ELEMENTARY DIFFERENTIAL EQUATIONS

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Preface

The book is designed to cover a certain portion of Differential Equations in the Under Graduate course of different Indian Universities. This book contains a number of solved examples and exercises to give students a chance to work on their own. An attempt has been made to present the subject in a clear, lucid and intelligible manner.

-Mr. Barometer Nongbri

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-Mr. Barometer Nongbri

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Chapter-1

Differential Equations: Formation and Solutions

Introduction

What are Differential Equations?

Differential equations are mathematical equations that relate some function with its derivatives. In simple terms, a differential equation describes a relationship between a function and the rate at which it changes. These equations are fundamental in describing various physical phenomena such as motion, heat, and sound.

Order of Differential Equations

The order of a differential equation is the order of the highest order derivative involved in the equation .

Example 1.01: The equation $\frac{dy}{dx} = f(x, y)$ is of first order while the equation $\frac{d^2y}{dx^2} + 3x\left(\frac{dy}{dx}\right)^3 + 3xy = 0$ is of second order .

Degree of Differential Equations

The degree of the differential equation is the power of the highest order derivative in the given differential equation. The differential equation must be a polynomial equation in derivatives for the degree to be defined.

Example 1.02: The equation $\left(\frac{d^2y}{dx^2}\right)^3 + 3x\left(\frac{dy}{dx}\right)^4 + 3xy = 0$ is of degree 3 as the highest order derivative is of degree 3 .

The degree of a differential equation $\tan\left(\frac{d^2y}{dx^2}\right) + \frac{dy}{dx} = \sin x$ is not defined as the equation is not a polynomial in the derivatives.

Solution: A function (or relation between a dependent and independent variable) that satisfies the differential equation is called a solution .

Example 1.03: Show that $y = a\cos(mx + b)$ is a solution of the differential equation $\frac{d^2y}{dx^2} + m^2y = 0$.

Sol: $y = a\cos(mx + b) \Rightarrow \frac{dy}{dx} = -ma\sin(mx + b)$

$\Rightarrow \frac{d^2y}{dx^2} = -m^2a\cos(mx + b) = -m^2y$

$\Rightarrow \frac{d^2y}{dx^2} + m^2y = 0$

#1.01-Formation of differential Equations

Differential equations often arise from practical problems in various scientific and engineering fields. The process of forming a differential equation involves translating a physical problem or a mathematical relationship into an equation involving derivatives.

Here are some common ways differential equations can be formed:

- **From Geometric Relationships:** Many differential equations originate from geometric properties, such as curves and their tangents.
- **From Physical Laws:** Physical laws, like Newton's laws of motion or the law of cooling, naturally lead to differential equations.
- **From Rate of Change Problems:** Many real-world problems involve rates of change, which are naturally described by derivatives.
- **From Mathematical Conditions:** Sometimes, mathematical conditions and constraints can be used to derive differential equations.

#1.02-Formation from Geometric Relationships

Consider a curve given by $y = f(x)$. The slope of the tangent to this curve at any point (x, y) is given by the derivative $\frac{dy}{dx}$. If the relationship between y and $\frac{dy}{dx}$ is known, a differential equation can be formed.

Example 1.04

Given that the slope of the tangent to a curve at any point (x, y) is equal to the product of the coordinates, we have:

$$\frac{dy}{dx} = xy$$

This is a first order differential equation representing the given geometric relationship.

#1.03-Formation from Physical Laws

Physical laws often describe how quantities change over time. These changes can be expressed as differential equations.

Example 1.05

Newton's Law of Cooling states that the rate of change of the temperature of an object is proportional to the difference between the object's temperature and the ambient temperature. If $T(t)$ represents the temperature of the object at time t and T_a is the ambient temperature, the law can be written as:

$$\frac{dT}{dt} = -k(T - T_a)$$

where k is a positive constant.

#1.04-Formation from Rate of Change Problems

Many problems involve rates of change, such as population growth, radioactive decay, and chemical reactions. These problems can often be modeled by differential equations.

Example 1.06

The rate of growth of a population $P(t)$ at time t is proportional to the current population. This can be written as:

$$\frac{dP}{dt} = kP$$

where k is the proportionality constant.

#1.04-Formation from Mathematical Conditions

Certain mathematical conditions or constraints can lead to the formation of differential equations.

Example 1.07

Consider a function $y(x)$ defined implicitly by the equation $x^2 + y^2 = r^2$, which represents a circle of radius r. Differentiating both sides with respect to x gives:

$$2x + 2y\frac{dy}{dx} = 0$$

Simplifying, we get: $\frac{dy}{dx} = -\frac{x}{y}$

This is a first order differential equation describing the slope of the tangent to the circle at any point (x, y)

#1.05-Parameter of a Function

A parameter is a constant in the function's formula, but its value can change within a specified range. Unlike the main variables, which are typically the inputs of the function, parameters modify the relationship between the inputs and outputs.

Examples

Consider the linear function $f(x) = mx + b$. Here, m and b are parameters.

#1.06-Family of Curves

Definition. An n-parameter family of curves is a set of points (x, y) defined by a relations of the form $f(x, y, c_1, c_2, \ldots, c_n) = 0$ where each c_i $(i = 1,2,,n)$ are parameters .

For example, the set of concentric circles defined by $x^2 + y^2 = a^2$ is one parameter when $c > 0$.

Again, the set of circles, defined by $(x - \alpha)^2 + (y - \beta)^2 = 4$ is a two-parameter family.

Definition

Let $g(x, y, y_1, y_2, \ldots, y_n) = 0$ be a given nth order ordinary differential equation. Then – A solution containing n independent arbitrary constants is called a **general solution** or **complete primitive**.

A solution obtained from a general solution by giving particular values to one or more of the n independent arbitrary constants is called a **particular solution**.

A solution which cannot be obtained from any general solution by any choice of the n independent arbitrary constants is called a **singular solution**.

#1.07-Method to form a differential Equation

To form a differential equation from a given function of the form $f(x, y, a_1, a_2 \ldots\ldots a_n) = 0$ or $y = f(x, a_1, a_2 \ldots\ldots a_n)$ where a_i , $i = 1,2,3 \ldots n$ are parameters , we differentiate the given function n times to get $(n+1)$ relations .

From these $(n+1)$ relations , we can eliminate all the parameters and the resulting equation with no parameters is the required differential equation.

Example 1.08: Form a differential equation of a family of circles $x^2 + y^2 = a^2$.

Sol : The Given relation in x and y consists of 1 parameter $'a'$ and hence we need to differentiate only once .

we have $x^2 + y^2 = a^2$

$$\Rightarrow x + y\frac{dy}{dx} = 0$$

The second equation is independent of the parameter $'a'$ and is the required differential equation .

Example 1.09 : Find the differential equation of the family of curves given by $y = A\cos x + B\sin x$

Sol: Given $y = A\cos x + B\sin x$ (2 parameters)

$$\Rightarrow \frac{dy}{dx} = -A\sin x + B\cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -(A\cos x + B\sin x)$$

or $\frac{d^2y}{dx^2} = -y$

#1.08-The Wronskian

Definition: The Wronskian of n functions $\phi_i(x)\ :\ i = 1\ to\ n$ is defined as

$$W(\phi_1, \phi_2, \dots . \phi_n)(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) \dots . . & \phi_n(x) \\ \phi_1{}'(x) & \phi_2'(x) \dots . . & \phi_n{}'(x) \\ \vdots & \vdots & \vdots \\ \phi_1^{n-1}(x) & \phi_2^{n-1}(x) \dots . & \phi_n^{n-1}(x) \end{vmatrix}$$

#1.09-Linearly dependent and independent set of functions

Definitions. The n functions $y_1(x), y_2(x), \dots, y_n(x)$ are linearly independent if $c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0 \Rightarrow \mathrm{c_i} = 0\ \ \forall\, i = 1, 2, \dots n$ and linearly dependent if there exist constants $c_1, c_2, \dots, c_n$ (not all zero),
such that $c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0$

Theorem 1.01: The two non-zero differentiable functions $f(x)$ and $g(x)$ defined on I are linearly dependent on I if and only if the Wronsteian $W(f, g) = 0$ for all $x \in I$.

Proof : Suppose $f(x)$ and $g(x)$ are linearly dependent on I, then there exists a real number $K \neq 0$ such that $\frac{f(x)}{g(x)} = K$

Therefore, $W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} f & \frac{1}{K} & f \\ f' & \frac{1}{K} & f' \end{vmatrix} = \frac{1}{K}\begin{vmatrix} f & f \\ f' & f' \end{vmatrix} = 0$

Conversely, suppose $W(f, g) = 0$

Then for all $x \in I$, we have $\begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = 0$

$\frac{f'(x)}{f(x)} = \frac{g'(x)}{g(x)}$

On integration, we get $\frac{f(x)}{g(x)} = c \Rightarrow f(x) - c\ g(x) = 0$

So that $f(x)$ and $g(x)$ are Linearly dependent .
The other side of the theorem can be stated (without further proof) as

Theorem 1.02: The two non-zero differentiable functions $f(x)$ and $g(x)$ defined on I are linearly independent on I if and only if the Wronsteian $W(f,g) \neq 0$ for all $x \in I$.

The above two theorems can be extended to 3 or more functions.

#1.10: Linear differential equations

The general linear ODE of order n is
$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = q(x)$ ………………..(1)

If $q(x) \neq 0$, the equation is inhomogeneous.
The Equation $y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0$ ……………..(2)

will be called the associated homogeneous equation of (1).

#1.11: Linear differential operators

If we take $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$ and so on , equation (1) above can be written as

$(D^n + a_1D^{n-1} + \cdots + a_n)y = q(x)$

or $p(D)y = q(x)$ or $L(y) = q(x)$

where $L = p(D) = D^n + a_1D^{n-1} + \cdots + a_n$ We call $p(D)$ a Linear differential operator in the sense that it possesses Linearity rule tobe seen below .

Operator rules

We will state below some of these rules and assume that the functions involved are sufficiently differentiable upto any order , so that the operators can be applied to them.

Sum rule: If $p(D)$ and $q(D)$ are polynomial operators, then for any function u, $[p(D) + q(D)]u = p(D)u + q(D)u$

Linearity rule: If u_1 and u_2 are functions, and c_i constants,
$p(D)(c_1u_1 + c_2u_2) = c_1p(D)u_1 + c_2p(D)u_2$

Multiplication rule: If $p(D) = g(D)h(D)$, as polynomials in D, then

$p(D)u = g(D)(h(D)u)$

Commutative Rule: For two differential operator $f(D)$ and $g(D)$ and a function $u(x)$, we have $f(D)g(D)\left(u(x)\right) = g(D)f(D)\left(u(x)\right)$

Substitution rule

$p(D)e^{ax} = p(a)e^{ax}$

The proof for all these are easy as they involve differentiation only

Theorem 1.03: Let $L(y) = 0$ be a homogeneous linear differential equation and let y_1 and y_2 be two solutions. Then $c_1y_1 + c_2y_2$ is also a solution for any pair or constants c_1 and c_2.

Proof : We shall prove the theorem for a linear differential equation of order 2, the result can be generalized to any order .

Let $L(y) = y'' + py' + qy$

Then $L(y_1) = L(y_2) = 0$

Now $\mathrm{L}(c_1y_1 + c_2y_2) = (c_1y_1 + c_2y_2)'' + p(t)(c_1y_1 + c_2y_2)' + q(t)(c_1y_1 + c_2y_2)$

$$= c_1y_1'' + c_2y_2'' + p(t)c_1y_1' + p(t)c_2y_2' + q(t)c_1y_1 + q(t)c_2y_2$$

$$= c_1y_1'' + p(t)c_1y_1' + q(t)c_1y_1 + q(t)c_2y_2'' + p(t)c_2y_2' + q(t)c_2y_2$$

$$= c_1(y_1'' + p(t)y_1' + q(t)y_1) + c_2\left(y_2'' + p(t)y_2' + q(t)y_2\right)$$

$$= c_1\,\mathrm{L}(y_1) + c_2\,\mathrm{L}(y_2) = 0 + 0 = 0$$

We state below another theorem (without proof) that will be important in the chapter to come .

Theorem 1.04: Existence and uniqueness Theorem.

If $a_1(x), a_2(x), \dots a_n(x), q(x)$ are continuous (real-valued) functions on some interval (a, b) containing x_0, then an initial value problem of the form

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = q(x),$$

$y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$ has a unique solution on (a, b).

#1.12: Some Geometrical Concepts

To start with , we first make an identification of any point $P(x, y, z)$ in space with a vector $\overrightarrow{OP} = x\,\hat{\imath} + y\hat{\jmath} + z\hat{k}$ where $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ are unit vectors along the $X - axis$,
$Y - axis$ and $Z - axis$ respectively.

If $\overrightarrow{OP}$ makes an angle α , β , γ with the coordinate axes , then taking the dot product of $\overrightarrow{OP}$ with $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ respectively , we obtain the direction cosines of $\overrightarrow{OP}$ as $\cos\alpha = \frac{x}{OP}$, $\cos\beta = \frac{y}{OP}$, $\cos\gamma = \frac{z}{OP}$ …………… (1)
and so x , y , z is one set of direction ratios .

Using the above concept and the addition of vectors for the two vectors $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, we can easily see that :

A line joining AB (where $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$) has direction cosines

$$\cos\alpha = \frac{x_2 - x_1}{AB}, \quad \cos\beta = \frac{y_2 - y_1}{AB}, \quad \cos\gamma = \frac{z_2 - z_1}{AB} \quad \dots\dots \quad (2)$$

and one set of direction ratios are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$

Taking $a = x_2 - x_1$, $b = y_2 - y_1$, $c = z_2 - z_1$

It is also clear that the line AB is parallel to a vector $\bar{v} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ which pass through the origin as they both have same direction ratios.

If two proper lines AB and CD have direction ratios (or direction cosines) a_1 , b_1 , c_1 and a_2, b_2, c_2 respectively , then the angle $'\theta'$ between them is same as the angle between the vectors parallel to them and passing through the origin . If further , AB and CD are **perpendicular** then using the dot product we can easily show that
$a_1a_2 + b_1b_2 + c_1c_2 = 0$ …………. (3)

Plane: If $P(x, y, z)$ and $A(x_0, y_0, z_0)$ be two points on the plane and $\bar{n}$ be a normal vector to the plane where $\bar{n} = n_x\hat{\imath} + n_y\hat{\jmath} + n_z\hat{k}$ then $\overline{PA}.\bar{n} = 0$
$\Rightarrow (x - x_0)n_x + (y - y_0)n_y + (z - z_0)n_z = 0$ ……… (4)
giving the plane in terms of point and directions of its normal normal .

Surfaces and Curves

The mathematical definition of surfaces and curves can be found in most topology books and as they are not part of our treatise , we shall bring up just a few concepts that we feelmaybe required in our chapters .

A relation of the form $F(x, y, z) = c$ .... (5)

always represent a family of surfaces .

If each points satisfying (5) can be described by a set of relations

$x = f_1(u, v)$, $y = f_2(u, v)$, $z = f_3(u, v)$ .... (6)

Then (6) is knows as the parametric equation of the surface (5) .
In solving the first two equations of (6) , we can obtain realtions of the form $u = \phi(x, y)$, $v = \psi(x, y)$ so that z can be expressed as a function of x and y say $z = f(x, y)$ .... (7)
and since (7) is just a change of form of (5) , it represents a surface .

Curves in Space

In general , the intersection of two surfaces $f(x, y, z) = 0 = g(x, y, z)$ represents a curve in space .

Let the curve C be represented by the parametric equations
$x = x(s)$, $y = y(s)$, $z = z(s)$ . . . . (8)

where s is the distance of a point $P\big(x(s), y(s), z(s)\big)$ from some fixed point P_0 on the the curve measured along the curve.

Let Q be a neighbouring point of P on on the curve whose straight distance $PQ = \delta c$.

If Q is at a distance δs along the curve from P, then the distance P_0Q along the curve will be $s + \delta s$, and the coordinates of Q are
$\{x(s + \delta s), y(s + \delta s), z(s + \delta s)\}$.

When the point Q approaches P i.e $\delta s \to 0$, the two distances δc and δs are almost identical (of course $\delta s > \delta c$) and we shall have
$\lim_{\delta s \to 0} \frac{\delta c}{\delta s} = 1$

Using the formulae from equation (2) , we see that the direction cosines of the chord PQ are $\left\{\frac{x(s+\delta s)-x(s)}{\delta c}, \frac{y(s+\delta s)-y(s)}{\delta c}, \frac{z(s+\delta s)-z(s)}{\delta c}\right\}$

Assuming the smoothness of the curve, we have

$$x(s+\delta s)-x(s) = \left(\frac{dx}{ds}\right)\delta s + \frac{d^2x}{ds^2}\frac{(\delta s)^2}{2} \dots\dots \quad \dots\dots \tag{9}$$

$$\Rightarrow \frac{x(s+\delta s)-x(s)}{\delta c} = \left(\frac{dx}{ds}\right)\frac{\delta s}{\delta c} + \frac{d^2x}{ds^2}\frac{(\delta s)^2}{2\delta c} + \dots$$

and since the chord PQ (when $\delta s \to 0$) becomes a tangent to the curve C at P and since $\lim_{\delta s\to 0}\frac{\delta c}{\delta s} = 1$, the direction cosines of the tangent PQ to the curve C at P becomes

$$\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) \quad . \dots\dots \tag{10}$$

and also $dx\,, dy\,, dz$ is one set of direction ratios of the tangent .

Suppose the curve C given by the equations (8) lies entirely on the surface S with equation $F(x,y,z) = 0$, then for each s , we have

$$F[x(s), y(s), z(s)] = 0 \quad \dots\dots\dots \tag{11}$$

Differentiating equation (11) with respect to s and suing the chain rule , we obtain

$$\frac{\partial F}{\partial x}\frac{dx}{ds} + \frac{\partial F}{\partial y}\frac{dy}{ds} + \frac{\partial F}{\partial z}\frac{dz}{ds} = 0 \quad \dots\dots \tag{12}$$

By the perpendicularity condition (see equation 3)
we see from (12) that the tangent to the curve C at the point P is perpendicular to the line whose direction ratios/cosines are

$$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) \dots\dots\dots \tag{13}$$

Also , the curve C above can be any arbitrary curve passing through the point P and lies on the surface S. Thus the line with direction ratios (13) is perpendicular to the tangent to every curve lying on S and passing through P. Hence the direction (13) is the direction of the normal to the surface S at the point P.

If the equation of the surface S is of the form $z = f(x, y)$, then taking
$F(x, y, z) = f(x, y) - z \quad , \frac{\partial z}{\partial x} = p \;, \frac{\partial z}{\partial y} = q$

we have $\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = p \;, \frac{\partial F}{\partial y} = q \quad , \frac{\partial F}{\partial z} = -1$

Therefore , normal to the surface $z = f(x, y)$ at $P(x, y, z)$ has direction ratios

$(p, q, -1)$ and direction cosines are $\left(\frac{p, q, -1}{\sqrt{p^2+q^2+1}}\right)$

Exercises

1. Find the differential equation of all circles which pass through the origin and whose centres are on the x-axis.

2. Find the differential equation of the curve $y = a \sin x + b \cos x + x \sin x$.

3. Form a differential equation of the following :
   (i) ellipses centered at the origin.
   (ii) parabolas with axis parallel to the axis of y.
   (iii) $y = a \cos mx + b \sin mx$.
   (iv) circles with center at (a, b) and radius r .
   (v) $xy = ae^{x} + be^{-1}$

4. Show that the given functions in each of the following are solutions to the given differential equations .
   (i) $y = c\,(x - c)^2$, $\left(\frac{dy}{dx}\right)^3 = 4y\,(x\frac{dy}{dx} - 2y)$.

