Moment Estimation and Weighted Approximation by modified-Bernstein Kantrovinch operators

Abstract

The present work deals with the Kantrovinch type modification of modified Bernstein operator. We discuss the rate of convergence of proposed operators by means of modulus of continuity and Peetre's K-functional for Hölder's class of functions. Further, we derive a Vornovskaya type asymptotic result and study weighted approximation with polynomial growth. Also, numerical examples illustrating the error functions and the approximation of the proposed operators for some continuous as well as piecewise continuous functions are given.

Keywords: Modulus of continuity, Kantrovinch operator, Bernstein operator, Moment estimates

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1. Introduction

Positive linear operators are widely used in various fields of science and engineering. A very famous polynomial in this regards, in the approximation theory of positive linear operators was studied by Bernstein [1]. Bernstein operator for every bounded function $\psi \in C[0,1]$, $n \geq 1$ and $t \in [0,1]$ is defined as

$$B_n(\psi;t) = \sum_{k=0}^{n} p_{n,k}(t)\psi\left(\frac{k}{n}\right),$$

and $p_{n,k}(t) = \binom{n}{k}t^{k-1}(1-t)^{n-k-1}$ is Bernstein basis function. F. Usta [2] presented a new modification for $\psi \in C[0,1],\ n \in \mathbb{N},\ t \in (0,1)$ as

$$\mathcal{B}_n(\psi;t) = \sum_{k=0}^{n} \binom{n}{k}(k-nt)^2 t^{k-1}(1-t)^{n-k-1}\psi\left(\frac{k}{n}\right). \tag{1.1}$$

Recently, M. Sofyalıoğlu [3] introduced a parametric generalization of (1.1). Thereafter, different modification of the above operator have become interest to many researchers. For more details on parametric genaralizations, we refer the readers to [4, 5, 6, 7, 8, 9]. Kantrovinch [10] introduced a modification involving integral for the class of Lebesgue integrable functions on [0,1] given by

$$K_n(\psi;t) = (n+1)\sum_{k=0}^{n} p_{n,k}(t)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \psi(u)du, \tag{1.2}$$

where $t \in (0,1)$.

In the present work, we introduce Kantrovinch modification of the operator given by equation 1.1 as follows:

$$\mathcal{K}_n(\psi;t) = n\sum_{k=0}^{n}\binom{n}{k}(k-nt)^2 t^{k-1}(1-t)^{n-k-1}\int_{k/n}^{(k+1)/n}\psi(u)du,\ t \in (0,1). \tag{1.3}$$

Also, We introduce some numerical example using MATLAB in order to show the theoretical approach for approximation by newly defined operators.

2. Preliminaries

Lemma 2.1. The modified-Bernstein operators $\mathcal{B}_n(.;t)$ [2], for $n \in \mathbb{N}$, satisfy the following identities:

(1) $\mathcal{B}_n(1;t) = 1$;

(2) $\mathcal{B}_n(y;t) = \left(\frac{n-2}{n}\right)t + \frac{1}{n}$;

(3) $\mathcal{B}_n(y^2;t) = \left(\frac{n^2-7n+6}{n^2}\right)t^2 + \left(\frac{5n-6}{n^2}\right)t + \frac{1}{n^2}$;

(4) $\mathcal{B}_n(y^3;t) = \left(\frac{n^3-15n^2+38n-24}{n^3}\right)t^3 + 12\left(\frac{n^2-4n+3}{n^3}\right)t^2 + \left(\frac{13n-14}{n^3}\right)t + \frac{1}{n^3}$.

3. Moment Estimation

Using the preliminaries, we can prove the following identities for Modified-Bernstein-Kantrovinch operators :

Lemma 3.1. For $n \in \mathbb{N}$ the operator $\mathcal{K}_n(\psi(y);t)$ satisfies the followings:

(1) $\mathcal{K}_n(1;t) = 1$;

(2) $\mathcal{K}_n(y;t) = \left(\frac{n-2}{n}\right)t + \frac{3}{2n}$;

(3) $\mathcal{K}_n(y^2;t) = \left(\frac{n^2-7n+6}{n^2}\right)t^2 + \left(\frac{6n-8}{n^2}\right)t + \frac{7}{3n^2}$;

(4) $\mathcal{K}_n(y^3;t) = \left(\frac{n^3-15n^2+38n-24}{n^3}\right)t^3 + \left(\frac{27n^2-117n+90}{2n^3}\right)t^2 + \left(\frac{42n-48}{n^3}\right)t + \frac{15}{4n^3}$.

