 1.$$

The Fourier-Laguerre series (1) is said to be (C, β, γ) summable to the definite number s if

$$C_m^{\beta,\gamma} = \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} s_h \to s \text{ as } m \to \infty.$$

Also, If

$$E_m^1 = \frac{1}{2^m} \sum_{h=0}^{m} \binom{m}{h} s_h \to s \quad as \quad m \to \infty,$$

then $\{s_m(g;y)\}$ converges to a definite value 's' by E_m^1 means (by Hardy [5]), and we write it as,

$$s_m \to s(E_m^1).$$

We now introduce the Cesàro-Euler product summability mean of order $(\beta, \gamma, 1)$ as follows.

Definition: The (C, β, γ) transform of the $(E, 1)$ transform defines $(C, \beta, \gamma)(E, 1)$ transform of order $(\beta, \gamma, 1)$ and we shall denote it by $(CE)_m^{1,\beta,\gamma}$. Moreover, if

$$\begin{aligned} t_m^{CE} &= (CE)_m^{q,\beta,\gamma} \\ &= \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} E_h^q \\ &= \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \\ &\quad . \frac{1}{(2)^h} \sum_{v=0}^{h} \binom{h}{v} s_v \to s \text{ as } m \to \infty. \end{aligned} \tag{4}$$

The regularity of (C, β, γ) and $(E, 1)$ methods implies the regularity of $(CE)_m^{1,\beta,\gamma}$ method.

2 Useful Lemmas

For the proof of the main theorem, we require following lemmas:

Lemma 1: Let δ is any real number ϵ are fixed +ve constant. Then

$$L_m^{\delta}(y) = O(m^{\delta}) \qquad if \quad 0<y<1/m, \tag{5}$$

$$= O(y^{-(2\delta+1)/4} m^{(2\delta-1)/4} \quad if\ 1/m < y < \epsilon, \qquad (6)$$
$$as\ m \to \infty.$$

Proof: On similar lines as given by Szegö (1975, p.177, Theorem 7.6.4) [6].

Lemma 2: Let α and δbe an arbitrary real no., and $0 < \xi < 4$ and $\epsilon > 0$, then

$$\max e^{-y/2} y^{\alpha} \mid L_m^{\delta}(y) \mid = O(m^Q)$$

where

$$Q = \max(\alpha - 1/2, \delta/2 - 1/4),\ \ \epsilon \leq y \leq (4-\xi)m, \qquad (7)$$
$$= \max(\alpha - 1/3, \delta/2 - 1/4),\ \ y > m. \qquad (8)$$

Proof: On similar lines as given by Szegö (1975, p.241, Theorem 8.91.7) [6].

We will also need the following outcomes:

Lemma 3: Let $\delta > -1$. If $q = 1$, then

$$\frac{1}{2^m} \sum_{h=0}^{m} \binom{m}{h} h^{(2\delta+1)/4} = O\Big(m^{(2\delta+1)/4}\Big), \qquad (9)$$

and if $\beta + \gamma > -1$, then

$$I = \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} (1+h)^{\delta} = O\Big((1+m)^{\delta}\Big). \qquad (10)$$

Proof: The first result is on similar lines as given by Lenski and Szal [10].Regarding the latter result, A. Zygmund [2][7, Vol. I (1.15) and Theprem 1.17] have stated that

$$A_m^{\beta+\gamma} = \binom{m+\beta+\gamma}{m} \equiv O\Big((m+1)^{\delta}\Big),$$

is positive for $\beta + \gamma > -1$. Moreover, $A_m^{\beta+\gamma}$ is decreasing for $-1 < \beta + \gamma < 0$ and increasing for $\beta + \gamma > 0$. Hence for $\delta < 0$,

$$I = \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{[m/2-1]} A_{m-h}^{\beta-1} A_h^{\gamma} (1+h)^{\delta}$$
$$+ \frac{1}{A_m^{\beta+\gamma}} \sum_{h=[m/2]}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} (1+h)^{\delta}$$