   (ii) $y = ce^{5x} + \frac{3}{5}$, $\frac{dy}{dx} = 5y - 3$

   (iii) $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$, $\left(\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y\right) = 0$

   (iv) $y = e^{-x}(c_1 \cos\, 4x + c_2 \sin\, 4x)$, $\left(\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 8y\right) = 0$

Chapter-2

First Order and First Degree Differential Equation

Introduction

A first-order differential equation is an equation that involves the first derivative of a function but no higher-order derivatives. When it is also of the first degree, it means that the highest power of the first derivative in the equation is one.
The general form of the first order and first degree equation is of the form –

$$\frac{dy}{dx} = f(x,y) \quad \text{or} \quad f(x,y)dx + g(x,y)dy = 0$$

#2.01: Geometrical Implication

Consider the equation $\frac{dy}{dx} = f(x,y)$ ……. (1)

Since $\frac{dy}{dx}$ represents the slope of the tangent to the curve $y = F(x)$ at any point $P(x,y)$ on the curve , therefore if $\phi(x,y) = c$ (for a constant c) is a solution curve or integral curve of the equation (1) , then $\frac{dy}{dx} = -\frac{\phi_x}{\phi_y}$ must satisfy equation (1) i,e the slope of the tangent to the curve ϕ (x,y)=c at any point P(x,y) on the curve must equal to f(x,y) .Also ϕ (x,y)=c is a family of curves lying at different heights depending on the value of 'c' .

Thus the general solution of the equation $\frac{dy}{dx} = f(x,y)$ is a family of curves called integral curves whose tangent at any point P(x,y) is f(x,y).

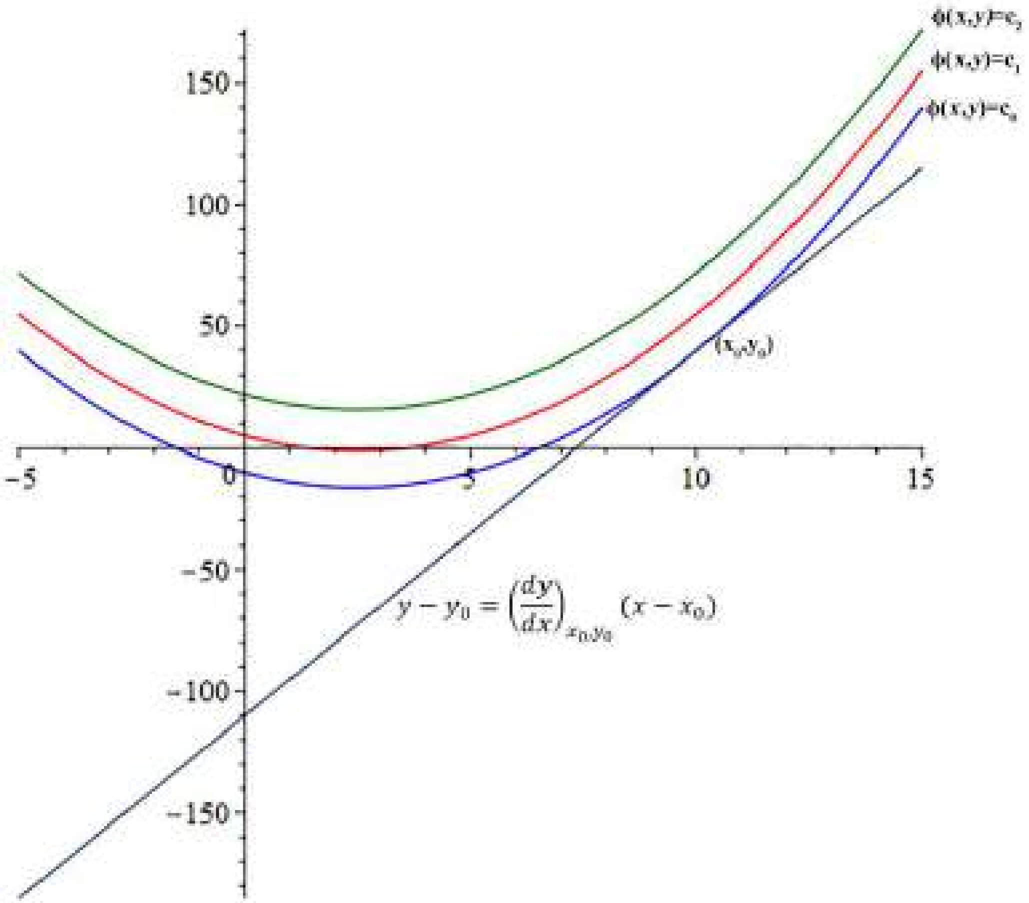

Figure 2.01

#2.02: There are several methods to solve such equations, including separation of variables, integrating factors, and exact equations. Here are some common types:

(A). Separation of Variables

This method is applied when an equation can be put in the form $f(y)dy = g(x)dx$ which can be solved by directly integrating both sides to get $\int f(y)dy = \int g(x)dx$

Example 1: Solve the differential equation:

$$2\frac{dy}{dx} = \frac{y(x+1)}{x}$$

Sol: Separating the variables we have

$$\frac{2dy}{y} = \frac{(x+1)dx}{x} \quad \text{or} \quad \frac{2dy}{y} = \left(1 + \frac{1}{x}\right)dx$$

Integrating both sides

$\int \frac{2dy}{y} = \int \left(1+\frac{1}{x}\right) dx$
$2\log y = x + \log x + K$

Example 2: Solve $xdy = y\log y \ dx$ given that $x = 2$ when $y = e$.

Sol: Separating variables we have

$$\frac{dy}{y\log y} = \frac{dx}{x}$$

Integrating: [For the y part, let $u = \log y$, then $du = \frac{dy}{y}$.

$$\int \frac{dy}{y\log y} = \int \frac{dx}{x}$$

$\log(\log y) = \log x + K$ ……….. (1)

Substituting $x = 2$ when $y = e$

we get $\log(\log e) = \log 2 + K$

$\Rightarrow K = -\log 2$

Substituting this (1)

we get $\log(\log y) = \log x - \log 2 = \log\left(\frac{x}{2}\right)$

$\Rightarrow \log y = \frac{x}{2}$ or $y = e^{x/2}$

(B) Reduction to Equation where Variables are seperable

Sometimes , there are equations where the variables cannot be separated but by some suitable substitutions , they are seperable . These Equations are mostly of the form

$$\frac{dy}{dx} = f(ax + by + c) \ .$$

Upon taking $(ax + by + c) = v$, they will be reduced to the the form
$dx = g(v)dv$

Example 3: Solve $\frac{dy}{dx} = \sin(x + y)$

Sol: Taking $(x + y) = v \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

Therefore $\frac{dv}{dx} - 1 = \sin v \Rightarrow dx = \frac{dv}{1+\sin v} = \frac{1}{2}\sec^2\left(\frac{v}{2}\right) dv$

Integrating $\int x\, dx = \int \frac{1}{2}\sec^2\left(\frac{v}{2}\right) dv$

$\Rightarrow x = \frac{1}{4}\tan\left(\frac{v}{2}\right) + C$

or $x = \frac{1}{4}\tan\frac{x+y}{2} + C$

Example 4: Solve $(x + y - 1)dx = (x + y + 1)dy$

Sol: We rewrite the equation as $\frac{dy}{dx} = \frac{x+y-1}{x+y+1}$. ……….. (1)

Let $x + y = v \Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx}$

so that

$\frac{dy}{dx} = \frac{dv}{dx} - 1.$

(1) becomes $\frac{dv}{dx} - 1 = \frac{v-1}{v+1}$

or $\frac{dv}{dx} = \frac{2v}{v+1}$

or $2dx = \left(1 + \frac{1}{v}\right) dv.$

∴ Integrating,

$2x + c = v + \log v$ or $x - y + c = \log(x + y)$

(C) Homogeneous Equations

Definition: A function $f(x, y)$ is said tobe homogeneous in x and y of degree n if $f(kx, ky) = k^n f(x, y)$

To Solve an equation of the form

$\frac{dy}{dx} = f(x, y)$ where $f(x, y)$ is a homogeneous function , we use a substitution $\boldsymbol{y = vx}$ so that $\boldsymbol{\frac{dy}{dx} = v + x\frac{dv}{dx}}$

Example: Solve $\frac{dy}{dx} = \frac{y-x}{y+x}$

Sol: Substitute $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Replace y in the given equation we get $v + x\frac{dv}{dx} = \frac{vx-x}{vx+x}$

or $v + x\frac{dv}{dx} = \frac{x(v-1)}{x(v+1)} = \frac{v-1}{v+1}$

or $x\frac{dv}{dx} = \frac{v-1}{v+1} - v$

or $x\frac{dv}{dx} = \frac{v-1-v(v+1)}{v+1}$

or $x\frac{dv}{dx} = \frac{v-1-v^2-v}{v+1}$

or $x\frac{dv}{dx} = \frac{-v^2-1}{v+1}$

or $x\frac{dv}{dx} = \frac{-(v^2+1)}{v+1}$

or $x\frac{dv}{dx} = -(v-1)$

Integrate both sides we get $\int \frac{v+1}{v^2+1} dv = -\int \frac{dx}{x}$

or $\int \frac{v}{v^2+1} dv + \int \frac{1}{v^2+1} dv = -\ln |x| + C$

The first integral is $\frac{1}{2}\ln |v^2 + 1|$, and the second is arctan (v) :

$\frac{1}{2}\log |v^2 + 1| + \tan v = -\ln |x| + C$

Back-substitute $v = \frac{y}{x}$:

$\frac{1}{2}\log \left|\left(\frac{y}{x}\right)^2 + 1\right| + \tan \left(\frac{y}{x}\right) = -\ln |x| + C$

Example 5: Solve $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

Sol. Taking $\frac{y}{x} = v$, i.e. $y = vx$, we get $\frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right)$.

The given equation becomes

$v + x\frac{dv}{dx} = v + \tan v$ or $\frac{dx}{x} = \frac{\cos v}{\sin v}dv$.

Integrating, $\log x + \log c = \log \sin v$, c being an arbitrary constant.
or $cx = \sin v$,
or $cx = \sin(y/x)$

(D) Equation reducible to homogeneous form

Equations of the form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$, where $\frac{a}{a'} \neq \frac{b}{b'}$, can be reduced to homogeneous form by expressing $c = ah + bk$ *and* $c' = a'h + b'k$ so that the given equation can be written as
$\frac{dy}{dx} = \frac{a(x+h)+b(y+k)}{a'(x+h)+b'(y+k)}$ which upon substitution $x + h = X$, $y + k = Y$, the R.H.S. will be homogeneous .

Example 6: Solve the equation $\frac{dy}{dx} = \frac{x+y-2}{x-y+4}$

Sol: We choose h, k such that

$$h + k = -2$$
$$h - k = 4$$

Therefore $h = 1$, $k = -3$

The given equation reduced to

$$\frac{dy}{dx} = \frac{(x+1)+(y-3)}{(x+1)-(y-3)}$$

Taking $x + 1 = X$, $y - 3 = Y$

The above equation becomes

$\frac{dY}{DX} = \frac{X+Y}{X-Y}$ (R.H.S is homogeneous)

On taking $Y = vX$

the last equation becomes $v + X\frac{dv}{dX} = \frac{1+v}{1-v}$

or $X\frac{dv}{dX} = \frac{1+v}{1-v} - v = \frac{1+v^2}{1+v}$

$\Rightarrow \frac{dX}{X} = \frac{1+v}{1+v^2}dv$, integrating we get $\int \frac{dX}{X} = \int \frac{1+v}{1+v^2}dv$

or $\int \frac{dX}{X} = \int \frac{1}{1+v^2}dv + \frac{1}{2}\int \frac{2v}{1+v^2}dv$

$\Rightarrow \log X = \tan^{-1} v + \frac{1}{2}\log(1+v^2) + C$

or $\log(x+1) = \tan^{-1}\frac{y-3}{x+1} + \frac{1}{2}\log\left(1 + \left(\frac{y-3}{x+1}\right)^2\right) + C$

Example 7: Solve $(x+y-10)\left(\frac{dy}{dx}\right) - 2x - y - 20 = 0.$

Sol: We rewrite the equation as $\frac{dy}{dx} = \frac{2x+y+20}{x+y-10}$ ……. (1)

Let h, k be such that

$2h + k = 20$ and $h + k = -10$

Then $h = 30, k = -40$

equation (1) can further be written as

$\frac{dy}{dx} = \frac{2(x+30)+(y-40)}{(x+30)+(y-40)}$

or $\frac{dY}{dX} = \frac{2X+Y}{X+Y}$ where $X = x + 30, Y = y - 40$

The R.H.S of the above equation is homogeneous and so we put $Y = vX$ to get

$v + X\frac{dv}{dX} = \frac{2+v}{1+v}$ or $X\frac{dv}{dX} = \frac{2-v^2}{1+v}$ or $\frac{dX}{X} = \frac{1+v}{1-v^2}\,dv$

integrating we get $\int \frac{dX}{X} = \int \frac{1+v}{1-v^2}\,dv = \int \frac{1}{1-v^2}\,dv - \frac{1}{2}\int \frac{-2v}{1-v^2}dv$

$\Rightarrow \log X = \frac{1}{2}\log\frac{1+v}{1-v} - \frac{1}{2}\log(1-v^2) + \log C$

$= \log \frac{\sqrt{1+v}}{\sqrt{(1-v)}\sqrt{1-v^2}} + \log C = \log \frac{C}{1-v}$

$\Rightarrow X = \frac{C}{1-v}$ or $X(1-v) = C$

or $X\left(1 - \frac{Y}{X}\right) = C$ or $X - Y = C$

or $x - y + 70 = C$ or $x - y = C_1$ where $C - 1 = C + 70$ a constant .

(E) Exact Differential Equation: An equation of the form

$Mdx + Ndy = 0$ …………(1) is said tobe exact if there exists a function $F(x, y)$ having continuous first order partial derivatives such that $M = \frac{\partial F}{\partial x}$, $N = \frac{\partial F}{\partial y}$.

If such a function exists , then (1) will become $\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0$ or $dF = 0$ and on integrating gives

$$\boldsymbol{F(x, y) = C}$$

which is a solution of (1) .

Theorem: The necessary and sufficient condition for differential equation

$$M.dx + N.dy = 0$$

to be an exact differential equation is that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Proof: (*P.G.Andhare , AIIRJ , Vol - V Issue-III MARCH 2018*)

Necessary Condition

Suppose M.dx + N.dy = 0 is an exact differential equation. Therefore there exist a function u of x and y such that $Mdx + Ndy = du$ By definition of total differentials $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$

From (1) and (2) we get

$$M \cdot dx + N \cdot dy = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

Comparing coefficients on both the sides of dx and Dy we obtain,

$M = \frac{\partial u}{\partial x}$ and $N = \frac{\partial u}{\partial y}$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\, \partial x}$$

But we know $\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial^2 u}{\partial x\, \partial y}$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial N}{\partial x}$$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Sufficient Condition: Conversely suppose $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

We Claim: The differential equation $Mdx + Ndy = 0$ is an exact differential equation.

We define $V = \int N.dy$, $x =$ constant
Differentiating V w. r.t. y we get

$\frac{\partial V}{\partial y} = N$ That is $N = \frac{\partial V}{\partial y}$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right) = \frac{\partial^2 V}{\partial x\, \partial y}$$

But we know that $\frac{\partial^2 V}{\partial x\, \partial y} = \frac{\partial^2 V}{\partial y\, \partial x}$ $\therefore \frac{\partial N}{\partial x} = \frac{\partial^2 V}{\partial y\, \partial x}$ .... (4)

From (3) and (4) $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 V}{\partial y\, \partial x}$

$\frac{\partial M}{\partial y} = \frac{\partial^2 V}{\partial y\, \partial x}$ or $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right)$

Integrating w. r. t. y treating x as constant

$$M = \frac{\partial V}{\partial x} + \text{ constant}$$

As we are treating x constant while integrating w. r. t. y .
Therefore constant of integration may contain the term in M not containing y . Hence it is function of x say $\varphi^1(x)$

$M = \frac{\partial V}{\partial x} + \varphi^1(x)$

Now

$Mdx + Ndy = \left[\frac{\partial V}{\partial x} + \varphi^1(x)\right] \cdot dx + \frac{\partial V}{\partial y} dy$

$= \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \varphi^1(x) dx$

But we know $\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy = dV$

$Mdx + Ndy = dV + \varphi^1(x) dx$

$= d[V + \varphi(x)]$

$= du$ where $u = [V + \varphi(x)]$

Therefore there exist a function u of x and y that is $u = V + \varphi(x)$ such that $Mdx + Ndy = du$

Hence the differential equation $Mdx + Ndy = 0$ is an exact differential equation.

Method to solve Exact Differential Equations

Method 1. Given the exact differential equation $Mdx + Ndy = 0$

The general solution is given by $F(x, y) = c$ where

$F(x, y)$ is a function satisfying $\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$.

This method is sometimes called **solution by inspection** .

Method 2. Given the exact differential equation $Mdx + Ndy = 0$
The general solution is given by

$\int Mdx + \int$ (Terms. N not containing $x)dy = c$
where in the first integral , we integrate w.r.t x treating y as constant and in the second integral , we integrate w.r.t y treating x as constant .

Method 3. Given the exact differential equation $Mdx + Ndy = 0$

The general solution is given by

$\int\ Ndy + \int$ (Terms. M not containing $y)dx$
where the integrals is tobe integrated similar as in the previous method .

Example 8: Solve $(3xy^2 - x^2)dx + (3x^2y - 6y^2 - 1)dx = 0$

Sol: Comparing the given equation with $Mdx + Ndy = 0$

we have $M = (3xy^2 - x^2)$, $N = (3x^2y - 6y^2 - 1)$

$\frac{\partial M}{\partial y} = 6xy$, $\frac{\partial N}{\partial x} = 6xy$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

∴ The differential equation is exact differential equation.
Integrating M w.r.t x treating y as constant ,
we have $\int(3xy^2 - x^2)dx = 3y^2\int(x)dx - \int x^2dx = \frac{3x^2y^2}{2} - \frac{x^3}{3}$
Integrating terms of N not containg x , w.r.t y we get

$\int(-6y^2 - 1)dy = -2y^3 - y$

Therefore the general solution is given by
$\frac{3x^2y^2}{2} - \frac{x^3}{3} - 2y^3 - y = c$

Example 9: Solve the equation $(y^2 + 2xy)dx + (x^2 + 2xy)dy = 0$

Sol: Taking the function $F(x, y) = x^2y + xy^2$

we have $\frac{\partial F}{\partial x} = y^2 + 2xy$, $\frac{\partial F}{\partial y} = x^2 + 2xy$

Therefore the given equation can be written as

$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$

or $dF = 0$

on integrating we get a solution as

$F(x, y) = C$
or $x^2y + xy^2 = C$

Example 10. Find the values of constant λ for which the equation

$$(2xe^y + 3y^2)\left(\frac{dy}{dx}\right) + (3x^2 + \lambda e^y) = 0 \text{ is exact.}$$

and hence solve it.