Proof. Using the linear property of $\mathcal{K}_n(\psi;t)$, we've

$$\mathcal{K}_n(y;t) = B_{n,a}(y;t) + \frac{1}{2n}B_{n,a}(1;t)$$

By using preliminaries, we can see part (2) is true. In a similar manner, we can prove other parts of above result. □

Let us denote the r^{th} order moment of $\mathcal{K}_n((y-t)^r;t)$ by $\gamma_{n,r}(t)$.

Lemma 3.2. For $n \in \mathbb{N}$, the r^{th} $(r = 1, 2, 4)$ ordered moments of $\mathcal{K}_n(.;t)$ are given by

(1) $\gamma_{n,1}(t) = \left(\frac{-2}{n}\right)t + \frac{3}{2n}$;

(2) $\gamma_{n,2}(t) = \left(\frac{-3n+6}{n^2}\right)t^2 + \left(\frac{3n-8}{n^2}\right)t + \frac{7}{3n^2}$;

Proof. Using the linear property of $\mathcal{K}_n(.;t)$ and lemma (3.1), above lemma can be derived easily. □

Corollary 3.1. For $n \in \mathbb{N}$, operator $\mathcal{K}_n(.;t)$ satisfies the followings:

(1) $\lim_{n\to\infty} n\mathcal{K}_n((y-t);t) = \frac{3}{2} - 2t$;

(2) $\lim_{n\to\infty} n\mathcal{K}_n((y-t)^2;t) = 3t(1-t)$;

4. Approximation Properties of $\boldsymbol{\mathcal{K}_n(.;t)}$

4.1. Local Approximation

Theorem 4.1. Let $\psi \in C(0,1)$, then

$$\lim_{n\to\infty} \mathcal{K}_n(\psi;t) = \psi(t).$$

uniformly on $(0,1)$.

Proof. Using lemma (3.1), we have

$$\lim_{n\to\infty} \mathcal{K}_n(y^k;t) = t^k,\ k = 0,\ 1,\ 2,$$

uniformly on (0,1). The required result is immediately given by Korovkin type theorem [11]. □

4.2. Rate of Convergence

For $\psi \in C(0,1)$, the modulus of continuity of ψ is defined as

$$\omega(\psi,\zeta) = \sup_{|y-t|\leq\zeta}\left\{\sup_{t\in(0,1)}|\psi(y)-\psi(t)|\right\}.$$

Also from [12], we can write

$$|\psi(y)-\psi(t)| \leq \left(1+\frac{(y-t)^2}{\zeta^2}\right)\omega(f,\zeta)$$

Also, the Peetre's K-functional is given by

$$K(\psi;\zeta) = \inf_{f\in C^2[0,1]}\{\|\psi-f\|+\zeta\,\|f"\|\},\ \zeta>0,$$

where $C^2[0,1]=\{\psi\in C[0,1]:\psi',\psi"\in C[0,1]\}$. By [13], $\exists$ a constant $M>0$ such that

$$K(\psi;\zeta) \leq M\omega_2(\psi,\sqrt{\zeta}),\ \zeta>0, \tag{4.1}$$

where $\omega_2(\psi,\sqrt{\eta}) = \sup_{0<|h|<\sqrt{\eta}}\ \sup_{t,t+2h\in(0,1)}|\psi(t+2h)-2\psi(t+h)+\psi(t)|$ is the second ordered modulus of continuity of ψ on (0,1).

Theorem 4.2. Let $t\in(0,1)$ and $\psi\in C[0,1]$. Then we have

$$|\mathcal{K}_n(\psi;t)-\psi(t)| \leq 2\omega(\psi,\sqrt{\gamma_{n,2}(t)}),$$

where $\gamma_{n,2}^2(t)=\mathcal{K}_n((y-t)^2;t)$, is the second ordered central moment of nth proposed operator.