$$
\begin{aligned}
&= O\left(\frac{(m+1)^{\beta-1}(m+1)^{\gamma}}{(m+1)^{\beta+\gamma}}\right)\sum_{h=0}^{[m/2-1]}(1+h)^{\delta} \\
&\quad + O\left((1+m)^{\delta}\right)\frac{1}{A_m^{\beta+\gamma}}\sum_{h=[m/2]}^{m}A_{m-h}^{\beta-1}A_h^{\gamma} \\
&= O\left((1+m)^{-1}\right)\sum_{h=0}^{m}(1+h)^{\delta}\int_h^{h+1}dz \\
&\quad + O\left((1+m)^{\delta}\right)\frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m}A_{m-h}^{\beta-1}A_h^{\gamma} \\
&= O\left((1+m)^{-1}\right)\sum_{h=0}^{m}\int_h^{h+1}z^{\delta}dz + O\left((1+m)^{\delta}\right) \\
&= O\left((1+m)^{-1}\right)\int_0^{m+1}z^{\delta}dz + O\left((1+m)^{\delta}\right) \\
&= O\left((1+m)^{-1}\right)\frac{(m+1)^{\delta+1}}{\delta+1} + O\left((1+m)^{\delta}\right) \\
&= O\left((1+m)^{\delta}\right)
\end{aligned}
$$

If $\delta > 0$, the outcome is obvious. Our proof is thus finished.

Additional Results:

Also, we will use

$$A_m^{\beta+\gamma}(y) = \frac{L_m^{(\beta+\gamma+1)}}{\Gamma(\beta+\gamma+1)}, \tag{11}$$

and also using this we can prove $\beta+\gamma > -1$, then

$$\frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m}A_{m-h}^{\beta-1}A_h^{\gamma}(h^{(2\delta+1)/4}) = O\left(m^{(2\delta+1)/4}\right). \tag{12}$$

3 Error Estimation Theorem

Let g be a lebesgue integrable function then the error estimation of g at $y=0$ by the Cesàro-Euler means of order $(\beta,\gamma,1)$ with $\beta+\gamma \geq -1,\ q=1$ of the Fourier-Laguerre series of g is given by

$$\mid (CE)_m^{1,\beta,\gamma}(g;0) - g(0) \mid = o(\varsigma(m)) \tag{13}$$

with conditions

$$\Psi(x) = \int_0^x \mid \psi(u) \mid du = o(x^{\delta+1}\varsigma(1/x)),\ x \to 0, \tag{14}$$

$$\int_{\epsilon}^{m} e^{u/2} u^{-(2\delta+3)/4} \mid \psi(u) \mid du = o(m^{(-2\delta+1)/4} \varsigma(m)), \tag{15}$$

and

$$\int_{m}^{\infty} e^{u/2} u^{-1/3} \mid \psi(u) \mid du = o(\varsigma(m)), \; m \to \infty, \tag{16}$$

where $\varsigma(x)$ is positive and monotonically increasing signal of x such that $\varsigma(m) \to \infty$ as $m \to \infty$.

Proof of theorem:

Based on the equality

$$L_m^{\delta}(0) = \binom{m+\delta}{\delta},$$

we obtain

$$\begin{aligned} s_m(0) &= s_m(g;0) \\ &= \sum_{h=0}^{m} c_h {L_h}^{\delta}(0) \\ &= \frac{1}{\Gamma(\delta+1)\binom{m+\delta}{m}} {L_m}^{\delta}(0) \int_0^{\infty} e^{-u} u^{\delta} g(u) \sum_{h=0}^{m} {L_h}^{\delta}(u) du \\ &= \frac{1}{\Gamma(\delta+1)} \int_0^{\infty} e^{-u} u^{\delta} g(u) {L_m}^{\delta+1}(u) du. \end{aligned}$$