Sol. Re-writing the given equation,

$(3x^2 + \lambda e^y)dx + (2xe^y + 3y^2)dy = 0.$ ……… (1)

Comparing with $Mdx + Ndy = 0$,

we have $M = 3x^2 + \lambda e^y$ and $N = 2xe^y + 3y^2$.

Now, for (1) to be exact we must have

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so that $\lambda e^y = 2e^y \Rightarrow \lambda = 2$.

$\therefore$ (1) becomes $(3x^2 + 2e^y)dx + (2xe^y + 3y^2)dy = 0$
Equation (3) in exact and hence its solution is its solution is
$\int Mdx + \int$ (terms in N not containing $x)dy = c$

$\int (3x^2 + 2e^y)dx + \int (3y^2)dy = c$

or $x^3 + 2xe^y + y^3 = c$

Example 11: Solve $[y^2 - x^2\sin(xy)]dy + [\cos(xy) - xy\sin(xy) + e^{2x}]dx = 0$

Sol: We have $M = \cos(xy) - xy\sin(xy) + e^{2x}$
$N = y^2 - x^2\sin(xy)$

$$\frac{\partial M}{\partial y} = -x^2 \text{ y}\cos(xy) - 2\text{ x}\sin(xy)$$

$$\frac{\partial N}{\partial x} = -x^2y\cos(xy) - 2x\sin(xy)$$

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

$$\int M(x,y)d = x\int \ (\cos(xy) - xy\sin(xy) + e^{2x})dx$$

$$= \frac{1}{y}\sin(xy) + x\cos(xy) - \frac{1}{y}\sin(xy) + \frac{1}{2}e^{2x} + f(y)$$

$$= x\cos(xy) + \frac{1}{2}e^{2x}$$

$\int (terms\ of\ N\ not\ containing\ \ x)\ dy = \int y^2 dy = \frac{y^3}{3}$

Therefore the general solution is given by

$x\cos(xy) + \frac{1}{2}e^{2x} + \frac{y^3}{3} = C$

Example 12: Solve: $(1 + 4xy + 2y^2)dx + (1 + 4xy + 2x^2)dy = 0$

Sol: Given $(1 + 4xy + 2y^2)dx + (1 + 4xy + 2x^2)dy = 0$

Compare with $Mdx + Ndy = 0$, we get

$\mathrm{M} = 1 + 4xy + 2y^2, N = 1 + 4xy + 2x^2$

$\Rightarrow \frac{\partial \mathrm{M}}{\partial y} = 4x + 4y \ \ , \frac{\partial N}{\partial x} = 4y + 4x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

∴ Equation (1) is exact.

$\int_y Mdx + \int$ (terms in N not containing $x)dy = c$

where c is an arbitrary constant of integration.

$$\Rightarrow \int_y (1 + 4xy + 2y^2)dx + \int \ (1)dy = c$$

$$\Rightarrow x + 4x\frac{y^2}{2} + 2y^2 \cdot x + y = c$$

$$\Rightarrow x + 2xy^2 + 2y^2x + y = c$$

$$\Rightarrow x + y + 2xy^2 + 2y^2x = c$$

$$\Rightarrow x + y + 2xy(y + x) = c$$

$$\Rightarrow (x + y)(1 + 2xy) = c$$

The Integrating Factor: If the given differential equation is not exact, it may sometimes be possible to multiply through by an integrating factor $\mu(x,y)$ to make it exact. An integrating factor $\mu(x,y)$ is a function that, when multiplied by the original differential equation, makes it exact. There is no universal method for finding integrating factors, but common strategies include looking for μ that depends only on x or y, or using specific forms based on the structure of the equation.

We shall list some of the commonly used strategies below:

(i) If $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = f(x)$ (a function of x alone) then the integrating factor of the equation $Mdx + Ndy = 0$ is given by $\mu = e^{\int f(x)dx}$

(ii) If $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = g(y)$ (a function of y alone) then the integrating factor of the equation $Mdx + Ndy = 0$ is given by $\mu = e^{\int g(y)dy}$

Example 13: Solve $(3xy - y^2)dx + x(x-y)dy = 0$
Sol: Here we have , $M = 3xy - y^2$, $N = x(x-y)$

$\frac{\partial M}{\partial y} = 3x - 2y$, $\frac{\partial N}{\partial x} = 2x - y$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact.

We have $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{N}[3x - 2y - (2x - y)] = \frac{x-y}{x(x-y)} = \frac{1}{x} = f(x)$

a function only of x .
Therefore , the integrating factor is

$\mu = e^{\int f(x)dx}$

$e^{\int (1/x)dx}$

$= e^{\log x}$

$= x$

Multiply the differential equation the I.F. we get

$x[(3xy - y^2)dx + x(x-y)dy = 0]$

or $(3x^2y - xy^2)dx + (x^3 - x^2y)dy = 0$ ………… (1)

Checking exactness of equation (1)

we have $M = 3x^2y - xy^2$, $N = x^3 - x^2y$

$\frac{\partial M}{\partial y} = 3x^2 - 2xy$, $\frac{\partial N}{\partial x} = 3x^2 - 2xy$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, equation (1) is exact and can be solved by the previous method .

Example 14. Solve $(x^2 + y^2 + 2x)dx + 2ydy = 0$

Sol. Given equation : $(x^2 + y^2 + 2x)dx + 2ydy = 0$ ……. (1)

Where $\mathrm{M} = x^2 + y^2 + 2x$, $\mathrm{N} = 2y \Rightarrow \frac{\partial \mathrm{M}}{\partial y} = 2y, \frac{\partial \mathrm{N}}{\partial x} = 0$

Since $\frac{\partial \mathrm{M}}{\partial y} \neq \frac{\partial \mathrm{N}}{\partial x}$, (1) is not an exact equation.

Also $\frac{1}{\mathrm{N}}\left(\frac{\partial \mathrm{M}}{\partial y} - \frac{\partial \mathrm{N}}{\partial x}\right) = \frac{1}{2y}(2y - 0) = \frac{1}{2y}(2y) = 1$ = real number

$\therefore$ I.F. $= \exp\left[\int f(x)dx\right] = \exp\left(\int 1dx\right) = e^x$

Multiplying (1) by e^x
we get $(x^2 + y^2 + 2x)e^x dx + 2ye^x dy = 0$…….. (2)

Now (2) is of the form $M_1 dx + N_1 dy = 0$
is an exact equation , where $\mathrm{M}_1 = (x^2 + y^2 + 2x)e^x$ and $\mathrm{N}_1 = 2ye^x$
$\therefore$ General solution of (2) is

$\int^x (x^2 + y^2 + 2x)e^x dx + \int 0dy = c$ ($\because$ no term in N_1 not containing x)

$\Rightarrow \int x^2 e^x dx + y^2 \int e^x dx + 2\int xe^x dx = c$

$\Rightarrow x^2 e^x - \int 2xe^x dx + y^2 e^x + \int 2xe^x dx = c$

$\Rightarrow (x^2 + y^2)e^x = c$

(iii) If the given equation $Mdx + Ndy = 0$ is homogeneous and $(Mx + Ny) \neq 0$, then $\frac{1}{Mx+Ny}$ is an integrating factor.

Example 15. Solve $\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$.

Sol. The given equation can be written as

$$(x^3 + y^3)dx + (xy^2)dy = 0 \ldots\ldots\ldots\ldots \quad (1)$$

Here $M = (x^3 + y^3)$ and $N = -(xy^2)$ which are homogeneous in x and y.
Also $Mx + Ny = (x^3 + y^3)x + (-xy^2)y = x^4 \neq 0 \quad if \ x \neq 0$

Thus $I.F = \frac{1}{Mx+Ny} = \frac{1}{(x^3+y^3)x+(-xy^2)y} = \frac{1}{x^4}$

Multiplying the given equation by the integrating factor to make it axact
we have $\frac{(x^3+y^3)}{x^4}dx + \frac{(xy^2)}{x^4}dy = 0$
The required solution is given by $\int \frac{(x^3+y^3)}{x^4}dx + \int [$ terms of $\frac{(xy^2)}{x^4}$ not containing $x]dy = c$

i.e The required solution is given by $\int \left(\frac{1}{x} + \frac{y^3}{x^4}\right) dx + \int (0)dy = c.$

⇒ The required solution is given by

$$\log\, x - \frac{y^3}{3x^3} = c.$$

(iv) If an equation is of the form $f(xy)ydx + g(xy)xdy$ where $M = yf(xy)$ and $N = xg(xy)$ then $\frac{1}{Mx-Ny}$ is an integrating factor .

(v) Inspection: Sometimes , the integrating factor can be found by inspection when the given differential equation is of familiar form . These form can be a differential of some standard functions as given below :

(i) $d(xy) = ydx + xdy$

(ii) $d(x^2 + y^2) = 2xdx + 2ydy$

(iii) $d\left(\frac{y^2}{x}\right) = \frac{2xydy-y^2dx}{x^2}$

(iv) $d\left(\frac{x^2}{y}\right) = \frac{2yxdx-x^2dy}{y^2}$

(v) $d\left(\frac{y^2}{x^2}\right) = \frac{2x^2ydy-2xy^2dx}{x^4}$

(vi) $d\left(\frac{x^2}{y^2}\right) = \frac{2y^2xdx-2yx^2dy}{y^4}$

(vii) $d[\log(xy)] = \frac{xdy+ydx}{xy}$ (viii) $d(xy) = xdy + ydx$

(ix) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy-ydx}{x^2+y^2}$ (x) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx-xdy}{x^2+y^2}$

(xi) $d\left[\log\left(\frac{y}{x}\right)\right] = \frac{xdy-ydx}{xy}$ (xil) $d\left[\log\left(\frac{x}{y}\right)\right] = \frac{ydx-xdy}{xy}$

(xiii) $d\left[\frac{1}{2}\log(x^2+y^2)\right] = \frac{xdx+ydy}{x^2+y^2}$ (xiv) $d\left(-\frac{1}{xy}\right) = \frac{xdy+ydx}{x^2y^2}$

(xv) $d\left(\frac{e^x}{y}\right) = \frac{ye^x dx-e^x dy}{y^2}$ (xvi) $d(\sin^{-1} xy) = \frac{xdy+ydx}{(1-x^2y^2)^{1/2}}$

(xvii) $d\left(\frac{y}{x}\right) = \frac{xdy-ydx}{x^2}$ (xviii) $d\left(\frac{x}{y}\right) = \frac{ydx-xdy}{y^2}$

Example 16: Solve $(x^3 + xy^2 + y)dx + (y^3 + yx^2 - x)dy = 0$.

Sol. Re-writing, the given equation,

$x(x^2+y^2)dx + y(x^2+y^2)dy + (ydx - xdy) = 0$

or $xdx + ydy + \frac{ydx-xdy}{x^2+y^2} = 0$ (the 3rd term being the differential of $\tan^{-1}\frac{x}{y}$)

or $xdx + ydy + d\left(\tan^{-1}\frac{x}{y}\right) = 0.$

Integrating we get , $\frac{x^2}{2} + y^2/2 + \tan^{-1}\left(\frac{x}{y}\right) = c/2$

or $x^2 + y^2 + 2\tan^{-1}\left(\frac{x}{y}\right) = c.$

(F) Linear Differential Equation of First order : An equation of the form

$\frac{dy}{dx} + p(x)y = q(x)$ …………….. (1)

(where $p(x), q(x)$ are functions of x) is called a linear equation of first order . If we multiply equation (1) by some Integrating Factor $\mu(x)$ we get

$\mu(x)\frac{dy}{dx} + \mu(x)p(x)\,y = \mu(x)q(x)$ ………. (2)

we assume that $\mu(x)p(x) = \mu'(x)$

so that $\frac{\mu'(x)}{\mu(x)} = p(x)$

on integrating we have $\log \mu(x) = \int p(x)dx$
$\Rightarrow \mu(x) = e^{\int p(x)dx}$

From (2) we get $\mu(x)\frac{dy}{dx} + \mu'(x)y = \mu(x)q(x)$

or $\frac{d}{dx}(\mu y) = \mu(x)q(x)$

Which can be integrated to get the general solution as
$\Rightarrow \mu y = \int \mu(x)q(x)dx + C$
or $\boldsymbol{ye^{\int p(x)dx} = \int e^{\int p(x)dx} q(x)dx + C}$

Example 17: Solve the equation $y' + \frac{3}{x}y = 4x - 3$

Sol: Here $p(x) = \frac{3}{x}$ and $q(x) = 4x - 3$.
The integrating factor is $\mu(x) = e^{\int (3/x)} dx = e^{3\ln x} = x^3$.
Multiplying both sides of the differential equation by $\mu(x)$ gives us

$$x^3y' + x^3\left(\frac{3}{x}\right) = x^3(4x - 3)$$

$$x^3y' + 3x^2y = 4x^4 - 3x^3$$

$$\frac{d}{dx}(x^3y) = 4x^4 - 3x^3.$$

$$\int \frac{d}{dx}(x^3y)dx = \int 4x^4 - 3x^3dx$$

$$x^3y = \frac{4x^5}{5} - \frac{3x^4}{4} + C$$

$$y = \frac{4x^2}{5} - \frac{3x}{4} + Cx^{-3}$$

Example 18: Solve the differential equation $\frac{dy}{dx} + 3x^2y = 6x^2$.

Sol: Comparing with the standard equation we have

$P(x) = 3x^2$ and $Q(x) = 6x^2$.

An integrating factor is $\mu(x) = e^{\int 3x^2 dx} = e^{x^3}$

Multiplying both sides of the differential equation by e^{x^3}, we get

$$e^{x^3}\frac{dy}{dx} + 3x^2 e^{x^3} y = 6x^2 e^{x^3}$$

Or

$$\frac{d}{dx}\left(e^{x^3} y\right) = 6x^2 e^{x^3}$$

Integrating both sides, we have

$$e^{x^3} y = \int 6x^2 e^{x^3} dx = 2e^{x^3} + C$$

$$y == 2 + Ce^{-x^3}$$

Example 19: Find the solution of the initial-value problem

$$x^2 y' + xy = 1 \quad , \quad x > 0 \quad , y(1) = 2$$

Sol: The given equation can be written as

$$y' + \frac{1}{x} y = \frac{1}{x^2} \quad \ldots\ldots.. \qquad (1)$$

The integrating factor is

$$\mu(x) = e^{\int (1/x) dx} = e^{\ln x} = x$$

Multiplication of Equation (1) by x gives

$$xy' + y = \frac{1}{x} \quad \text{or} \quad \frac{d}{dx}(xy) = \frac{1}{x}$$

Integrating we get ,

$$xy = \int \frac{1}{x} dx = \log x + C$$

Since $y(1) = 2$, we have

$$2 = \log 1 + C \;\Rightarrow\; C = 2$$

Hence the required solution is $xy = \log x + 2$

Example 20: Solve $\frac{dy}{dx} + \frac{3x^2}{1+x^3} y = \frac{\sin^2 x}{1+x^3}$

Sol: Given

$$\frac{dy}{dx} + \frac{3x^2}{1+x^3} y = \frac{\sin^2 x}{1+x^3}$$

Comparing with $\frac{dy}{dx} + Py = Q$ we have $P = \frac{3x^2}{1+x^3}$, $Q = \frac{\sin^2 x}{1+x^3}$

Therefore $I.F = e^{\int Pdx} = e^{\int \frac{3x^2}{1+x^3} dx} = e^{\log(1+x^3)} = 1 + x^3$

Hence the solution is given by

$$y(1 + x^3) = \int \frac{\sin^2 x}{1+x^3}(1 + x^3) \cdot dx + c = \int \sin^2 xdx + c$$

$$= \frac{1}{2}\int 2\sin^2 xdx + c = \frac{1}{2}\int (1 - \cos 2x)dx + c = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c$$

Example 21: Solve $(1 + y^2)dx = (\tan^{-1} y - x)dy$

Sol: The given equation can be written as

$$\frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{\tan^{-1} y}{1 + y^2}$$

Comparing with the standard equation $\frac{dx}{dy} + Py = Q,$

we get $P = \frac{1}{1+y^2}$, $Q = \frac{\tan^{-1} y}{1+y^2}$

Therefore , $I.F = e^{\int Pdy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$

Hence the solution is given by

$$= \int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y} dy + c \text{ ....} \qquad (1)$$

Put $\tan^{-1} y = z$ so that $\frac{1}{1+y^2} dy = dz$

Equation (1) becomes

$$xe^{\tan^{-1} y} = \int e^z \cdot zdz + c = ze^z - \int 1 \cdot e^z dz + c = ze^z - e^z + c$$

$$= \tan^{-1} y e^{\tan^{-1} y} - e^{\tan^{-1} y} + c$$

$$\Rightarrow x = \tan^{-1} y - 1 + ce^{\tan^{-1} y}$$

Which is the general solution of the given equation.

(G) Equation reducible to Linear Form

Equations of the form

(i) $f'(y)\frac{dy}{dx} + p(x)f(y) = q(x)$

Equations of these form can be reduced to linear form by a substitution
$\boldsymbol{v = f(y)}$: $\boldsymbol{\frac{dv}{dx} = f'(y)\frac{dy}{dx}}$

(ii) $\frac{dy}{dx} + p(x)y = q(x)y^{(n)}$ (**Bernoulli's equation)** where $n \in N$ and $y^{(n)}$ denotes the n^{th} order derivative of y w.r.t. x .
These kind of equations can be reduced to linear form by putting $\boldsymbol{v = \frac{1}{y^{(n-1)}}}$

(iii) A first-order homogeneous differential equation is of the form:
$$\frac{dy}{dx} = \frac{F(y/x)}{G(y/x)}$$

Let $v = \frac{y}{x}$, which implies $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Substituting these into the original equation:
$v + x\frac{dv}{dx} = \frac{F(v)}{G(v)}$ or $\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x}\frac{F(v)}{G(v)}$

which is linear .
This can be rewritten as: $x\frac{dv}{dx} = \frac{F(v)}{G(v)} - v$
which is separable and can be integrated to find $v(x)$.

Example 22: Solve the Bernoulli Differential Equation $\frac{dy}{dx} + 3y = 2y^2$

Sol: Divide by y^2 we get

$$y^{-2}\frac{dy}{dx} + 3y^{-1} = 2$$

Let $v = y^{-1}$, so $\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$.

The given equation becomes

$-\frac{dv}{dx}+3v=2$

or $\frac{dv}{dx}-3v=-2$

which is now a linear differential equation in v.

Example 23: Solve the equation $x\frac{dy}{dx}+y\log\, y=xye^x$.

Sol. Dividing by xy, the given equation reduces to

$\frac{1}{y}\frac{dy}{dx}+\frac{1}{x}\log\, y=e^x$ ………… (1)

Let $\log\, y=v$ so that $\frac{1}{y}\frac{dy}{dx}=\frac{dv}{dx}$

$\left(\frac{dv}{dx}\right)+\left(\frac{1}{x}\right)v=e^x$ ………. (2)

Here , we have $p(x)=\frac{1}{x}$ and $q(x)=e^x$.