Proof. For $\psi\in C[0,1]$, we obtain

$$\begin{aligned}|\mathcal{K}_n(\psi;t)-\psi(t)| &= n\sum_{k=0}^{n}p_{n,k}(t)\int_{k/n}^{(k+1)/n}|\psi(y)-\psi(t)|\,dy\\ &\leq n\sum_{k=0}^{n}p_{n,k}(t)\int_{k/n}^{(k+1)/n}\left(1+\frac{(y-t)^2}{\zeta^2}\right)\omega(f,\zeta)dy\\ &= \left(1+\frac{1}{\zeta^2}\mathcal{K}_n((y-t)^2;t)\right)\omega(\psi,\zeta)\end{aligned}$$

By taking $\zeta^2=\gamma_{n,2}(t)$, we reach the required result. □

Next, we define Hölder's class of functions for $\alpha\in(0,1]$ as follows

$$\mathcal{H}_\alpha(0,1)=\{\psi\in C(0,1):|\psi(y)-\psi(t)|\leq M_\psi\,|y-t|^\alpha;\ y,t\in(0,1)\}$$

The following theorem gives the convergence rate for Hölder's class of functions:

Theorem 4.3. Let $t \in (0,1)$ and $\psi \in \mathcal{H}_\alpha(0,1)$. Then we have

$$|\mathcal{K}_n(\psi;t) - \psi(t)| \leq M\sqrt{\gamma_{n,2}^{\alpha}(t)},$$

where $\gamma_{n,2}(t)$ is the second ordered central moment of nth proposed operator.

Proof. For $\psi \in \mathcal{H}_\alpha(0,1)$, consider

$$|\mathcal{K}_n(\psi;t) - \psi(t)| = n\sum_{k=0}^{n} p_{n,k}(t)\int_{k/n}^{(k+1)/n} |\psi(y) - \psi(t)|\, dy$$

On applying Hölder inequality with $p = \frac{2}{\alpha},\ q = \frac{2}{2-\alpha}$ twice, we are led to

$$\begin{aligned}|\mathcal{K}_n(\psi;t) - \psi(t)| &\leq \left\{n\sum_{k=0}^{n} p_{n,k}(t)\int_{k/n}^{(k+1)/n} |\psi(y) - \psi(t)|^{\frac{2}{\alpha}}\, dy\right\}^{\frac{\alpha}{2}}\\ &\leq M\left\{n\sum_{k=0}^{n} p_{n,k}(t)\int_{k/n}^{(k+1)/n} |y - t|^{2}\, dy\right\}^{\frac{\alpha}{2}}\\ &= M\mathcal{K}_n((y-t)^2;t)^{\frac{\alpha}{2}},\end{aligned}$$

which completes the result. □

Theorem 4.4. Let $\psi \in C[0,1]$ and $0 < t < 1$. Then for all $n \in \mathbb{N}$, $\exists$ an absolute constant M such that

$$|\mathcal{K}_n(\psi;t) - \psi(t)| \leq M\omega_2\left(\psi;\sqrt{\left\{\gamma_{n,2}(t) + \frac{1}{2}\gamma_{n,1}^2(t)\right\}}\right) + 2\omega\Big(\psi, |\gamma_{n,1}(t)|\Big).$$

Proof. Firstly, we define an auxiliary operator

$$\mathcal{K}_n^*(g;t) = \mathcal{K}_n(g;t) - g\left(\frac{n-2}{n}t + \frac{3}{2n}\right) + g(t) \tag{4.2}$$

Then, we have $\mathcal{K}_n^*(1;t) = 1$ and $\mathcal{K}_n^*(y-t;t) = 0$. Now Taylor's expansion for $g \in C^2[0,1]$ is given by

$$g(y) = g(t) + (y-t)g'(t) + \int_t^y (y-u)g"(u)du,\ t \in (0,1).$$

Applying auxiliary operator to both sides of above expansion, we obtain

$$\mathcal{K}_n^*(g;t) - g(t) = \mathcal{K}_n\left(\int_t^y (y-u)g"(u)du;t\right) - \int_t^{\frac{n-2}{n}t+\frac{3}{2n}}\left(\frac{n-2}{n}t + \frac{3}{2n} - u\right)g"(u)du \tag{4.3}$$