Now

$$\begin{aligned} (CE)_m^{1,\beta,\gamma}(g;0) &= \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{2^h} \sum_{v=0}^{h} \binom{h}{v} s_v(0) \\ &= \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{2^h} \sum_{v=0}^{h} \binom{h}{v} \\ &\cdot \frac{1}{\Gamma(\delta+1)} \int_0^{\infty} e^{-u} u^{\delta} g(u) {L_v}^{\delta+1}(u) du. \end{aligned}$$

Therefore using (3), we have

$$\begin{aligned} \mid (CE)_m^{1,\beta,\gamma}(g;0) - g(0) \mid &= \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{2^h} \\ &\cdot \sum_{v=0}^{h} \binom{h}{v} \int_0^{\infty} \mid \psi(u) \mid {L_v}^{\delta+1}(u) du \\ &= \left(\int_0^{1/m} + \int_{1/m}^{\epsilon} + \int_{\epsilon}^{m} + \int_{m}^{\infty} \right) \frac{1}{A_m^{\beta+\gamma}} \end{aligned}$$

$$
\begin{aligned}
&\quad . \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{2^h} \sum_{v=0}^{h} \binom{h}{v} \\
&\quad . \mid \psi(u) \mid {L_v}^{\delta+1}(u) du \\
&= J_1 + J_2 + J_3 + J_4. \qquad (17)
\end{aligned}
$$

Using orthogonal property (14), Lemma [1][condition 5] and Lemma [3] we get

$$
\begin{aligned}
J_1 &= \int_0^{1/m} \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{2^h} \sum_{v=0}^{h} \binom{h}{v} \\
&\quad . \mid \psi(u) \mid {L_v}^{\delta+1}(u) du \\
&= \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{2^h} \sum_{v=0}^{h} \binom{h}{v} O(m^{\delta+1}) \\
&\quad . \int_0^{1/m} \mid \psi(u) \mid du \\
&= \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} O(m^{\delta+1}) o\Bigg(\varsigma(m)/m^{\delta+1} \Bigg) \\
&= O(m^{\delta+1}) o\Bigg(\varsigma(m)/m^{\delta+1} \Bigg) \\
&= o(\varsigma(m)). \qquad (18)
\end{aligned}
$$

Further using the orthogonal property (15), Lemma [1][condition 6], Lemma (3) and using the argument as in Nigam and Sharma [7] and Krasniqi [17] then integrating by parts, we get

$$
\begin{aligned}
J_2 &= \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{2^h} \sum_{v=0}^{h} \binom{h}{v} \int_{1/m}^{\epsilon} \mid \psi(u) \mid {L_v}^{\delta+1}(u) du \\
&= \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{2^h} \sum_{v=0}^{h} \binom{h}{v} \\
&\quad . O(v^{(2\delta+1)/4}) \int_{1/m}^{\epsilon} \mid \psi(u) \mid u^{-(2\delta+3)/4} du \\
&= \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} O(h^{(2\delta+1)/4}) \int_{1/m}^{\epsilon} \mid \psi(u) \mid u^{-(2\delta+3)/4} du \\
&= O(m^{(2\delta+1)/4}) \int_{1/m}^{\epsilon} \mid \psi(u) \mid u^{-(2\delta+3)/4} du \\
&= o(\varsigma(m)). \qquad (19)
\end{aligned}
$$

Using (15), Lemma [2][condition 7] and Lemma [3] we get

$$
\begin{aligned}
J_3 &= \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{2^h} \sum_{v=0}^{h} \binom{h}{v} \\
&\quad . \int_{\epsilon}^{m} \mid \psi(u) \mid L_v{}^{\delta+1}(u) du \\
&\leq \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{2^h} \sum_{v=0}^{h} \binom{h}{v} \int_{\epsilon}^{m} e^{u/2} \\
&\quad .u^{-(2\delta+3)/4} \mid \psi(u) \mid e^{-u/2} u^{(2\delta+3)/4} L_v{}^{\delta+1}(u) du \\
&= \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{2^h} \sum_{v=0}^{h} \binom{h}{v} O(v^{(2\delta+1)/4}) \\
&\quad . \int_{\epsilon}^{m} e^{u/2} u^{-(2\delta+3)/4} \mid \psi(u) \mid du \\
&= O(m^{(2\delta+1)/4}) o\bigg(m^{-(2\delta+1)/4} \varsigma(m) \bigg) \\
&= o(\varsigma(m)). \qquad (20)
\end{aligned}
$$