Now $\int p(x)dx=\int\left(\frac{1}{x}\right)dx=\log\, x$

Therefore $\mu(x)=e^{\int p(x)dx}=e^{\log\, x}=x$.

Hence solution of (2) is $v.\mu\,(x)=\int\mu(x)q(x)\,dx+c$ or $vx=\int xe^x dx+c$ or $vx=xe^x-\int e^x dx+c=xe^x-e^x+c$ or $x\log\, y=e^x(x-1)+c$.

Example 24: Solve : $x^2\frac{dy}{dx}=\frac{y(x+y)}{2}$

Sol: Given equation is $\frac{dy}{dx}=\frac{xy+y^2}{2x^2}$

$\because f(kx,ky)=f(x,y)$, the given equation is a homogeneous equation.

Put $y=vx$, $\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$

The given equation can be written as

$v+x\frac{dv}{dx}=\frac{vx^2+v^2x^2}{2x^2}=\frac{v^2+v}{2}$

$\Rightarrow x\frac{dv}{dx}=\frac{v^2+v}{2}-v=\frac{v^2+v-2v}{2}=\frac{v^2-v}{2}$

Separating variables : $2\frac{dv}{v(v-1)} = \frac{dx}{x}$

$$\Rightarrow 2\int \frac{dv}{v(v-1)} = \int \frac{dx}{x} + \log c$$

or $2\int \left(\frac{1}{v-1} - \frac{1}{v}\right) dv = \log x + \log c \Rightarrow 2[\log(v-1) - \log v] = \log cx$

or $\log\left(\frac{v-1}{v}\right)^2 = \log(cx) \Rightarrow \left(\frac{v-1}{v}\right)^2 = cx.$

Putting $v = \frac{y}{x}$, the general solution of is
$\left[\frac{(y/x)-1}{y/x}\right]^2 = cx \Rightarrow (y-x)^2 = cxy^2$

Example 25. Solve : $\frac{dy}{dx} = 2y\tan x + y^2\tan^2 x$

Sol. Given equation can be written as

$\frac{dy}{dx} - (2\tan x)y = y^2\tan^2 x \Rightarrow \frac{1}{y^2}\frac{dy}{dx} - (2\tan x)\frac{1}{y} = \tan^2 x$

Let $-\frac{1}{y} = z \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}.$

Then the above equation becomes :
$\frac{dz}{dx} + (2\tan x)z = \tan^2 x.$

This is a linear equation in z where $\text{P} = 2\tan x.$
Now I.F. $= e^{\int 2\tan x dx} = e^{2\log\sec x} = e^{\log\sec^2 x} = \sec^2 x.$

∴ General solution is
$z\sec^2 x = \int \tan^2 x\sec^2 x dx - c$
$\Rightarrow z\sec^2 x = \int \tan^2 x d(\tan x) - c = (\tan^3 x)/3 - c$
∴ The general solution of the given equation is
$-\frac{1}{\text{y}}\sec^2 x = \frac{1}{3}\tan^3 x - c$

Example 26: Solve $x\frac{dy}{dx} + y = y^2\log x$

Sol. Given equation can be written as

$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{x}\cdot\frac{1}{y}=\frac{\log x}{x}$ ………….. (1)

Let $u=\frac{1}{y}$ $\Rightarrow\frac{du}{dx}=-\frac{1}{y^2}\frac{dy}{dx}$ $\Rightarrow\frac{1}{y^2}\frac{dy}{dx}=-\frac{du}{dx}$

Equation (1) becomes $\frac{du}{dx}-\frac{1}{x}u=\frac{-\log x}{x}$ ………… (2)

which is a linear equation in u and x where $P=-\frac{1}{x}$, $Q=\frac{-\log x}{x}$

The I.F. $=\exp\left(\int Pdx\right)=\exp\left(\int -\frac{1}{x}dx\right)=e^{-\log x}=e^{\log x^{-1}}=\frac{1}{x}$

The General solution of (2) is

$u\,(\text{I.F.})=\int Q(\text{I.F})dx+c$

$\Rightarrow\frac{u}{x}=\int\frac{-\log x}{x}\frac{1}{x}dx+c=\int -\frac{1}{x^2}\log x dx+c$

$\Rightarrow\frac{u}{x}=\frac{1}{x}\log x-\int\frac{1}{x}\cdot\frac{1}{x}dx+c=\frac{1}{x}\log x+\frac{1}{x}+c$

Putting $u=\frac{1}{y}$ above ,

The general solution of the given equations is :

$\frac{1}{xy}=\frac{1}{x}\log x+\frac{1}{x}+c$ or $1=y\log x+y+cxy$

Example 27: Solve the equation : $\frac{dy}{dx}-\frac{1}{x}y=(1+\log x)y^3$.

Sol: Dividing by y^3, we get $y^{-3}\frac{dy}{dx}-\frac{1}{x}y^{-2}=(1+\log x)$,

or $-\frac{1}{2}\frac{d}{dx}(y^{-2})-\frac{1}{x}y^{-2}=(1+\log x)$ ……. (1)

Let $y^{-2}=z$, (1) becomes $\frac{dz}{dx}+\frac{2}{x}z=(1+\log x)$ ……… (2)

The integrating factor is $\mu(x)=e^{\int\frac{2}{x}dx}=x^2$,

Solution of (2) is : $zx^2=\int -x^2(1+\ln x)dx=-\frac{2}{3}x^3\left(\frac{2}{3}+\ln x\right)+c$.

The general solution of (1) is : $\frac{x^2}{y^2}=-\frac{2}{3}x^3\left(\frac{2}{3}+\ln x\right)+c$.

Exercise

1. Solve the initial valued problem $y' = \frac{xy^3}{\sqrt{1+x^2}}$, $y(0) = -1$

2. Solve the following by expressing in the form $\frac{dy}{dx} =$ a homogeneous function .

   (i) $(x^2 + xy)dy = (x^2 + y^2)dx.$ (ii) $\frac{dy}{dx} = \frac{y}{x} + \sin\left(\frac{y}{x}\right).$

   (iii) $(x^2 - y^2)dy = 2xydx.$ (iv) $(x^3 - y^3)dx + xy^2dy = 0.$

   (v) $x^2ydx - (x^3 + y^3)dy = 0.$ (vi) $x(x - y)dy = y(x + y)dx.$

3. Solve the following by reducing to the form as in Question 2.

   (i) $\frac{dy}{dx} = \frac{2x+2y+3}{4x+3y+4}.$ (ii) $\frac{dy}{dx} = \frac{2x+2y-2}{x+3y-5}$.

   (iii) $\frac{dy}{dx} = \frac{2x-y+1}{x+2y-3}.$ (iv) $(2x + y - 5)\left(\frac{dy}{dx}\right) + (3x + 2y - 2) = 0.$

   (v) $(3x - 2y - 7)dx = (2x + 3y - 6)dy.$

4. Test the exactness of the following equations and solve when the equation is exact.

   (a) $2xydx + (x^2 + 3y^2)dy = 0$
   (b) $(x^2 - ay)dx = (ax - y^2)dy.$
   (c)$e^xdx + (e^y(y + 1))dy = 0$
   (d) $\cos x\cos^2 ydx - \sin x\sin 2ydy = 0$
   (e) $(e^y + 1)\cos xdx + e^y\sin xdy = 0.$
   (f) $(x + y)dx + (x - y)dy = 0$
   (g) $(3x^2 + 6xy^2)dx + (6x^2y^2 + 4y^3)dy = 0$
   (h) $(3\,x^2\log|x| + x^2 + y)dx + xdy = 0$

5. Solve the following linear differential equation.

   (i) $4x^3y + x^4y' = \sin^3 x$

   (ii) $y' + y = \sin(e^x)$

   (iii) $\sin x\frac{dy}{dx} + (\cos x)y = \sin(x^2)$

   (iv) $x\frac{dy}{dx} - 4y = x^4e^x$

(v) $(1+x^2)\left(\frac{dy}{dx}\right)+y=e^{\tan^{-1} x}$.

(vi) $(dy/dx)+2y\tan\ x=\sin\ x$, given that $y=0$ when $x=\pi/3$.

(vii) $(1+y^2)+\left(x-e^{\tan^{-1} y}\right)(dy/dx)=0$.

(viii) $\frac{dy}{dx}+\frac{4x}{x^2+1}y=\frac{1}{(x^2+1)^2}$.

Chapter-3

Higher Degree Equations of First Order

Introduction: In this chapter we shall discuss some of the standard methods in solving the equations of first order and higher degree. The general form of these equations is -

$$P_0\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2} + \cdots \ldots\ldots\; P_{n-1}y' + P_n = 0$$

or $\boldsymbol{P_0p^n + P_1p^{n-1} + P_2p^{n-2} \ldots\ldots + P_{n-1}p + P_n = 0}$**............(1)**

where P_i are constants or functions of x and y and $p = \frac{dy}{dx}$

(A) Equations solvable for p

Suppose the equation $P_0p^n + P_1p^{n-1} + P_2p^{n-2} \ldots\ldots + P_{n-1}p + P_n = 0$ ....(1)can be written as the product of linear factor of p as $(p - q_1(x,y))(p - q_2(x,y))$ $(p - q_n(x,y)) = 0$ Equating each factor to zero we get a set of n first order and first degree equations $p - q_i(x,y) = 0$, $i = 1,2,\ldots.n$ which can be solved to get a solutions as $f_{\mathrm{i}}(\mathrm{x},\mathrm{y},\mathrm{c_i}) = 0, \mathrm{i} = 1,2,3 \ldots\ldots..\mathrm{n}$ These n solutions form the general solution of equation (1) which can be written combinely as $\boldsymbol{f_1(\mathbf{x},\mathbf{y},\mathbf{c})f_2(\mathbf{x},\mathbf{y},\mathbf{c}) \ldots\ldots..f_{\mathbf{n}}(\mathbf{x},\mathbf{y},\mathbf{c}) = 0}$.

Example 1: Solve the following differential equations: $x^2p^2 + xyp - 6y^2 = 0.$

Sol. Given equation is $x^2p^2 + xyp - 6y^2 = 0$

or $x^2p^2 + 3xyp - 2xyp - 6y^2 = 0$

or $xp(xp + 3y) - 2y(xp + 3y) = 0$

or $(xp + 3y)(xp - 2y) = 0.$

Equating each factor to zero we get

$x\left(\frac{dy}{dx}\right) + 3y = 0$ and $x\left(\frac{dy}{dx}\right) - 2y = 0$

or $\left(\frac{1}{y}\right) dy + 3\left(\frac{1}{x}\right) dx = 0$ and $\left(\frac{1}{y}\right) dy - 2\left(\frac{1}{x}\right) dx = 0.$

Integrating, $\log\, y + 3\log\, x = \log\, c$, i.e., $yx^3 = c$

and $\log\, y - 2\log\, x = \log\, c$, i.e., $\frac{y}{x^2} = c$

Therefore , the general solution is $(yx^3 - c)\left(\frac{y}{x^2} - c\right) = 0$.

Example 2. Solve $xy\left(\frac{dy}{dx}\right)^2 + (x^2 + y^2)\frac{dy}{dx} + xy = 0$

Sol: This is first-order differential equation of degree 2. Let $p = \frac{dy}{dx}$

The given equation can be written as

$xyp^2 + (x^2 + y^2)p + xy = 0$
$(xp + y)(yp + x) = 0$

$\Rightarrow\ xp + y = 0, yp + x = 0$

$\Rightarrow \frac{dx}{x} + \frac{dy}{y} = 0$, $ydy + xdx = 0$

Integrating we get $\log xy = \log c_1$ *or* $xy = c_1$ and $x^2 + y^2 = c_2$ respectively.The general solution can be written in the form
$(x^2 + y^2 - \mathrm{c})(xy - \mathrm{c}) = 0$

(B) Equations Solvable for x

If the differential equation $P_0p^n + P_1p^{n-1} + P_2p^{n-2} \dots \dots . + P_{n-1}p + P_n = 0$ be solvable for x . Then x can be expressed as a function y and p, that is, $x = f(y, p)$ ………….. (1)

Differentiating (1) with respect to y we get

$\frac{1}{p} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p} \cdot \frac{dp}{dy}$ which is a linear equation of first order in y and p and can be solved to get a solution of the form $\phi(y, p, c) = 0$ ………….. (2)

The general solution of (1) can be found by substituting the value of p from (2) in (1) or by eliminating p between (1) and (2)

If the elimination of p between (1) and (2) is not possible, then we solve (1) and (2) to express x and y in terms of p and c in the form

$$x = f_1(p, c),\ y = f_2(p, c).$$

These two equations together form the general solution of (1) in the parametric form .

(C) Equations Solvable for y

If the differential equation

$P_0 p^n + P_1 p^{n-1} + P_2 p^{n-2} \dots\dots + P_{n-1} p + P_n = 0$ be solvable for y . Then y can be expressed as a function x and p, that is, $y = f(x, p)$ ………….. (1)

Differentiating (1) with respect to x we get
$\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p} \cdot \frac{dp}{dx}$ …………. (2)

The last equation is a linear first order differential equation of first degree in x and p.

It may be solved by previous methods .

After getting a solution of (2) in the form
$\psi(x, p) = 0$ ……….. (3)

Then p can be eliminated between (1) and (3) to get the solution of (1) .
If the elimination of p between (1) and (3) is not possible, then we solve (1) and (3) to express x and y in terms of p and c in the form
$x = f_1(p, c),\ y = f_2(p, c).$

These two equations together form the general solution of (1) in the parametric form.

Example 3: Solve $y = 2px + p^2 y$.

Sol. The given equation is easily seen tobe solvable for x .
Solving for x, we get
$2x = -py + y/p.$ ……….. (1)

Differentiating (1) w.r.t. y, we get

$\frac{2}{p} = -p - y\frac{dp}{dy} + \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy}$,

or $p + \frac{1}{p} = -y\left(\frac{dp}{dy}\right)\left(1 + \frac{1}{p^2}\right)$

or $p\left(1 + \frac{1}{p^2}\right) + y\left(\frac{dp}{dy}\right)\left(1 + \frac{1}{p^2}\right) = 0$

or
$\left(1 + \frac{1}{p^2}\right)[p + y\left(\frac{dp}{dy}\right)] = 0.$

The first factor leads to $p = \pm i$ which is a singular solution and hence we can omit .

We shall be taking the second factor

$p + y\left(\frac{dp}{dy}\right) = 0$ or $\frac{dp}{p} + \frac{dy}{y} = 0$

Integrating $\log\, p + \log\, y = \log\, c$

or $py = c$.
$\Rightarrow$ $p = c/y$. Putting this value of p in (1), we get $2x = -c + \frac{y^2}{c}$
or $2xc - y^2 + c^2 = 0$

Example 4: Solve $x\left(\frac{dy}{dx}\right)^3 - 12\frac{dy}{dx} - 8 = 0$

Sol: Letting $p = \frac{dy}{dx}$, the given eequation becomes
$xp^3 - 12p - 8 = 0$

which is solvable for x to get

$x = \frac{12p+8}{p3} = \frac{12}{p^2} + \frac{8}{p^3}$ …………… (1)

Differentiating with respect to y, we get

$\frac{dx}{dy} = -2\frac{12}{p^3}\frac{dp}{dy} - 3\frac{8}{p^4}\frac{dp}{dy}$ or or $\frac{1}{p} = -\frac{24}{p^3}\frac{dp}{dy} - \frac{24}{p^4}\frac{dp}{dy}$

or $dy = \left(-\frac{24}{p^2} - \frac{24}{p^3}\right) dp$

Integrating we get

$y = \frac{24}{p} + \frac{12}{p^2} + c$ ……… (2)

As p cannot be eliminated between (1) and (2) , therefore (1) and (2) constitutes the general solution of the given equation in parametric form .

Example 5: Find the general and singular solution of $y^2 - 1 - p^2 = 0$

Sol: It is clear that the equation is solvable for y, to get

$y = \sqrt{1 + p^2}$ …………. (1)

By differentiating with respect to x we get

$$\frac{dy}{dx} = \frac{1}{2}\frac{1}{\sqrt{1+p^2}} \cdot 2p\frac{dp}{dx}$$

or $p = \frac{p}{\sqrt{1+p^2}}\frac{dp}{dx}$

or $p\left[1 - \frac{1}{\sqrt{1+p^2}}\frac{dp}{dx}\right] = 0$

giving $\mathrm{p} = 0$ or $1 - \frac{p}{\sqrt{1+p^2}}\frac{dp}{dx} = 0$ ………….. (2)

Taking the second factor we have

$1 - \frac{1}{\sqrt{1+\mathrm{p}^2}}\frac{\mathrm{dp}}{\mathrm{dx}} = 0$ or $\frac{dp}{dx} = \sqrt{1+p^2}$ or $\frac{dp}{\sqrt{1+p^2}} = dx$

Integrating we get

$\sinh^{-1} p = x + \mathrm{c}$

or $p = \sinh(x + c)$

Putting this value of p in (1) we get

$y = \sqrt{1 + \sinh^2(x+c)} = \cosh(x + c)$ ……….(3) which is is a general solution of (1)

(D) Lagrange's Equation

An equation of the form $y = xf(p) + g(p)$ where f and g are functions of p only are known as Lagrange's equation .

Being the special case of the form described in (C) , we can follow the same process.
Given $y = xf(p) + g(p)$ ……… (1)

Differentiating w.r.t x we get

$$p = f(p) + xf'(p)\frac{dp}{dx} + g'(p)\frac{dp}{dx}$$

or $\left(x\,f'(p) + g'(p)\right)\frac{dp}{dx} + f(p) - p = 0$ which is linear in x and p and can be solved to get the solution as

$x = \varphi(p, c)$ …………….. (2)

The general solution of (1) can be found by eliminating p between (1) and (2) .

If p cannot be eliminated between (1) and (2) then from (2) we replace the value of x in (1) to get $y = \psi\,(\,p.c\,)$ ……….(2)

In this case (2) and (3) constitute the general solution in parametric form.

we state below a special case of Lagrange's equation namely

(E) Clairaut's equation: An equation of the form $y = px + f(p)$

where $f(p)$ is a function of p only . The method to solve these kind of equations will be followed with the same method as in Lagrange's Equation . Given $y = px + f(p)$ …… (1)

Differentiating w.r.t x we get we get

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$\Rightarrow \frac{dp}{dx}\left(x + f'(p)\right) = 0$ ……… (2)

Taking $\frac{dp}{dx} = 0$ gives $p = c$

and putting in (1) gives the general solution as

$\boldsymbol{y = cx + f(c)}$ …….. (3)

which is same as replacing p in the original equation with an arbitrary constant c .
From (2) if we take $x + f'(p) = 0$ ……… (4)

and eliminate p between (1) and (4) , we shall arrive at a singular solution which cannot be obtained from the general solution (3) .

……………………………………………………….

Example 6: Solve the equation $y = px + (p^2 - 1)$.