Now,

$$\left|\int_t^y (y-u)g"(u)du\right| \leq \|g"\|\,(y-t)^2$$

and

$$\left|\int_t^{\frac{n-2}{n}t+\frac{3}{2n}}\left(\frac{n-2}{n}t+\frac{3}{2n}-u\right)g"(u)du\right| \le \frac{1}{2}\|g"\|\left(\frac{-2}{n}t+\frac{3}{2n}\right)^2$$

Rewriting equation 4.3, we obtain

$$\begin{aligned}|\mathcal{K}_n^*(g;t)-g(t)| &\le \|g"\|\,\mathcal{K}_n((y-t)^2;t)+\frac{1}{2}\|g"\|\left(\frac{-2}{n}t+\frac{3}{2n}\right)^2\\ &=\|g"\|\left\{\gamma_{n,2}(t)+\frac{1}{2}\gamma_{n,1}^2(t)\right\}\end{aligned} \tag{4.4}$$

Also,

$$|\mathcal{K}_n^*(g;t)| \le 3\|g\| \tag{4.5}$$

In the view of equations 4.4 and 4.5, we get

$$\begin{aligned}|\mathcal{K}_n(\psi;t)-\psi(t)| =& \left|\, \mathcal{K}_n^*(\psi;t)+\psi\left(\frac{n-2}{n}t+\frac{3}{2n}\right)-\psi(t)-\psi(t)+g(t)\right.\\ &\left.-\,g(t)+\mathcal{K}_n^*(g;t)-\mathcal{K}_n^*(g;t)\,\right|\\ &\le |\mathcal{K}_n^*(\psi-g;t)-(\psi-g)(t)|\\ &+|\mathcal{K}_n^*(g;t)-g(t)|+\left|\psi\left(\frac{n-2}{n}t+\frac{3}{2n}\right)-\psi(t)\right|\\ &\le 4\|\psi-g\|+\|g"\|\left\{\gamma_{n,2}(t)+\frac{1}{2}\gamma_{n,1}^2(t)\right\}\\ &+\omega(\psi,\zeta)\left(1+\frac{1}{\zeta}\left|\frac{-2}{n}t+\frac{3}{2n}\right|\right)\end{aligned}$$

Taking infimum to RHS of above equation over $g \in C^2[0,1]$ and $\zeta = |\gamma_{n,1}(t)|$, we are led to

$$|\mathcal{K}_n(\psi;t)-\psi(t)| \le 4K\left(\psi;\left\{\gamma_{n,2}(t)+\frac{1}{2}\gamma_{n,1}^2(t)\right\}\right)+2\omega\Big(\psi,|\gamma_{n,1}(t)|\Big).$$

We reach the required result immediately by using equation 4.1 □

4.3. Voronovskaya-type Asymptotic Result

In this subsection, we derive an asymptotic formula for the proposed operator as follows:

Theorem 4.5. Let $\psi \in C^2[0,1]$. and $t \in (0,1)$. Then, we have

$$\lim_{n\to\infty} n(\mathcal{K}_n(\psi;t)-\psi(t)) = \frac{1}{2}\big\{(3-4t)\psi'(t)+3t(1-t)\psi"(t)\big\}.$$

Proof. From Peano form of remainder of Taylor's expansion, we can write

$$\psi(y) = \psi(t) + (y-t)\psi'(t) + \frac{1}{2}(y-t)^2\psi"(t) + (y-t)^2\epsilon(y,t), \tag{4.6}$$

where $\epsilon(y,t) = \frac{\psi"(\xi)-\psi"(t)}{2}$ for some ξ lying between t and y. Also, $\lim_{y\to t}\epsilon(y,t) = 0$. Now, operating the equation 4.6 by $\mathcal{K}_n(.;t)$, we get

$$\mathcal{K}_n(\psi;t) - \psi(t) = \mathcal{K}_n((y-t);t)\psi'(t) + \frac{1}{2}\mathcal{K}_n((y-t)^2;t)\psi"(t) + \mathcal{K}_n(\epsilon(y,t)(y-t)^2;t).$$