Finally, using (16), Lemma [2][condition 7] and Lemma [3], we get

$$
\begin{aligned}
J_4 &= \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{2^h} \sum_{v=0}^{h} \binom{h}{v} \int_{m}^{\infty} e^{u/2} \\
&\quad .u^{-(3\delta+5)/6} \mid \psi(u) \mid e^{-u/2} u^{(3\delta+5)/6} L_v{}^{\delta+1}(u) du \\
&= \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{(2^h} \sum_{v=0}^{h} \binom{h}{v} O(m^{(\delta+1)/2}) \\
&\quad . \int_{m}^{\infty} \frac{e^{u/2} u{-1/3} \mid \psi(u) \mid}{u^{(\delta+1)/2}} du \\
&= O(m^{(\delta+1)/2}) o(m^{-(\delta+1)/2} \varsigma(m)) \\
&= o(\varsigma(m)). \qquad (21)
\end{aligned}
$$

Combining (17), (18), (19), (20) and (21), we get

$$
\mid (CE)_m^{1,\beta,\gamma}(g;0) - g(0) \mid = o(\varsigma(m)).
$$

4 Corollary

- If we take $\beta = 1, \gamma = 0$ *and* $q = 1$, Our findings reduces to the results given by Krasniqi [17] for $q = 1$.
- If we take $\beta = 2, \gamma = 0$ *and* $q = 1$, Our findings reduces to the results given by Sonker [13] for $q = 1$.

- If we take $\beta = 0, \gamma = 0$ *and* $q = 1$, Our findings reduces to the results given by Nigam and Sharma [7] and many other.

5 Examples

Here, we consider the function

$$g(y) = y^6,$$

with its Fourier Laguerre series

$$g(y) = \sum_{m=0}^{\infty} (-1)^m \binom{6}{m} \Gamma(7) L_m^{\beta}(y).$$

Here, $\{c_m\}$ is the cofficient sequence in Fourier Laguerre expansion. $(CE)_m^{1,\beta,\gamma}$ is the proposed mean about the point y=0. We are plotting g and $(CE)_m^{1,\beta,\gamma}$ verses Number of terms. Here, we discuss our results in following cases:

Case 1: When $\beta + \gamma < 0$ *and* $q = 1$, we interpret that after applying $(CE)_m^{1,\beta,\gamma}$ mean the Fourier-Laguerre polynomial is approximating $g(y)$ from negative side and larger the value of $\beta + \gamma$ better will be the approximation.

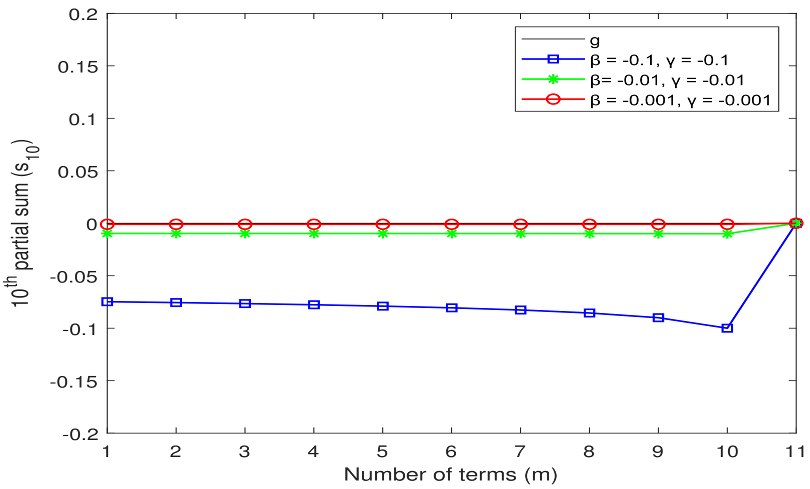

Fig. 1 Approximation of function when $\beta + \gamma < 0$

Case 2: When $\beta + \gamma > 0$ *and* $q = 1$, we interpret that after applying $(CE)_m^{1,\beta,\gamma}$ mean the Fourier-Laguerre polynomial is approximating $g(y)$ from positive side and smaller the value of $\beta + \gamma$ better will be the approximation.