Sol: Given $y = px + (p^2 - 1)$
The general solution is given by $y = cx + (c^2 - 1)$

Example 7: Solve the equation : $y = 2xy' - 3(y')^2$

Sol. Let $y' = p$, the equation is written in as :

$y = 2xp - 3p^2$ ………….(1)
Which is in Lagrange's form .

Differentiating both sides,

$$\frac{dy}{dx} = 2x\frac{dp}{dx} + 2p - 6p\frac{dp}{dx}$$

$$p = 2x\frac{dp}{dx} + 2p - 6p\frac{dp}{dx}$$

$\Rightarrow \frac{dx}{dp} + \frac{2}{p}x - 6 = 0$ ………..(2)

which is a linear equation in x and p . The integrating factor is

$$u(p) = \exp\left(\int \frac{2}{p}dp\right) = \exp\,(2\log p) = \exp\,(\log p^2) = p^2$$

The general solution of (2) is given by

$$xp^2 = \int p^2 \cdot 6dp + C = 2p^3 + C$$

$\Rightarrow x = 2p + \frac{C}{p^2}$ ………………. (3)

As p cannot be eliminated between (1) and (3) , therefore we replace the value of x from (3) and put in (1) , we get

$y = p^2 + \frac{2C}{p}$ ……………… (4)

Equation (3) and (4) together form the general solution of the given equation .

Example 8: Find the general solution of the equation

$y = \frac{3}{2}xp + \mathrm{e}^{\mathrm{p}}$ …………….. (1)

Sol. The given equation is a Lagrange equation

Differentiating both sides with respect to x and putting $\frac{dy}{dx} = p$

we have $p = \frac{3}{2}p + \frac{3}{2}x\frac{dp}{dx} + e^p\frac{dp}{dx}$

or $-\frac{1}{2}p = \left(\frac{3}{2}x + e^p\right)\frac{dp}{dx}$

or $\frac{dx}{dp} + \frac{3}{p}x = -\frac{2}{p}e^p, \quad p \neq 0.$ ………….. (2)

The integrating factor for linear equation is

$\mu(p) = \exp\int\frac{3}{p}dp = \exp 3\log p = \exp\log p^3 = p^3.$

Therefore, the general solution of (2) is

$xp^3 = \int -\frac{2}{p}e^p \times p^3\,\mathrm{dp} = -2\int p^2e^pdp$

$= -2\mathrm{e}^{\mathrm{p}}(p^2 - 2p + 2) + \mathrm{c}.$ after integrating by parts twice .

$\Rightarrow x = -\frac{2e^p}{p^3}(p^2 - 2p + 2) + \frac{c}{p^3}.$ …………. (3)

As we cannot eliminate p between (1) and (3)
we substitute the value of x from (3) in (1)

we get $y = \frac{3}{2}p\left(-\frac{2e^p}{p^3}(p^2 - 2p + 2) + \frac{c}{p^3}.\right) + e^p$ ………… (4)
Equation (3) and (4) together form the general solution of (1)

(F) Singular Solution

A singular solution is a solution not obtainable by assigning particular values to the arbitrary constants of the general solution. It is the equation of an envelope of the family of curves represented by the general solution.
Let $\boldsymbol{f(x, y, p) = 0}$ be the given differential equation and let $\boldsymbol{\phi(x, y, c) = 0}$ be its general solution .

(i) A relation $\psi(x, y) = 0$ obtained by eliminating p between the equations

$f(x, y, p) = 0$ and $\frac{\partial}{\partial p} f(x, y, p) = 0$ is the $\boldsymbol{p}$ **−discriminat** of the given equation .

(ii) If the given equation is a quadratic equation in p of the form
$Ap^2 + Bp + C = 0$ then

$\psi(x, y) = B^2 - 4AC = 0$ is the $\boldsymbol{p}$ **−discriminat** of the given equation

(iii) A relation $\chi(x, y) = 0$ obtained by eliminating c between the equations
$\phi(x, y, c) = 0$ and $\frac{\partial}{\partial c} \phi(x, y, c) = 0$ is the $\boldsymbol{c}$ **−discriminat** of the given equation.

(**iv**) If the general solution $\phi(x, y, c) = 0$ is a quadratic equation in c of the form $Ac^2 + Bc + C = 0$ then $\chi(x, y) = B^2 - 4AC = 0$ is the $\boldsymbol{c}$ **−discriminat** of the given equation

Determination of Singular Solution

(v) If the p-discriminant $\psi(x, y) = 0$ satisfy the given differential equation $f(x, y, p) = 0$ then $\psi(x, y) = 0$ is a singular solution . If it does not satisfy, then resolving $\psi(x, y) = 0$ into simpler factor , the part that satisfy the differential equation is a singular solution .

If the differential equation can be written as the product of linear factors in p or If the differential equation is of first degree in p , then there will be no singular solution.

(vi) If the c-discriminant $\chi(x,y) = 0$ satisfy the given differential equation $f(x,y,p) = 0$ then $\chi(x,y) = 0$ is a singular solution . If it does not satisfy , then resolving $\chi(x,y) = 0$ into simpler factor , the part that satisfy the differential equation is a singular solution .

Example 9: Find the general and singular solution of the equation

$$p^2 + 4x\text{p} - 4y = 0$$

Sol : The given equation can be written as

$y = px + \frac{1}{4}p^2$ …………… (1)

which is in the Clairaut's form and hence the general solution is given by

$\text{y} = \text{cx} + \frac{\text{c}^2}{4}$ ………………. (2)

To find a singular solution , differentiating (1) partially w.r.t p we get

$\frac{p}{2} + x = 0$ or $p = -2x$ ……………. (3)

Eliminating p between (1) and (3) we get

$$y = -2x^2 + x^2 = -x^2$$

which is clearly a solution of (1) . Since it contains no arbitrary constant , $y = -2x$ is a singular solution .

Example 10: Find the general and singular solution of $y^2 - 1 - p^2 = 0$

Sol: It is clear that the equation is solvable for y, to get

$y = \sqrt{1 + p^2}$ …………. (1)

By differentiating with respect to x we get

$$\frac{dy}{dx} = \frac{1}{2}\frac{1}{\sqrt{1+p^2}} \cdot 2p\frac{dp}{dx}$$

or $p = \frac{p}{\sqrt{1+p^2}}\frac{dp}{dx}$

or $p\left[1 - \frac{1}{\sqrt{1+p^2}}\frac{dp}{dx}\right] = 0$

giving $p = 0$ or $1 - \frac{p}{\sqrt{1+p^2}}\frac{dp}{dx} = 0$ ………….. (2)

Taking the second factor we have

$$1 - \frac{1}{\sqrt{1+p^2}}\frac{dp}{dx} = 0 \quad \text{or} \quad \frac{dp}{dx} = \sqrt{1+p^2} \quad \text{or} \quad \frac{dp}{\sqrt{1+p^2}} = dx$$

Integrating we get
$\sinh^{-1} p = x + c$
or $p = \sinh(x + c)$
Putting this value of p in (1) we get

$y = \sqrt{1 + \sinh^2(x + c)} = \cosh(x + c)$ ………. (3)

which is is a general solution of (1)
From (2) , if we take the first factor we get $p = 0$
and from (1) we get $y = 1$ which is clearly a solution of (1)
which cannot be obtained by giving a particular value to c in (3).
Hence $y = 1$ is a singular solution .

Example 11: Solve the equation $p^2 + y^2 = 1$.

Sol. Given $p^2 + y^2 = 1$.
We rewrite the above equation as
$p^2 + 0 \cdot p + (y^2 - 1) = 0$ ………….. (1)

which is quadratic in p .
Solving we get $p = \frac{dy}{dx} = \pm(1 - y^2)^{\frac{1}{2}}$ or $dx = \pm\frac{1}{\sqrt{1-y^2}}dy$.

Integrating, $x + c = \pm\sin^{-1} y$ or $\sin^{-1} y = \pm(x + c)$
or $y = \sin(x + c)$ which is the general solution .
From (1), the p-discriminant relation is
$0 - 4 \cdot 1 \cdot (y^2 - 1) = 0$ or $y^2 - 1 = 0$ or $(y - 1)(y + 1) = 0$.
Now, $y - 1 = 0 \Rightarrow p = 0$.

Substituting $y = 1$ and $p = 0$ in (1) we have
$L.H.S = R.H.S = 0$
Hence $y = 1$ is a singular solution.

Similarly we see that $y = -1, p = 0$ satisfy equation (1).
i.e $y = -1$ is also a singular solution .

Hence $y = \sin(x + c)$ is a general solution and $y = \pm 1$ are singular solutions.

Example 12: Solve the equation $4p^2 = 9x$

Sol. From the given equation we have $p = \frac{dy}{dx} = \pm\frac{3}{2}\sqrt{x}$ …….. (1)

$\Rightarrow dy = \pm\frac{3}{2}\sqrt{x}dx$

Integrating we get $y + c = \pm x^{\frac{3}{2}}$

or $(y + c)^2 = x^3$ or $c^2 + 2yc + (y^2 - x^3) = 0$ ………….. (2)

which is is a quadratic equation in c.

Now the c-discriminant relation is of (2) is

$4y^2 - 4 \times 1 \times (y^2 - x^3) = 0$ or $x^3 = 0$ or $x = 0$

$\Rightarrow \frac{1}{p} = \frac{dx}{dy} = 0$

The given equation an be written as

$$\frac{x}{p^2} = \frac{4}{9}$$

Putting $x = 0$, $\frac{1}{p} = 0$ we see that $L.H.S \neq R.H.S$

i.e $x = 0$ does not satisfy the given equation .

Hence there is no singular solution to the given equation .

Exercises

Find the general and singular solutions (if any) of the following equations .

1. $p^3 = \text{p}e^{2x}$

2. $4xp^2 = (3x - 4)^2$

3. $y(y - 2)p^2 - (y - 2x + xy)p + x = 0$

4. $p^2(4 - x^2) = 1 - y^2$

5. $3p^2e^y + 1 = px$

6. $xp^2 = yp + y$

7. $yp^2 + (x - y)p - x = 0$

8. $xp^2 - 2yp + 2x = 0$

9. $y^2(1 + 4p^2) - 2pxy - 1 = 0$

10. $p^2 = y - x$

11. $xp^2 - 2yp + 4x = 0$

12. $xy\left(y - x\frac{dy}{dx}\right) = x + y\frac{dy}{dx}$

13. $y = xp - p^3$

14. $y = xp + 5p^2$

Chapter-4

Linear Differential Equations with Constant Coefficients

Introduction: Linear differential equations with constant coefficients are a special class of linear differential equations where the coefficients of the derivatives are constants . These equations are of the form:

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots \ldots \ldots \ldots + \frac{a_{n-1} dy}{dx} + a_n y = q(x) \quad \ldots\ldots\ldots\ldots\ldots(1)$$

where $q(x)$ is a function of x .
Writing $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, ...

Equation (1) can be re-written as
$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots .. a_{n-1} D + a_n) y = q(x) \ \ldots \qquad (2)$$

or $f(D)y = q(x)$ where
$f(D) = (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots .. a_{n-1} D + a_n)$ is (as had been seen earlier) a linear operator satisfying the following :

Sum rule: If $p(D)$ and $q(D)$ are polynomial operators, then for any function u, $[p(D) + q(D)]u = p(D)u + q(D)u$

Linearity rule: If u_1 and u_2 are functions, and c_i constants,
$p(D)(c_1 u_1 + c_2 u_2) = c_1 p(D) u_1 + c_2 p(D) u_2$

Multiplication rule: If $p(D) = g(D)h(D)$, as polynomials in D, then
$p(D)u = g(D)(h(D)u)$

Substitution rule:
$$p(D)e^{ax} = p(a)e^{ax}$$

If $q(x) = 0$ then equation (2) reduces to
$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots .. a_{n-1} D + a_n) y = 0 \quad \ldots\ldots\ldots\ldots \qquad (3)$$

or $f(D)y = 0$ which is called a homogeneous part of equation (2) .
Before we proceed to the general solution of (3) , we shall discuss first a few particular type of homogeneous equation below :

(i) Solution of $(\boldsymbol{D-m})\boldsymbol{y}=\mathbf{0}$.

We have $(D-m)y=0 \Rightarrow \frac{dy}{dx}=my$

$\Rightarrow \frac{1}{y}dy=mdx$ and on integrating both sides we get
$\log y=mx+k$

$\Rightarrow y=e^{(mx+k)}=e^k e^{mx}=ce^{mx}$

(ii) Solution of $(\boldsymbol{D-m})^{\mathbf{2}}\boldsymbol{y}=\mathbf{0}$
We rewrite the above equation as

$(D-m)(D-m)y=0$ ……..(4)

Let $(D-m)y=v$ …………(5)

Then (1) becomes $(D-m)v=0$
Using (i) we get $v=c_2e^{mx}$ and putiing in (5)
we get $(D-m)y=c_2e^{mx}$

$\Rightarrow \frac{dy}{dx}-my=c_2e^{mx}$ …………………. (6)

which is linear with integrating factor $\mu(x)=\exp\int -mdx=e^{-mx}$
The solution of (6) is then given by

$ye^{-mx}=c_2\int e^{-mx}e^{mx}dx+c_1=c_2x+c_1$
$\Rightarrow y=(c_1+c_2x)e^{mx}$

we shall state the generalized form (without further proving) that

(iii) The general solution of the equation $(\boldsymbol{D-m})^{\boldsymbol{n}}\boldsymbol{y}=\mathbf{0}$ is given by

$$\boldsymbol{y=(c_1+c_2x+c_3x^2+\cdots.c_nx^{n-1})e^{mx}}$$

where the $c_i{}'s$ are arbitrary constants .

Complimentary Function

The general solution of the reduced homogeneous equation
$(D^n+a_1D^{n-1}+a_2D^{n-2}+\cdots..a_{n-1}D+a_n)y=0$ ………. (7)
or $f(D)y=0$ is called the **Complimentary Function** (C.F)
where $f(D)=(D^n+a_1D^{n-1}+a_2D^{n-2}+\cdots..a_{n-1}D+a_n)$ Consider the equation $m^n+a_1m^{n-1}+a_2m^{n-2}+\cdots..a_{n-1}m+a_n=0$ ……………. (8)

called the **auxiliary/characteristic equation** of (7) or (2) .

Equation (8) being a polynomial equation of degree n will be having n roots .

Suppose that $m_1, m_2, m_3, \ldots \ldots m_n$ be the roots of (8).

Then $f(D) = 0$ can be written as
$f(D) = (D - m_1)(D - m_2) \ldots \ldots (D - m_n) = 0$ and hence equation (7) can be written as

$$(D - m_1)(D - m_2) \ldots \ldots (D - m_n)y = 0 \quad \ldots\ldots\ldots\ldots \quad (9)$$

First we notice that the factors in equation (8) are obviously commutative since $D = \frac{d}{dx}$, (*for example* $\left(\frac{d}{dx} - 2\right)\left(\frac{d}{dx} + 3\right)y = \left(\frac{d}{dx} + 3\right)\left(\frac{d}{dx} - 2\right)y$) and hence can be shuffle in any order .

Now $y = c_i e^{m_i x}$ is a solution of $(D - m_i)y = 0$
$\Rightarrow (D - m_i)c_i e^{m_i x} = 0$

Also $f(D)c_i e^{m_i x} = (D - m_1)(D - m_2) \ldots \ldots (D - m_n)c_i e^{m_i x}$

$= (D - m_1)(D - m_2) \ldots . (D - m_{i-1})(D - m_{i+1}) \ldots (D - m_n)(D - m_i)c_i e^{m_i x}$
$= 0 \quad \ldots\ldots\ldots\ldots\ldots(10)$

Case I: If all the roots of equation (8) are equal then the given equation takes the form (iii) as discussed in the previous section .

Case II: Suppose all roots are distinct . Let $c_1 e^{m_1 x}$, $c_2 e^{m_2 x}$, $\ldots \ldots . c_n e^{m_n x}$ be the individual solution to each of the equation $(D - m_i)y = 0$

Let $y = (c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots \ldots . + c_n e^{m_n x})$

Now $f(D)y = f(D)(c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots \ldots . + c_n e^{m_n x})$
$= \sum_{i=1}^{n} f(D)c_i e^{m_i x} = 0$ as each term is zero as seen from (10)

Also $c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots \ldots . + c_n e^{m_n x}$ consists of n arbitrary constants.

Hence $C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots \ldots . + c_n e^{m_n x}$ is the general solution of the homogeneous equation (7)

Case III: If some of the roots are repeated then equation (7) can be written as $f(D)y = (D - m_1)^{r_1}(D - m_2)^{r_2} \ldots\ldots (D - m_k)^{r_k} y = 0$ where $\sum r_i = n$. As seen from (iii) in the previous section , each individual equation $(D - m_i)^{r_i}\, y = 0$ will have solution of the form $\left(c_{i1} + c_{i2}x + c_{i3}x^2 \ldots + c_{ir_i}x^{r_i - 1}\right)e^{m_i x}$ and the general solution of (7) is given by

$$C.F = \sum_{i=1}^{k}\left(c_{i1} + c_{i2}x + c_{i3}x^2 \ldots + c_{ir_i}x^{r_i-1}\right)e^{m_i x}$$

Particular Integral (P.I)

The solution of the equation $f(D)y = q(x)$ which is not part of the complimentary function is called the particular integral (P.I) . Let $y = v$ be as particular solution .

Then $f(D)v = q(x)$
$\Rightarrow v = \frac{1}{f(D)} q(x)$
or $P.I = \frac{1}{f(D)} q(x)$
where $\frac{1}{f(D)} q(x)$ is defined to be that function of x which when operated upon by

$f(D)$ gives $q(x)$.
For instant **,** $\boldsymbol{\frac{1}{D} q(x) = \int q(x) dx}$

General Solution of the equation $f(D)y = q(x)$:

Given an equation $f(D)y = q(x)$ …………….. (11)

We have just seen the solution of the reduced homogeneous equation $f(D)y = 0$ and the particular integral .

Let the complimentary function of (11) be $C.F = u$ and the particular integral be $P.I = v$.
Therefore $f(D)u = 0$ and $f(D)v = q(x)$
Consider the relation $y = u + v$

We have , $f(D)y = f(D)(u + v) = f(D)u + f(D)v = 0 + q(x) = q(x)$
Since $(u + v)$ consists of n arbitrary parameters , it becomes the general solution of equation (11) .

Summary: The general solution of the equation $f(D)y = q(x)$ is given by $y = C.F + P.I$

……………………………..

Examples 1: Find the complimentary function of the equation $(D-3)^2(D-1)y = \sin x$

Sol: The auxiliary equation of the given equation is given by

$$(m-3)^2(m-1) = 0$$

whose roots are $m_1 = 3$, $m_2 = 3$, $m_3 = 1$

Therefore $C.F = (c_1 + c_2 x)e^{3x} + c_3 x e^x$ where c_1, c_2, c_3 are arbitrary constants .

Examples 2: Find the complimentary function of the equation

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} - 8y = e^x$$

Sol: The given equation can be written as

$$(D^3 - 6D^2 + 12D - 8)y = e^x$$

The characteristic equation of the given equation is given by

$$(m^3 - 6m^2 + 12m - 8) = 0$$

or $(m-2)^3 = 0$

whose roots are $m_1 = 2$, $m_2 = 2$, $m_3 = 2$

Therefore $C.F = (c_1 + c_2 x + c_3 x^2)e^{2x}$

where c_1, c_2, c_3 are arbitrary constants .