Using corollary 3.1 and Cauchy-Schwartz inequality, we can deduce

$$\begin{aligned}\lim_{n\to\infty} n(\mathcal{K}_n(\psi;t) - \psi(t)) &= \psi'(t)\lim_{n\to\infty} n\mathcal{K}_n((y-t);t) + \frac{1}{2}\psi"(t)\lim_{n\to\infty} n\mathcal{K}_n((y-t)^2;t) \\ &\quad + \lim_{n\to\infty}\big(n\mathcal{K}_n\big((y-t)^2\epsilon(y,t);t\big)\big) \\ &\leq (3-4t)\psi'(t) + \frac{3}{2}t(1-t)\psi"(t) \\ &\quad + \lim_{n\to\infty}\sqrt{n^2\mathcal{K}_n\big((y-t)^4;t\big)}\sqrt{\mathcal{K}_n\big(\epsilon^2(y,t);t\big)}.\end{aligned} \tag{4.7}$$

By theorem 4.1, we have

$$\lim_{n\to\infty}\mathcal{K}_n\big(\epsilon^2(y,t);t\big) = \epsilon^2(t,t) = 0.$$

Using above equation in 4.7, we are led to the required result. □

4.4. Weighted Approximation

Consider a weight function $\sigma(t) = 1+t^2$ on (0,1). Let $B_\sigma(0,1)$ denotes the space of all functions φ on (0,1) such that

$$|\varphi(t)| \leq M_\varphi\sigma(t)$$

and $C_\sigma(0,1)$ be the subspace of all continuous functions in $B_\sigma(0,1)$ endowed with norm $\|.\|_\sigma$ given by

$$\|\varphi\|_\sigma = \sup_{t\in(0,1)}\frac{\varphi(t)}{\sigma(t)}$$

Next, we prove an inequality and convergence for the operator $\mathcal{K}_n(.;t)$ in weighted space as follows:

Lemma 4.1. Let $\psi \in C_\sigma(0,1)$. Then following inequality holds for $\mathcal{K}_n(\psi;t)$

$$\|\mathcal{K}_n(\psi;t)\|_\sigma \leq \frac{7}{3}\|\psi\|_\sigma .$$

Proof. By using definition of proposed operator, we may write

$$\begin{aligned}\|\mathcal{K}_n(\psi;t)\|_\sigma &= \sup_{t\in(0,1)} \frac{|\mathcal{K}_n(\psi;t)|}{\sigma(t)} \\ &\leq \|\psi\|_\sigma \sup_{t\in(0,1)} \frac{n}{1+t^2}\sum_{k=0}^{n} b_{n,k}(t)\int_{k/n}^{(k+1)/n}(1+u^2)du \\ &= \|\psi\|_\sigma \sup_{t\in(0,1)} \frac{1}{1+t^2}\{1+\mathcal{K}_n(y^2;t)\} \leq \frac{7}{3}\|\psi\|_\sigma .\end{aligned}$$

□

Theorem 4.6. For $\psi \in C_\sigma(0,1)$, the newly modified operator $\mathcal{K}_n(.;t)$ satisfies

$$\lim_{n\to\infty} \|\mathcal{K}_n(\psi;t)-\psi(t)\|_\sigma = 0.$$

Proof. From lemma 3.1, we obtain

$$\|\mathcal{K}_n(y;t)-t\|_\sigma = \sup_{t\in(0,1)} \frac{|\mathcal{K}_n(y;t)-t|}{1+t^2} = \left|\frac{1}{2n}\right| \sup_{t\in(0,1)} \frac{|3-4t|}{1+t^2} \leq \frac{1}{n}.$$

Also,

$$\begin{aligned}\left\|\mathcal{K}_n(y^2;t)-t^2\right\|_\sigma &= \sup_{t\in(0,1)} \frac{|\mathcal{K}_n(\psi;t)-\psi(t)|}{1+t^2} \\ &\leq \frac{-35}{3n^2}+\frac{13}{n}.\end{aligned}$$

Thus, in limiting condition, we can write

$$\lim_{n\to\infty} \left\|\mathcal{K}_n(y^j;t)-t^j\right\|;\ j=0,1,2.$$

Then, the weighted convergence holds for all $\psi \in C_\sigma(0,1)$ from the results given by Gadjiev [14]. □

5. Graphical Analysis

Now, we introduce some simulation results in order to substantiate the convergence behavior of $\mathcal{K}_n(\psi;t)$ for continuous as well as piece-wise continuous functions ψ by using MATLAB.