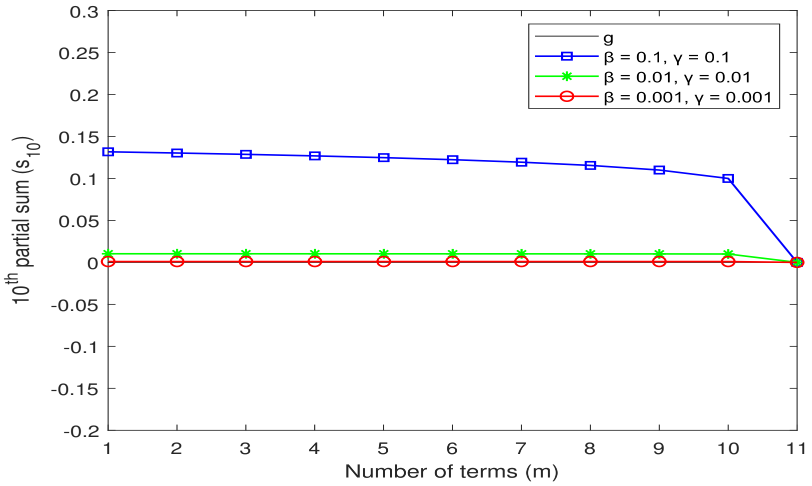

Fig. 2 Approximation of function when $\beta + \gamma > 0$

Comparison with existing methods: From the graph given below it can be analysed that the rate of convergence of proposed method is much faster than the existing methods given by Krasniqi[17] and Sonker [13] for $q = 1$.

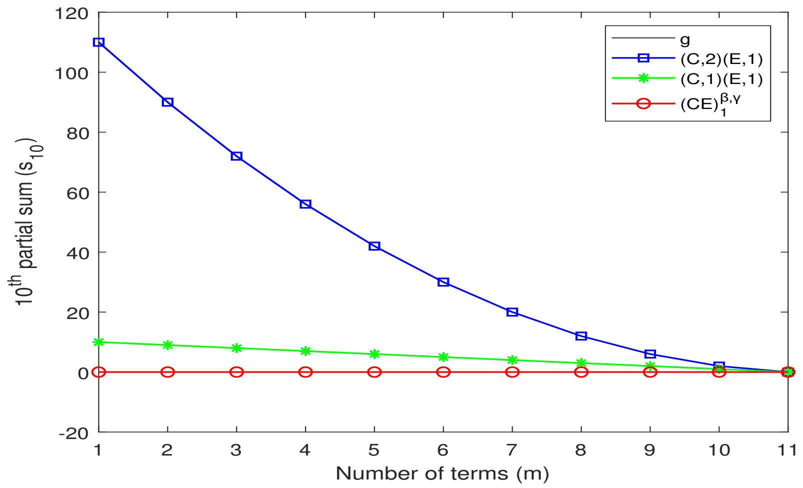

Fig. 3 Comparison with existing method

From above graphical interpretation, we can say that $(CE)_m^{1,\beta,\gamma}$ product summability method is much efficient. Also, the change in the value of β and γ changes the behaviour of approximation.

Conclusion

The use of $(CE)_m^{1,\beta,\gamma}$ product summability of order $(\beta,\gamma,1)$ generalised the results discussed in corollary and add flexibility to convergence as with the change in values of β,γ changes the behaviour of approximation. Also, the rate of convergence is improved with the help of proposed method. We can infer that our result is much efficient and useful.

Conflict of interest: The authors declare that they have no conflict of interest.

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