Examples 3: Find the complimentary function of the equation

$$\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 9y = x^2 + 1$$

Sol: The given equation can be written as

$$(D^3 + 5D^2 + 3D - 9)y = e^x$$

The characteristic equation of the given equation is given by

$$(m^3 + 5m^2 + 3m - 9) = 0$$

or $(m+3)^2(m-1) = 0$ whose roots are $m_1 = -3$, $m_2 = -3$, $m_3 = 1$

Therefore $C.F = (c_1 + c_2 x)e^{-3x} + c_3 e^x$

where c_1, c_2, c_3 are arbitrary constants .

Example 4: Solve the differential equation: $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$

Sol: Given $(D^2 - 8D + 15)y = 0$

The Characteristic equation is: $m^2 - 8m + 15 = 0$
$\Rightarrow (m-3)(m-5) = 0$

$\Rightarrow m = 3{,}5$
$\therefore$ C.F. $= c_1 e^{3x} + c_2 e^{5x}$

Therefore , the general solution is given by
$y =$ C.F
or $y = c_1 e^{3x} + c_2 e^{5x}$

Example 5 Find the Complimentary Function of the differential equation
$$\frac{d^4y}{dx^4} - 2\frac{d^2y}{dx^2} + y = 0$$

Sol: Given $(D^4 - 2D^2 + 1)y = 0$

The Characteristic equation is given by
$m^4 - 2m^2 + 1 = 0$
$\Rightarrow (m^2 - 1)^2 = 0$
$\Rightarrow (m+1)^2(m-1)^2 = 0$
whose roots are $1, 1, -1, -1$
$\therefore$ C.F. $= (c_1 + c_2 x)e^{-x} + (c_3 + c_4 x)e^{x}$

Note on Complex roots: We know that complex root of an equation and its conjugate always occur together .

Suppose $c_1 e^{m_1 x}$, $c_2 e^{m_2 x}$ be parts of the complimentary function , where m_1 is a complex number and m_2 is the conjugate of m_1 .

Then $m_1 = a + ib$, $m_2 = a - ib$ where $i^2 = -1$.
Now $(c_1 e^{m_1 x} + c_2 e^{m_2 x}) = \left(c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}\right)$
$= e^{ax}\left(c_1 e^{ibx} + c_2 e^{-ibx}\right)$

Using Euler's formulas :

$e^{imx} = \cos(mx) + i\sin(mx)$
$e^{-imx} = \cos(mx) - i\sin(mx)$

$\cos(mx) = \frac{e^{imx} + e^{-imx}}{2}$
$\sin(mx) = \frac{e^{imx} - e^{-imx}}{2i}$

We have $(c_1e^{m_1x} + c_2e^{m_2x}) = e^{ax}\left(c_1e^{ibx} + c_2e^{-ibx}\right)$

$$= e^{ax}[c_1(\cos bx + i\sin bx) + c_2(\cos bx - i\sin bx)]$$
$$= e^{ax}\{(c_1 + c_2)\cos bx + (c_1i - c_2i)\sin bx\}$$
$$= e^{ax}(A\cos bx + B\sin bx)$$

where $A = c_1 + c_2$, $B = c_1i - c_2i$ are arbitrary constants .
If $a = 0$, then m_1 , m_2 are purely imaginary
i.e $m_1 = ib$, $m_2 = -ib$

i.e The equation is of or contains the form $(D^2 + b^2)$.
In this case , the C.F is given by (or contains the part) $(A\cos bx + B\sin bx)$.

Example 6: Find the C.F of the equation $(D^2 + 4)y = x$.

Sol: The auxiliary equation of the given equation is given by
$m^2 + 4 = 0$ whose roots are $\pm 2i$
Therefore $C.F = A\cos 2x + B\sin 2x$

Example 7: Find the C.F of the equation $(D^2 + 25)(D - 3)y = x$.

Soln: The auxiliary equation of the given equation is given by

$(m^2 + 25)(m - 3) = 0$ whose roots are $5i$, $-5i$, 3
Therefore $C.F = A\cos 5x + B\sin 5x + Ce^{3x}$

Method to find the Particular Integral

Theorem: For a function $q(x)$ of x , we have
$\frac{1}{D-a}q(x) = e^{ax}\int q(x)\,e^{-ax}dx$.

Proof : Taking $y = \frac{1}{D-a}q(x)$
$\Rightarrow (D - a)y = q(x) \quad \Rightarrow \frac{dy}{dx} - ay = q(x)$ ……….. (1)

which is a linear equation of first order . Its integrating factor is
$\mu(x) = e^{\int -a dx} = e^{-ax}$

So the solution of (1) is given by
$ye^{-ax} = \int q(x)\,e^{-ax}dx$

or $y = e^{ax}\int q(x)e^{-ax}dx$

i.e $\frac{1}{D-a}q(x) = e^{ax}\int q(x)\, e^{-ax}dx$

Theorem: For a function $q(x)$ of x , we have
$\frac{1}{D+a}q(x) = e^{-ax}\int q(x)\, e^{ax}dx$.

Proof : As above .

A repeated application of the above formulas gives

Theorem: For any positive integer r we have , $\frac{1}{(D-a)^r}e^{ax} = \frac{x^r}{r!}e^{ax}$

Theorem: If $f(a) \neq 0$ then $\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}$

Proof. Let $f(D) = D^n + c_1D^{n-1} + c_2D^{n-2} + \cdots + c_{n-1}D + c_n$.
We have,

$$De^{ax} = ae^{ax},\ D^2e^{ax} = a^2e^{ax},\ \ldots,\ D^{n-1}e^{ax} = a^{n-1}e^{ax},\ D^ne^{ax} = a^ne^{ax}.$$

$\therefore f(D)e^{ax} = (D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots..a_{n-1}D + a_n)e^{ax}$

$= (a^n + a_1a^{n-1} + \cdots + a_{n-1}a + c_n)e^{ax} = f(a)e^{ax}$

$f(D)e^{ax} = f(a)e^{ax}$

$\Rightarrow e^{ax} = \frac{1}{f(D)}f(a)e^{ax}$

or $e^{ax} = f(a)\frac{1}{f(D)}e^{ax}$

$\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$

Theorem: If $f(D)$ can be expressed as a function of D^2 say $f(D) = \phi(D^2)$ and $\phi(-a^2) \neq 0$ Then

$$\frac{1}{\phi(D^2)}\sin ax = \frac{1}{\phi(-a^2)}\sin ax \quad , \frac{1}{\phi(D^2)}\cos ax = \frac{1}{\phi(-a^2)}\cos ax \quad .$$

Theorem: If $f(D)$ can be expressed as a function of D^2 say $f(D) = \phi(D^2)$

with $\phi(-a^2) = 0$ Then $f(D)$ has a factor of the form $(D^2 + a^2)$
and **(i)** $\frac{1}{D^2+a^2}\sin ax = \frac{x}{2}\int \sin ax\, dx = -\frac{x}{2a}\cos ax$
(ii) $\frac{1}{D^2+a^2}\cos ax = \frac{x}{2}\int \cos ax\, dx = \frac{x}{2a}\sin ax$

The proof of the above theorem is done by taking $\sin ax = \frac{e^{iax}-e^{-iax}}{2i}$, $\cos ax = \frac{e^{iax}+e^{-iax}}{2}$ and use the previous two theorems.

Using the above theorems and more that we shall stated if needed , we list below a quick summary on the methods to find the particular integrals of some of the standards functions $q(x)$.

Summary

(A) To find the P.I of the equation $f(D)y = q(x)$ where $q(x) = e^{ax}$.

Case I: If $f(a) \neq 0$ then $P.I = \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$

Case II: If $f(a) = 0$ then $f(D) = (D-a)^r\phi(D)$ where $\phi(a) \neq 0$
In this case , $P.I = \frac{1}{f(D)}e^{ax} = \frac{1}{(D-a)^r\phi(D)}e^{ax}$
$= \frac{1}{\phi(a)} \times \frac{1}{(D-a)^r}e^{ax} = \frac{1}{\phi(a)} \times \frac{x^r}{r!}\ e^{ax}$

(B) To find the P.I of the equation $f(D)y = q(x)$ where $q(x) = \sin ax$ or $\cos ax$. and $f(D) = \phi(D^2)$ with $\phi(-a^2) \neq 0$ Then
$P.I = \frac{1}{\phi(D^2)}\sin ax = \frac{1}{\phi(-a^2)}\sin ax$ $\ or\ P.I = \frac{1}{\phi(D^2)}\cos\ ax = \frac{1}{\phi(-a^2)}\cos ax$.

(C) To find the P.I of the equation $f(D)y = q(x)$ where $q(x) = \sin ax$ or $\cos ax$. and $f(D)$ cannot be expressed as a function of D^2 .

In this case , we expressed $D^3 = DD^2$, $D^4 = D^2D^2$, $D^5 = D^4D$ and so on , so that $f(D) = \phi(D^2, D)$. replacing D^2 by $-a^2$ will make the denominator linear in D.

Now rationalize the denominator and substitute $D^2 = -a^2$. the numerator will become a linear factor in D .

i.e $\phi(-a^2, D) = k(D+m) \;\; or \;\; k(D-m)$

$$P.I = \frac{1}{\phi(-a^2,D)}\sin ax = \frac{1}{k(D\pm m)}\sin ax = \frac{1}{k}\frac{D\mp m}{(D^2-m^2)}\sin ax = \frac{1}{k}\frac{D\mp m}{(-a^2-m^2)}\sin ax$$

$$or\ P.I = \frac{1}{\phi(D^2)}\cos\ ax = \frac{1}{k(D\pm m)}\cos ax = \frac{1}{k}\frac{D\mp m}{(-a^2-m^2)}\sin ax$$

which can be completed by differentiation .

(D) To find the P.I of the equation $f(D)y = q(x)$ where $q(x)$ is a polynomial in x .

In this case , we express $f(D) = k\big(1 \pm \phi(D)\big)$

so that $P.I = \frac{1}{f(D)}q(x) = \frac{1}{k\big(1\pm\phi(D)\big)}q(x) = \frac{1}{k}\big(1\pm\phi(D)\big)^{-1}q(x)$

Expanding $\big(1\pm\phi(D)\big)^{-1}$ by binomial theorem and differentiate will complete the $P.I$.

Note: $(1+x)^{-1} = 1 - x + x^2 - x^3 \ldots..$
$(1-x)^{-1} = 1 + x + x^2 + x^3 \ldots..$

(E) To find the P.I of the equation $f(D)y = q(x)$ where $q(x)$ is of the form $e^{ax}v(x)$ for some function $v(x)$.

In this case , we use $P.I = \frac{1}{f(D)}e^{ax}v(x) = e^{ax}\frac{1}{f(D+a)}v(x)$ and $\frac{1}{f(D+a)}v(x)$ can be completed by some other methods .

(F) To find the P.I of the equation $f(D)y = q(x)$ where $q(x)$ any other form not mention above .

In this case , resolve $\frac{1}{f(D)}$ into partial fractions with each denominators as linear factor in D and use the rule

$$\frac{1}{D+a}q(x) = e^{-ax}\int q(x)e^{ax}dx \;\; or \;\; \frac{1}{D-a}q(x) = e^{ax}\int q(x)e^{-ax}dx$$

(G) To find the P.I of the equation $f(D)y = xq(x)$,

In this case , Use the rule:

$\frac{1}{f(D)}\left(xq(x)\right) = x\frac{1}{f(D)}q(x) - \frac{f'(D)}{\left(f(D)\right)^2}q(x)$ where $f'(D) = \frac{d}{dD}f(D)$

Example 8: Solve $(D^2 + 4D + 3)y = e^{-2x}$.

Sol: Here Characteristic equation is

$m^2 + 4m + 3 = 0$
$\Rightarrow m^2 + 3m + m + 3 = 0$
$\Rightarrow m(m + 3) + 1(m + 3) = 0$
$\Rightarrow (m + 3)(m + 1) = 0$

Therefore the roots are $m_1 = -3$, $m_2 = -1$
$\Rightarrow$ $C.F = C_1e^{-x} + C_2e^{-3x}$, where C_1 are C_2 are arbitrary constants.

Particular Integral (P.I.) is given by
$P.I = \frac{1}{f(D)}q(x) = \frac{1}{D^2+4D+3}e^{-2x} = \frac{1}{(-2)^2+4(-2)+3}e^{-2x} = \frac{1}{4-8+3}e^{-2x} = -e^{-2x}$

The general solution of the given equation is given by
$y = C.F + P.I$
or $y = C_1e^{-x} + C_2e^{-3x} - e^{-2x}$

Example 9.Solve $(D^2 + 6D + 9)y = 3e^{3x}$

Sol: The auxiliary equation is : $m^2 + 6m + 9 = 0$
or $(m + 3)^2 = 0 \Rightarrow m = -3 , -3$
$\therefore$ $C.F = (c_1x + c_2)e^{-3x}$

$$P.I = \left(\frac{1}{(D^2 + 6D + 9)}\right)3e^{3x}$$

$$= \left(\frac{1}{(3)^2+6(3)+9}\right)e^{3x} = \frac{1}{12}e^{3x}$$

The general solution is : y = C.F + P.I
or $y = (c_1x + c_2)e^{-3x} + \frac{1}{12}e^{3x}$

Example 10: Solve $(D^2 - 4D + 4)y = e^{2x}$.

Sol: Here Characteristic equation is
$m^2 - 4m + 4 = 0$
or $(m - 2)^2 = 0$

The roots are $2, 2$
Therefore $C.F = (c_1 + c_2 x)e^{2x}$
Here $f(D) = (D^2 - 4D + 4) \Rightarrow f(2) = 0$

$$P.I = \frac{1}{f(D)} q(x) = \frac{1}{(D-2)^2} e^{2x} = \frac{x^2}{2!} e^{2x} = \frac{x^2}{2} e^{2x}$$

Therefore , the genral solution of the given equation is given by
$y = C.F + P.I$
or $y = (c_1 + c_2 x)e^{2x} + \frac{x^2}{2} e^{2x}$

Example 11 . Solve the differential equation: $(D^2 + D - 2)y = \sin\ x$

Sol: The Auxiliary equation is given by $m^2 + m - 2 = 0$

or $(m+2)(m-1) = 0$
$\Rightarrow m = -2,1$
C.F. $= c_1 e^{-2x} + c_2 e^{x}$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{f(D)} \sin\ x = \frac{1}{D^2 + D - 2} \sin\ x$

putting $D^2 = -1^2 = -1$
P.I $= \frac{1}{D-3} \sin\ x = \frac{D+3}{D^2 - 9} \sin\ x$, Rationalizing the denominator

$= \frac{(D+3)\sin\ x}{-10}$, Putting $D^2 = -1$

$= \frac{-1}{10}(\text{D}\sin x + 3\sin x) = \frac{-1}{10}(\cos x + 3\sin x)$: $d = \frac{d}{dx}$

Example 12: Solve the differential equation : $(D^2 - 4)y = \sin\ 4x$

Sol: The Characteristic equation is $m^2 - 4 = 0$ whose roots are
$m = -2\ , 2$

Therefore complementary function (C.F.) is
$C.F = C_1 e^{-2x} + C_2 e^{2x}$
Particular Integral (P.I.) ,
$P.I = \frac{1}{D^2 - 4} \sin\ 4x$
$= \frac{1}{-4^2 - 4} \sin\ 4x = -\frac{1}{20} \sin\ 4x$
Hence the general solution is given by

$y = C.F + \boldsymbol{P.I} = \boldsymbol{C_1 e^{-2x} + C_2 e^{2x} - \frac{1}{20}\sin\ 4x}$

Example 13: Solve $(D^2 + D + 1)y = \sin\ 2x$

Sol:

The auxiliary equation is : $m^2 + m + 1 = 0$

$$\Rightarrow m = \frac{-1 \pm i\sqrt{3}}{2} = \left(-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right)$$

Therefore C.F $= e^{\frac{-x}{2}}\left[c_1\cos\left(\frac{\sqrt{3}x}{2}\right) + c_2\sin\left(\frac{\sqrt{3}x}{2}\right)\right]$

P.I $= \left(\frac{1}{(D^2+D+1)}\right)\sin\ 2x = \left(\frac{1}{(-4+D+1)}\right)\sin\ 2x = \left(\frac{1}{D-3}\right)\sin\ 2x$

$= \left(\frac{D+3}{D^2-9}\right)\sin\ 2x = \left(\frac{D+3}{-13}\right)\sin\ 2x = -\frac{2\cos\ 2x}{13} - \frac{3\sin\ 2x}{13}$

The general solution is given by : $y = C.F +$ P.I

or $y = e^{\frac{-x}{2}}\left[c_1\cos\left(\frac{\sqrt{3}x}{2}\right) + c_2\sin\left(\frac{\sqrt{3}x}{2}\right)\right] - \frac{2\cos\ 2x}{13} - \frac{3\sin\ 2x}{13}$

Example 14: Solve $(D^2 - 3D + 2)y = \sin\ 3x$.

Sol. The characteristic equation is :

$m^2 - 3m + 2 = 0$ giving $m = 1.2$

C.F. $= c_1 e^x + c_2 e^{2x}$, c_1, c_2 being arbitrary constants.

P.I $= \frac{1}{D^2-3D+2}\sin\ 3x = \frac{1}{-3^2-3D+2}\sin\ 3x$

$= -\frac{1}{3D+7}\sin\ 3x = -\frac{(3D-7)}{(3D-7)(3D+7)}\sin\ 3x$

$= -\frac{(3D-7)}{9D^2-49}\sin\ 3x = -\frac{(3D-7)}{9(-3^2)-49}\sin\ 3x$

$= \frac{1}{130}(3D-7)\sin\ 3x = \frac{1}{130}(9\cos\ 3x - 7\sin\ 3x)$

$\therefore$ Solution is $y = c_1 e^x + c_2 e^{2x} + \frac{1}{130}(9\cos\ 3x - 7\sin\ 3x)$

Example 15: Solve $(D^2-6D+8)y=(e^{2x}-1)^2+\sin\ 3x$.