To test the approximation behavior of newly defined operators, let us consider a polynomial function $\psi(t)=t^3-t^2+\frac{t}{10}+0.1$ and a piesewise continuous function ϕ given by

$$\phi(t)=\begin{cases} e^{2t}, & 0\leq t\leq 0.4 \\ \cos(10t), & 0.4<t\leq 1 \end{cases}.$$

As the new sequence of operators is defined on $(0,1)$, so for that we will consider approximation over equally spaced grids in $[0.0005, 0.9995]$. Figure 1 and 2 shows the approximation and error in the approximation by proposed operator to $\psi(t)$ respectively for n=20, 50 and 100. On the other hand, behavior of proposed operators towards $\phi(t)$ is shown in figure 3 and we can observe from the graph that error of approximation near point of discontinuity is gradually increasing here.

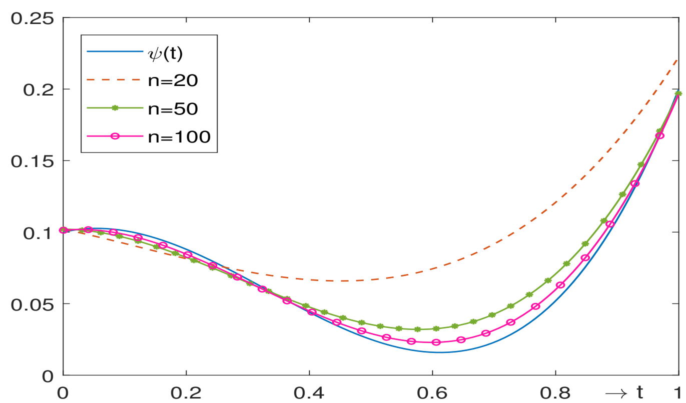

Figure 1: Approximation by proposed operator $\mathcal{K}_n(\psi;t)$ to ψ at different values of n.

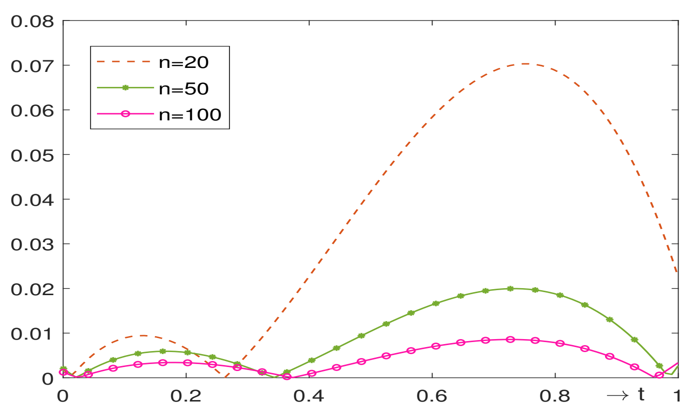

Figure 2: Error in the approximation by proposed operator $\mathcal{K}_n(\psi;t)$ to ψ at different values of n.

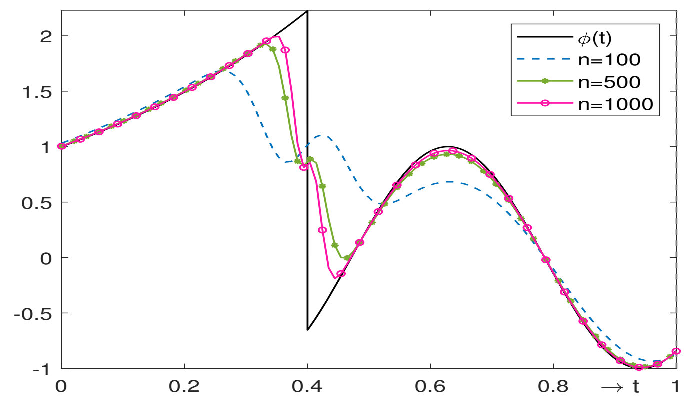

Figure 3: Approximation by proposed operators $\mathcal{K}_n(\phi; t)$ to discontinuous function $\phi(t)$ at different values of n.

6. Closing Comments

In this manuscript, we presented modified-Bernstein-Kantrovinch operators and discussed the rate of convergence, asymptotic formula and weighted approximation of these operators. Also, we included some numerical simulations in order to test the newly defined operators.

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