Sol. The auxiliary equation is

$D^2-6D+8=0$ whose roots are $m=2,4$

$\therefore$ C.F. $=c_1e^{2x}+c_2e^{4x}$

P.I. corresponding to $(e^{2x}-1)^2=\frac{1}{(D-4)(D-2)}(e^{4x}-2e^{2x}+1)$

$$=\frac{1}{(D-4)(D-2)}e^{4x}-2\frac{1}{(D-2)(D-4)}e^{2x}+\frac{1}{(D-4)(D-2)}e^{0.x}$$

$$=\frac{1}{D-4}\frac{1}{(4-2)}e^{4x}-2\frac{1}{(D-2)(2-4)}e^{2x}+\frac{1}{(0-4)(0-2)}e^{0.x}=\frac{1}{2}xe^{2x}+xe^{2x}+\frac{1}{8}$$

P.I. corresponding to $\sin\ 3x=\frac{1}{D^2-6D+8}\sin\ 3x=\frac{1}{-3^2-6D+8}\sin\ 3x$

$$=-\frac{1}{6D+1}\sin\ 3x=-\frac{6D-1}{36D^2-1}\sin\ 3x=-\frac{(6D-1)\sin\ 3x}{36(-3^2)-1}$$

$$=\frac{1}{325}(18\cos\ 3x-\sin\ 3x)$$

$\therefore$ The general solution is

$y=C.F+P.I$

$$y=c_1e^{2x}+c_2e^{4x}+\frac{x}{2}e^{4x}+xe^{2x}+\frac{1}{8}+\frac{1}{325}(18\cos\ 3x-\sin\ 3x)$$

Example 16: Solve the differential equation $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+10y=e^{2x}$

Sol : $\Rightarrow (D^2-2D+10)y=e^{2x}$

The characteristic equation is: $m^2-2m+10=0$

whose roots are $m=\frac{2\pm\sqrt{4-40}}{2}=1\pm 3i$

C.F. $=e^x(c_1\cos\ 3x+c_2\sin\ 3x)$

P.I. $=\frac{1}{f(D)}F(x)=\frac{1}{f(D)}e^{2x}=\frac{1}{f(2)}e^{2x}$, by putting $D=2$

$=\frac{1}{2^2-2(2)+10}e^{2x}=\frac{1}{10}e^{2x}$

Complete solution is: $y=$ C.F. + P.I

$\Rightarrow y = e^{x}(c_1\cos\ 3x + c_2\sin\ 3x) + \frac{1}{10}e^{2x}$

Example 17: Solve the differential equation: $(D^2 + 2D + 1)y = \cos^2\ x$

Sol: The auxiliary equation is: $m^2 + 2m + 1 = 0$

or $(m+1)^2 = 0$ whose roots are $m = -1, -1$

Therefore C.F. $= e^{-x}(c_1 + c_2 x)$

P.I. $= \frac{1}{f(D)}F(x) = \frac{1}{f(D)}\cos^2\ x = \frac{1}{D^2+2D+1}\left(\frac{1+\cos\ 2x}{2}\right)$

$= \frac{1}{2}\frac{1}{(D^2+2D+1)}1 + \frac{1}{2}\frac{1}{(D^2+2D+1)}\cos\ 2x$

$= \frac{1}{2}\frac{1}{(D^2+2D+1)}e^{0x} + \frac{1}{2}\frac{1}{(D^2+2D+1)}\cos\ 2x$ …………..(1)

Now $\frac{1}{2}\frac{1}{(D^2+2D+1)}e^{0x} = \frac{1}{2}\frac{1}{0+0+1)}e^{0x} = \frac{1}{2}$

and $\frac{1}{2}\frac{1}{(D^2+2D+1)}\cos\ 2x = \frac{1}{2}\frac{1}{(-4+2D+1)}\cos\ 2x$

$=\frac{1}{2}\frac{1}{(2D-3)}\cos\ 2x$

$= \frac{1}{2}\frac{2D+3}{(4D^2-3^2)}\cos\ 2x$, Rationalizing the denominator

$\frac{1}{2}\frac{2D+3}{(4(-4)-3^2)}\cos\ 2x$, Putting $D^2 = -4$

$= \frac{(2D+3)\cos\ 2x}{-50}$

$\therefore$ P. I. $= \frac{1}{2} - \frac{1}{50}(-4\sin 2x + 3\cos 2x)$

Now $y =$ C.F. + P.I

$\Rightarrow y = e^{-x}(c_1 + c_2 x) + \frac{1}{2} - \frac{1}{50}(-4\sin\ 2x + 3\cos\ 2x)$

Example 18: Solve the differential equation: $(D^2 - 6D + 9)y = 1 + x^2$

Sol: The characteristic equation is: $m^2 - 6m + 9 = 0$
or $(m - 3)^2 = 0$ whose roots are $m = 3,3$
Therefore C.F. $= e^{3x}(c_1 + c_2 x)$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2 - 6D + 9}(1 + x^2)$

$= \frac{1}{9\left(1 - \frac{2D}{3} + \frac{D^2}{9}\right)}(x + x^3)$

$= \frac{1}{9}\left(1 - \left(\frac{2D}{3} - \frac{D^2}{9}\right)\right)^{-1} (1 + x^2)$

$= \frac{1}{9}\left[1 + \left(\frac{2D}{3} - \frac{D^2}{9}\right) + \left(\frac{2D}{3} - \frac{D^2}{9}\right)^2 + \cdots\right](1 + x^2)$

$= \frac{1}{9}\left[1 + \frac{2D}{3} - \frac{D^2}{9} + \frac{4D^2}{9} + \cdots\right](1 + x^2)$ (neglecting higher degree of D)

$= \frac{1}{9}\left[1 + \frac{2D}{3} + \frac{D^2}{3} + \cdots\right](1 + x^2)$

$= \frac{1}{9}\left[1 + x^2 + \frac{4x}{3} + \frac{2}{3}\right] = \frac{1}{27}[3x^2 + 4x + 5]$

∴ The general solution is given by
$y = C.F + P.I = e^{3x}(c_1 + c_2 x)\frac{1}{27}[3x^2 + 4x + 5]$

Example 19: Solve the differential equation: $(D^2 + 4)y = x^2 e^{3x}$

Sol: The auxiliary equation is: $m^2 + 4 = 0$
whose roots are $m = \pm 2i$
Therefore $C.F = c_1 \cos 2x + c_2 \sin 2x$

$\text{P.I} = \frac{1}{f(D)} x^2 e^{3x} = \frac{1}{D^2 + 2} x^2 e^{3x}$ ***(of the form*** $\boldsymbol{\frac{1}{f(D)} e^{ax} v(x)}$ ***)***

$= e^{3x} \frac{1}{(D+3)^2 + 4} x^2$

$= e^{3x} \frac{1}{D^2 + 6D + 13} x^2$

$$= \frac{e^{3x}}{13} \frac{1}{\left(1+\frac{6D}{13}+\frac{D^2}{13}\right)} x^2$$

$$= \frac{e^{3x}}{13} \left(1 + \left(\frac{6D}{13} + \frac{D^2}{13}\right)\right)^{-1} x^2$$

$$= \frac{e^{3x}}{13} \left[1 - \left(\frac{6D}{13} + \frac{D^2}{13}\right) + \left(\frac{6D}{13} + \frac{D^2}{13}\right)^2 + \cdots\right] x^2$$

$$= \frac{e^{3x}}{13} \left[1 - \frac{6D}{13} - \frac{D^2}{13} + \frac{36D^2}{169} + \cdots\right] x^2$$

$$= \frac{e^{3x}}{13} \left[1 - \frac{6D}{13} + \frac{23D^2}{169} + \cdots\right] x^2 = \frac{e^{3x}}{13} \left[\frac{46}{169} - \frac{12x}{13} + x^2\right]$$

Therefore , the complete solution of the given equation is
$y = C.F + P.I$
or $y = c_1 \cos 2x + c_2 \sin 2x + \frac{e^{3x}}{13} \left[\frac{46}{169} - \frac{12x}{13} + x^2\right]$

Example 20: Solve $(D^2 - 4D + 3)y = e^x \cos\ 2x$

Sol: The characteristic equation is :

$$m^2 - 4m + 3 = 0$$

or $(m-1)(m-3) = 0 \Rightarrow m = 1\ , 3$

$\therefore C.F = c_1 e^x + c_2 e^{3x}$

P.I $= \left(\frac{1}{(D^2-4D+3)}\right) e^x \cos\ 2x$ (*of the form* $\frac{1}{f(D)} e^{ax} v(x)$)

$$= \left(\frac{e^x}{(D+1)^2-4(D+1)+3}\right) \cos\ 2x = \left(\frac{e^x}{-4-2D}\right) \cos 2x$$

$$= -\frac{1}{2} \left(\frac{e^x}{D+2}\right) \cos\ 2x = -\frac{e^x}{2} \left(\frac{D-2}{D^2-4}\right) \cos\ 2x$$

$$= -\frac{e^x}{2} \left[\frac{(D-2)\cos\ 2x}{-8}\right] = -\frac{e^x}{16} (-2\sin\ 2x - 2\cos\ 2x)$$

$$= \frac{e^x}{8} (\sin\ 2x + \cos\ 2x)$$

The general solution is : $\mathrm{y = C.F + P.I}$
or $y = c_1 e^x + c_2 e^{3x} - \frac{e^x}{8} (\sin\ 2x + \cos\ 2x)$

Example 21: Solve the differential equation: $(D^2+16)y = x\cos\ 5x$

Sol: The auxiliary equation is: $m^2+16=0$
whose roots are $m = \pm 4i$
Therefore C.F. $= (c_1\cos\ 4x + c_2\sin\ 4x)$
P.I. $= \frac{1}{D^2+16} x\cos 5x$
$= x\frac{1}{D^2+16}\cos 5x + \frac{-2D}{(D^2+16)^2}\cos 5x$

$= x\frac{1}{-25+16}\cos 5x + \frac{-2D}{(-25+16)^2}\cos\ 5x,$ Putting $D^2 = -25$

$= \frac{x\cos\ 5x}{-9} - \frac{2D\cos\ 5x}{81}$

$\therefore$ P.I. $= \frac{x\cos\ x}{-9} + \frac{10\sin\ x}{81}$

The general solution is given by $y =$ C.F. + P.I
or $y = c_1\cos\ 4x + c_2\sin\ 4x + \frac{x\cos\ x}{-9} + \frac{10\sin\ x}{81}$

Method of Variation of Parameters

Let $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$…………………….. (1)

be the given equation . where P, Q are constants . The case where P, Q are functions of x will be seen later .

Suppose $y = u\ and\ y = v$ be independent solutions of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ (i.e $u\ and\ v$ are part of the complimentary solution of (1))

Therefore $u_2 + Pu_1 + Qu = 0$, $v_2 + Pv_1 + Qv = 0$…….(*)

Then $y = au + bv$ --------------------- (2)

is also the general solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

Let $y = Au + Bv$ -------------------------- (2)

be the general solution of equation (1) , where A, B are functions of x.
Differentiating we get

$y_1 = Au_1 + Bv_1 + (uA_1 + vB_1)$ ------------- (3)
where $A_1 = \frac{dA}{dx}$.

In order to simplify the process , we take one extra condition called the Auxiliary Condition namely -

$$uA_1 + vB_1 = 0 \quad \text{------------------} \quad (4)$$

so that (3) becomes $y_1 = Au_1 + Bv_1$----------- (5)

$\Rightarrow y_2 = Au_2 + Bv_2 + A_1u_1 + B_1v_1$ ------------ (6)

Putting the value of y, y_1 *and* y_2 from (2) , (5) and (6) in (1)

we get $(Au_2 + Bv_2 + A_1u_1 + B_1v_1) + P(Au_1 + Bv_1) + Q(Au + Bv) = R$
or $A(u_2 + Pu_1 + Qu) + B(v_2 + Pv_1 + Qv) + A_1u_1 + B_1v_1) = R$
$\Rightarrow A_1u_1 + B_1v_1 = R$ ---------------------- (7)
using (*)
Now , A_1 and B_1 can be solved from (4) and (7) to get
$A_1 = \frac{dA}{dx} = h(x)$, $B_1 = \frac{dB}{dx} = g(x)$ say
So that A *and* B can be obtained by integration .

A quick method to find A and B

Solving (4) and (7) we get get

$$A_1 = \frac{dA}{dx} = -\frac{vR}{uv_1 - u_1v} = -\frac{vR}{W} \quad , \quad B_1 = \frac{dB}{dx} = \frac{uR}{uv_1 - u_1v} = \frac{uR}{W}$$

(Where $W = (uv_1 - u_1v) = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix}$ called the Wronskian of u and v)

from which we shall get

$$A = -\int \frac{vR}{W}\, dx + C_1 \quad , \quad B = \int \frac{uR}{W} dx + C_2$$

The general solution is then given by $y = Au + Bv$

or $\boldsymbol{y = -u\int \frac{vR}{W}\, dx + v\int \frac{uR}{W} dx}$

Example 22: Solve the equation $\frac{d^2y}{dx^2} + 9y = \sec x$ by the method of variation of parameter .

Sol: The auxiliary equation : $m^2 + 9 = 0 \Rightarrow m = \pm 3i$
Therefore $C.F = c_1 \cos 3x + c_2 \sin 3x$

We have , $u = \cos x$, $v = \sin x$ are parts of the complimentary function .
The wroskian of u and v is given by

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

Let $y = Au + Bv$ be the complete solution .

Then $A = -\int \frac{vR}{W}\, dx + C_1 = -\int \sin x \sec x \; dx + c_1 = \log \cos x + c_1$
$B = \int \frac{uR}{W} dx + c_2 = \int \cos x \sec x \; dx + c_2 = x + c_2$

Therefore , the complete solution of the given equation is given by
$y = (\log \cos x + c_1) \cos x + (\, x + c_2) \sin x$

Equations Reducible to Linear Form with Constant Coefficients

Cauchy – Euler's / Homogeneous Linear Differential Equation

The differential equation of the form:

$$x^n \frac{d^n y}{dx^n} + c_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + - - - - + c_{n-1} x \frac{dy}{dx} + c_n y = Q(x) \quad \ldots\ldots\ldots\ldots(1)$$

where the c_i 's are constants is called Cauchy – Euler's linear equation These kind of equations can be reduced to linear differential equations with constant coefficients by following substitutions:

$$x = e^z \;\Rightarrow log \;\; x = z$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\,\frac{dy}{dz}$$

$\Rightarrow x \frac{dy}{dx} = \frac{dy}{dz}$ or $xDy = D_1 y$ i.e $xD = D_1$ where $D = \frac{d}{dy}$, $D_1 = \frac{d}{dz}$

In a similar way we will find that

$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} \text{ or } x^2 D^2 = D_1(D_1 - 1)$$

$x^3 D^3 = D_1(D_1 - 1)(D_1 - 2)$ etc .
Equation (1) will then reduce to the form

$(D_1^n + a_1 D_1^{n-1} + a_2 D_1^{n-2} \dots\dots\dots . + a_n)y = q(z)$ .... (2)

Which can be solved as in previous section .

Example 23: Solve $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 0$

Sol. The given equation can be written as
$(x^2D^2 - 3xD + 4)y = 0 \dots\dots\dots\dots$ (1)

where $D = \frac{d}{dx}$
Let $x = e^z$ so that $z = \log\, x$
Then, $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1)$where $D_1 = \frac{d}{dz}$,

with these substitution , (1) reduces to
$\{D_1(D_1 - 1) - 3D_1 + 4\}y = 0$

or $(D_1 - 2)^2 y = 0$ .... (2)

The characteristic equation is $(m - 2)^2 = 0$
with roots 2 , 2.

As R.H.S of (1) is 0 , The general solution is same as C.F.
Therefore The general solution is given by
$y = (c_1 + c_2 z)e^{2z} = (c_1 + c_2 \log\, x)x^2$

Example 24: Solve the differential equation:

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log\, x,\ x > 0$$

Sol: The given equation can be written as

$(x^2D^2 - 2x\mathrm{D} - 4)y = x^2 + 2\log\, x$ .... (1)

Let $x = e^z \therefore \log\, x = z$
Then $x^2D^2 = D_1(D_1 - 1)$, $xD = D_1$ where $D = \frac{d}{dx}$, $D_1 = \frac{d}{dz}$
With these substitutions , equation (1) becomes

$(D_1(D_1 - 1) - 2D_1 - 4)y = e^{2z} + 2z$
or $(D_1^2 - 3D_1 - 4)y = e^{2z} + 2z$ .... (2)

Auxiliary equation of (2) is : $m^2 - 3m - 4 = 0$
$(m + 1)(m - 4) = 0$ whose roots are $m = -1,\ 4$

Therefore C.F. $= c_1 e^{-z} + c_2 e^{4z} = \frac{c_1}{x} + \frac{c_2}{x^4}$

$$\text{P.I} = \frac{1}{(D_1^2 - 3D_1 - 4)}(e^{2z} + 2z)$$

$$= \frac{1}{(D_1^2 - 3D_1 - 4)} e^{2z} + \frac{1}{(D_1^2 - 3D_1 - 4)} 2z$$

$$= \frac{1}{-6} e^{2z} + \frac{1}{-4\left(1 - \left(\frac{1}{4}D_1^2 - \frac{3}{4}D_1\right)\right)} 2z$$

$$= -\frac{e^{2z}}{6} - \frac{1}{2}\left(1 - \left(\frac{D_1^2}{4} - \frac{3}{4}D_1\right)\right)^{-1} z$$

$$= -\frac{e^{2z}}{6} - \frac{1}{2}\left(1 + \left(\frac{D_1^2}{4} - \frac{3D_1}{4}\right) + \cdots.\right) z$$

$$= -\frac{e^{2z}}{6} - \frac{1}{2}\left(z - \frac{3}{4}.\right) = -\frac{x^2}{6} - \frac{1}{2}\log x + \frac{3}{8}$$

The general solution is

$$y = \frac{c_1}{x} + \frac{c_2}{x^4} - \frac{x^2}{6} - \frac{1}{2}\log x + \frac{3}{8}$$

Example 25: Solve $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} - 3y = x^2$

Sol: Taking $D = \frac{d}{dx}$, the given equation can be written as

$(x^3 D^3 + 2x^2 D^2 + 3xD - 3)y = x^2$ …….. (1)

Let $x = e^z \quad \therefore \quad z = \log x$

Let $D_1 = \frac{d}{dz}$

Then (1) becomes

$[D_1(D_1 - 1)(D_1 - 2) + 2D_1(D_1 - 1) + 3D_1 - 3]y = e^{2z}$

or $(D_1^3 - D_1^2 + 3D_1 - 3)y = e^{2z}$ …………. (2)

A. E of (2) is : $m^3 - m^2 + 3m - 3 = 0$

or $(m^2 + 3)(m - 1) = 0$

whose roots are $m = 1, +i\sqrt{3}$

C.F of (2) $= c_1 e^z + c_2 \cos(\sqrt{3}z) + c_3 \sin(\sqrt{3}z)$

and P.I $= \frac{1}{D_1^3 - D_1^2 + 3D_1 - 3} e^{2z}$

$$= \frac{1}{D_1^3 - D_1^2 + 3D_1 - 3} e^{2z} = \frac{1}{8 - 4 + 6 - 3} e^{2z} = \frac{1}{7} e^{2z}$$

Therefore , the complete solution of (2) is

$$y = C.F + PI = c_1 e^{z} + c_2 \cos(\sqrt{3}z) + c_3 \sin(\sqrt{3}z) + \frac{e^{2z}}{7}$$

Hence the complete solution of the given equation is

$$y = c_1 x + c_2 \cos(\sqrt{3}\log x) + c_3 \sin(\sqrt{3}\log x) + \frac{x^2}{7}$$

Example 26: Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$

Sol: Taking $D = \frac{d}{dx}$, the given equation can be written as

$$(x^2 D^2 - xD - 3)y = x^2 \log x \ldots\ldots\ldots\ldots.. \quad (1)$$

Let $x = e^{z}$ $\therefore z = \log x$ and let $D_1 = \frac{d}{dz}$

Then (1) becomes

$$[D_1(D_1 - 1) - D_1 - 3]y = e^{2z}z$$

$$(D_1^2 - 2D_1 - 3)y = ze^{2z} \ldots\ldots\ldots \quad (2)$$

The auxiliary equation of (2) is :

$m^2 - 2m - 3 = 0$ or $(m+1)(m-3) = 0$

$\Rightarrow m = -1, \ 3$

$\therefore$ C.F $= c_1 e^{-z} + c_2 e^{3z}$

$$\text{P.I} = \frac{1}{D_1^2 - 2D_1 - 3} ze^{2z} = e^{2z} \frac{1}{(D_1+2)^2 - 2(D_1+2) - 3} z$$

$$= e^{2z} \frac{1}{D_1^2 + 2D_1 - 3} z = -\frac{e^{2z}}{3}\left[1 - \frac{2}{3}D_1 - \frac{1}{3}D_1^2\right]^{-1} z$$

$$= -\frac{1}{3}e^{2z}\left[1 + \frac{2}{3}D_1\right] z = -\frac{1}{3}e^{2z}\left(z + \frac{2}{3}\right)$$

Therefore ,

$$y = C.F + P.I = c_1 e^{-z} + c_2 e^{3z} - \frac{e^{2z}}{3}\left(z + \frac{2}{3}\right)$$

$$= \frac{c_1}{x} c_2 x^3 - \frac{x^2}{3}\left(\log x + \frac{2}{3}\right)$$

Which is the complete solution of (1)

Legendre's Linear Differential Equation

The differential equation of the form:

$$(ax+b)^n \frac{\mathrm{d^n y}}{\mathrm{dx^n}} + c_1(ax+b)^{n-1}\frac{\mathrm{d^{n-1}y}}{\mathrm{dx^{n-1}}} + \cdots + c_{n-1}(ax+b)\frac{\mathrm{dy}}{\mathrm{dx}} + c_n y = Q(x)$$

….(1) is called Legendre's linear equation .

These equations can be reduced to Cauchy-Euler's form by a substitution

$$(ax+b) = z$$

or reduce to linear differential equations with constant coefficients by a substitutions - $(ax+b) = e^z \Rightarrow z = \log(ax+b)$

So that $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\left(\frac{a}{ax+b}\right)$

$\Rightarrow (ax+b)Dy = (ax+b)\frac{dy}{dx} = a\frac{dy}{dz} = aD_1 y,$ where $D_1 = \frac{d}{dt}$

$$\text{or}\quad (ax+b)\frac{dy}{dx} = (ax+b)Dy = aD_1 y$$

Similarly $(ax+b)^2\frac{d^2y}{dx^2} = (ax+b)^2 D^2 y = a^2 \mathrm{D_1(D_1-1)y}$

$$(ax+b)^3\frac{d^3y}{dx^3} = (ax+b)^3 D^3 y = a^3 \mathrm{D_1(D_1-1)(D_1-2)y}$$

Equation (1) will then reduce to linear form with constant co-efficient of the form $(D_1^n + a_1 D_1^{n-1} + a_2 D_1^{n-2} \ldots\ldots\ldots + a_n)y = q(z)$. which can be solved by the method discussed before .

Example 27: Solve $2(2x+1)^2\frac{d^2y}{dx^2} - (2x+1)\frac{dy}{dx} + 3y = \frac{1}{(2x+1)^2}$

Sol: The given equation can be written as
$\{2(2x+1)^2D^2 - (2x+1)D + 3\}y = (1+2x)^{-2}$ ……….. (2)
where $D = \frac{d}{dx}$
Let $(2x+1) = e^z$ or $\log(2x+1) = z,$
Then, we have $(2x_1+1)D = 2D_1,\ (2x+1)^2D^2 = 2^2D_1(D_1-1)$

Using These , equation (2) becomes
$\{8D_1(D_1-1) - 2D_1 + 3\}y = e^{-2z}$
or $(8D_1^2 - 10D_1 + 3)y = e^{-2z}$……………… (3)

Here auxiliary equation for (3) is
$8m^3 - 10m^2 + 3 = 0$
whose roots are $m = \frac{10 \pm \sqrt{100-96}}{16} = \frac{3}{4}\ , \frac{1}{2}$

Therefore $C.F = c_1\, e^{\frac{3}{4}z} + c_2 e^{\left(\frac{1}{2}\right)z} = c_1(2x+1)^{\frac{3}{4}} + c_2(2x+1)^{\frac{1}{2}}$
$P.I = \frac{1}{8D_1^2 - 10D_1 + 3} e^{-2z} = \frac{e^{-2z}}{55} = \frac{1}{55}(2x+1)^{-2}$
the general solution is then given by

$$y = c_1(2x+1)^{\frac{3}{4}} + c_2(2x+1)^{\frac{1}{2}} + \frac{1}{55}(2x+1)^{-2}$$

Example 28: Solve the equation $\left[(x+2)^2 \frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + 1\right] y = 3x + 4$

Sol. The given equation can be written as
$[(x+2)^2 D^2 - (x+2)D + 1]y = 3x + 4$ ………… (1)

Let $x + 2 = e^z \Rightarrow z = \log(x+2)$
Therefore $(x+2)\frac{dy}{dx} = D_1$, $(x+2)^2 \frac{d^2y}{dx^2} = D_1(D_1 - 1)$

Equation (1) becomes

$[D_1(D_1 - 1) - 2D_1 + 1]y = 3e^z - 2$
or $[D_1^2 - 2D_1 + 1]y = 3e^z - 2$

Its auxiliary equation is
$m^2 - 2m + 1 = 0$ whose roots are $m = 1\ ,1$
Therefore $C.F = (c_1 + c_2 z)e^z = (c_1 + c_2 \log(x+2))\,(x+2)$
P.I $= \frac{1}{D_1^2 - 2D_1 + 1}(\,3e^z - 2) = \frac{1}{D_1^2 - 2D_1 + 1} 3e^z - \frac{2}{D_1^2 - 2D_1 + 1} e^{0z}$
$= \frac{1}{(D-1)^2} 3e^z - 2$
$= \frac{3}{2} z^2 e^z - 2 = \frac{3}{2}(\log(x+2))^2 (x+2) - 2$

The general solution is given by $y = C.F + P.I$

Exercices

1. Solve $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^4 \sin x$, *Ans* : $y = c_1 x^2 + c_2 x^3 - x^2 \sin x$

2. Solve $(D-1)^2(D+1)^2 = \sin^2\frac{x}{2} + e^x + x$

   Ans **:** $y = (c_1 + c_2x)e^x + (c_3 + c_4x)e^{-x} - \frac{1}{8}\cos\ x + \frac{x^2}{8}e^x + x + \frac{1}{2}$

3. $\frac{D^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \log x$ *Ans* : $y = (c_1 + c_2x)e^x + \frac{x^2}{2}e^x\left(\log x - \frac{3}{2}\right)$

4. $(D^2 - 2D + 1)y = xe^x \cos x$, *Ans* : $y = (c_1 + c_2x)e^x + e^x[-\mathrm{x}\cos x + 2\sin x]$

5. $(D^2 - 4D + 4)y = x^2$, Ans: $y = c_1e^{2x} + c_2e^{-3x} - \frac{1}{36}6x + 1)$

6. $(D^2 + 2D + 1)y = 2x + x^2$.

7. $(D-2)^3y = xe^{2x}$.

   Ans. $y = (c_1 + c_2x + c_3x^2)e^{2x} + \frac{\mathrm{x}^4}{24}e^{2x}$

8. $(D^2 - 4D + 1)y = e^{2x}\sin\ x$.

   Ans. $y = c_1e^x + c_2e^{-x} + \frac{1}{5}e^x(2\sin\ x - \cos\ x)$

9. $(D^2 + 1)y = e^{-x} + \cos\ x$.

   Ans. $y = c_1\cos x + c_2\sin x + \frac{1}{2}e^{-x} + \frac{1}{2}\mathrm{x}\sin x$

10. $(2D^2 - \mathrm{D} - 6)y = e^{-\frac{3\mathrm{x}}{2}} + \sin x$

    Ans. $y = c_1e^{2x} + c_2e^{-\left(\frac{3x}{2}\right)} + \frac{\cos x - 8\sin x}{65} - \frac{x}{7}e^{-\left(\frac{3x}{2}\right)}$

Find the solution of the given initial-value problem.

11. $\mathrm{y}'' + \mathrm{y}' - 2\mathrm{y} = \mathrm{x}$; $\mathrm{y}(0) = 0, \mathrm{y}'(0) = 1$.

12. $\mathrm{y}'' + 4\mathrm{y} = \mathrm{x}^2 + \mathrm{e}^\mathrm{x}$; $\mathrm{y}(0) = 0, \mathrm{y}'(0) = 1$.

13. $\mathrm{y}'' - 5\mathrm{y}' + 6\mathrm{y} = \sin 2\mathrm{x}$; $\mathrm{y}(0) = 1, \mathrm{y}'(0) = -1$.

14. $\mathrm{y}'' - 2\mathrm{y}' + \mathrm{y} = \mathrm{xe}^\mathrm{x}$; $\mathrm{y}(0) = 1, \mathrm{y}'(0) = 1$.

Chapter-5

Simultaneous Linear Differential Equations

These kind of equations arise from the case where there are two or more functions of the same variables , i.e there is one independent variable and two or more than two dependent variables. To solve such equations completely, there must be as many equations as there are dependent variables.

In this unit we shall be considering only the case where there are two dependent variables and one independent variable .

Suppose that x and y are functions of $'t'$. Then the differential quations that arise take the form :

$f_1(D)x + f_2(D)y = F(t)$ ……………. (1)

$g_1(D)x + g_2(D)y = G(t)$ …………….. (2)

where $D = \frac{d}{dt}$

Skippinng aside the case where f_1, f_2 are proportional to g_1, g_2 , the above equations can be solved by the process **elimination** ,the general idea where we eliminate one variable x or y by equation the co-efficients of either x or y in both the equations and subtract to get rid of one of them . (the way we solve linear equations in two variables).

Example 1: Solve the eequations

$$\frac{dy}{dt} + y = x + \mathrm{e}^{\mathrm{t}}$$
$$\frac{dx}{dt} + x = y + \mathrm{e}^{\mathrm{t}}$$

Sol. the given equations can be written as

$(D + 1)y - x = e^t$ ………….. (1)

and $-y + (D + 1)x = e^t$ ………… (2)

where $D = \frac{d}{dt}$

Operating (2) by $(D+1)$, we get
$-(\mathrm{D}+1)y+(D+1)^2x=(D+1)e^t=2e^t$ ………… (3)

Adding (1) and (3), we get

$D(D+2)x=3e^t$ …………. (4)

We first solve equation (4) to get the value of x .
The auxillary equation of (4) is

$m(m+2)=0$ so that

$C.F=c_1e^{0t}+c_2e^{-2t}=c_1+c_2e^{-2t}$

$P.I=\frac{1}{D(D+2)}3e^t=e^t$

Therefore the general solution of (4) is given by
$x=c_1+c_2e^{-2t}+e^t$ …………. (5)

Putting this value of x in (2) we get
$y=(D+1)x-e^t=(D+1)(c_1+c_2e^{-2t}+e^t)-e^t$
$=c_1-c_2e^{-2t}+e^t$ ………….. (6)

The general solution of the given equations are given by (5) and (6) .
(*notice the choice of putting the value of* x *in equation (2) , if we put the value of* x *in (1) , then to get the value of* y *we will have to follow the method of finding*

$P.I$, *which may take a bit longer*)

Example 2. Solve the equations

$$\frac{dx}{dt}+2x+3y=0\ ,$$

$$\frac{dy}{dt}+3x+2y=2e^{2t}.$$

Sol. Let $D=\frac{d}{dt}$,

the given equations become

$(D+2)x+3y=0$ ……………… (1)

$3x+(D+2)y=2e^{2t}$ ………… (2)

Operating (2) by $(D+2)$ and multiplying (1) by 3 and subtracting,
we get $[(D+2)^2-9]y=(D+2)2e^{2t}$

or $(D^2+4D-5)y=8e^{2t}$……….. (3)
We now solve equation (3) to get the value of y .

The auxiliary equation of (3) is
$m^2+4m-5=0 \Rightarrow m=1,-5$

∴ C.F. of (3) $=C_1e^t+C_2e^{-5t}$, c_1 and c_2 being arbitrary constants

P.I. $=\frac{1}{D^2+4D-5}8e^{2t}=8\frac{1}{2^2+4\cdot2-5}e^{2t}=\frac{8}{7}e^{2t}$

∴ The general solution of (3) is given by
$y=c_1e^t+\text{c}e^{-5t}+\frac{8}{7}e^{2t}$ …………………… (4)

Differentiating w.r.t. t we get

$\frac{dy}{dt}=c_1e^t-5c_2e^{-5t}+\frac{16}{7}e^{2t}$ ……………… (5)

From (2),
$3x=2e^{2t}-2y-\frac{dy}{dt}$

or $3x=2e^{2t}-2\left\{c_1e^t+c_2e^{-5t}+\frac{8}{7}e^{2t}\right\}-\left\{c_1e^t-5c_2e^{-5t}+\frac{16}{7}e^{2t}\right\}$
(using (5))

or $3x=-3c_1e^t+3c_2e^{-5t}-\frac{18}{7}e^{2t}$
$x=-C_1e^t+C_2e^{-5t}-(6/7)e^{2t}$ …………………. (5)

The required solution is given by (4) and (5).

Example 3: Solve the equations :
$\frac{dx}{dt}=3x+8y$; $\frac{dx}{dt}=-x+-3y$

Also find the solution given that $x(0)=6$ and $y(0)=-2$.

Sol: Taking $\frac{d}{dt} \equiv D$, the given equations can be written as
$Dx - 3x - 8y = 0 \Rightarrow (D-3)x - 8y = 0$ ………. (1)

and $Dy + x + 3y = 0 \Rightarrow (D+3)y + x = 0$ ………….. (2)

Multiplying (1) by $(D+3)$ and (2) by 8 adding we get

$(D^2 - 1)x = 0$ ………….. (3)
The auxiliary equation of (3) is :

$m^2 - 1 = 0 \Rightarrow m^2 = 1$
$\Rightarrow m = \pm 1$
$\therefore\ C.F. = C_1e^t + C_2e^{-t}$, P.I. = 0

Therefore , $x = C_1e^t + C_2e^{-t}$ …….. (4)

From (1) we get $(D-3)[C_1e^x + C_2e^{-x}] = 8y$

$\Rightarrow 8y = -C_1e^t - C_2e^{-t} - 3C_1e^t - 3C_2e^{-t}$

$\Rightarrow 8y = -2C_1e^t - 4C_2e^{-t}$
$\Rightarrow y = -\frac{1}{2}(C_1e^t + 2C_2e^{-t})$ ……….. (5)

The general solution is given by (4) and (5)
Initially when $t = 0$ then $x = 2$.

From (4) we get $2 = C_1e^0 + C_2e^0 \Rightarrow C_1 + C_2 = 2$ …….. (6)
Also when $t = 0, y = -2$.

From (5), $-2 = -\frac{1}{4}(C_1e^0 + 2C_2e^0)$
$\Rightarrow 8 = C_1 + 2C_2$ …………. (7)

Solving (6) and (7), we get $C_1 = -4$ and $C_2 = 6$
Hence, the required solution is : $x = -4e^t + 6e^t$ and $y = -\frac{1}{4}(-4e^t + 12e^t)$

Exercise

1. Solve : $\frac{dx}{dt} + x = y + e^t, \frac{dy}{dt} + y = x + e^t$

Ans : $x = c_1 + c_2e^{-2t} + e^t \;\; , y = c_1 - c_2e^{-2t} + e^t$

2. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + \sin\ 2x$

Ans **:** $y = c_1 + e^{-x}(c_2 + c_3x) + \frac{1}{18}e^{2x} + \frac{1}{100}(6\cos\ 2x - 8\sin\ 2x)$

3. Solve the following set of simultaneous differential equations $\frac{dx}{dt} - 7x + y = 0, \frac{dy}{dt} - 2x - 5y = 0$

Ans : $x = e^{6t}(c_1\cos\ t + c_2\sin\ t)\;\; , \;\; y = e^{6t}(c_1 - c_2)\cos\ t + (c_1 + c_2)\sin\ t)$

4. $(D + 1)x + (2D + 1)y = e^t \;; \quad (D - 1)x + (D + 1)y = 1$

Ans: $x = c_1e^t + c_2e^{-2t} + 2e^{-t} \;\; , \;\; y = 3c_1e^t + 2c_2e^{-2t} + 3e^{-t}$

5. Solve $\frac{dx}{dt} - 3x + 4y = e^{-2t}, \frac{dy}{dt} - x + 2y = 3e^{-2t},$
given that $x = 12, y = 7$ for $t = 0$.

Ans**:** $x = 6e^{2t} + 9e^{-t} - 3e^{-2t}, y = \frac{3}{2}e^{2t} + 9e^{-t} - \frac{7}{2}e^{-2t}$

6. Solve $\frac{dx}{dt} + 2x + 3y = 0 \;\; , \;\; 3x + \frac{dy}{dt} + 2y = 2e^{2t}$

Ans. $x = c_1e^t + c_2e^{-5t} - \frac{6}{7}e^{2t}, y = -c_1e^t + c_2e^{-5t} + \frac{8}{7}e^{2t}$

7. $\frac{dx}{dt} + 5x + y = e^t, \frac{dy}{dt} + 3y - x = e^{2t}$

Ans. $x = (c_1 + c_2t)e^{-4t} + \frac{4}{25}e^t - \frac{1}{36}e^{2t}, y = -(c_1 + c_2 + c_2t)e^{-4t} + \frac{1}{25}e^t + \frac{7}{36}e^{2t}$

8. $\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = 0, \frac{dy}{dt} + 5x + 3y = 0$

Ans. $x = c_1\cos\ t + c_2\sin\ t, y = \frac{1}{2}(c_2 - 3c_1)\cos\ t - \frac{1}{2}(c_1 - 3c_2)\sin\ t$

9. $\frac{\mathrm{dx}}{\mathrm{dt}} = 4\mathrm{x} - 2\mathrm{y}, \frac{\mathrm{dy}}{\mathrm{dt}} = 3\mathrm{x} - \mathrm{y}$

Ans. $x = \frac{2}{3}c_1e^t + c_2e^{2t}, y = c_1e^t + c_2e^{2t}$

10. $\frac{\mathrm{dx}}{\mathrm{dt}} = \mathrm{x} + 2\mathrm{y}, \frac{\mathrm{dy}}{\mathrm{dt}} = \mathrm{y}$

Ans. $x = c_2e^t + 2c_1te^{2t}, y = c_1e^t$

11. $\frac{dx}{dt} = 7x - y, \frac{dy}{dt} = 2x + 5y$

Ans. $x = e^{6t}(c_1\cos\ t + c_2\sin\ t), y = e^{6t}\{(c_1 - c_2)\cos\ t + (c_1 + c_2)\sin\ t\}$

12. $\frac{dx}{dt} + 2x + 3y = 0, \frac{dy}{dt} + 3x + 2y = 2e^{2t}$

Chapter-6

Exact Linear Differential Equation of Higher Order

Definition

A differential equation is said tobe exact if it is obtained by differentiating the next lower order equation .

For Example: An equation

$$x\frac{d^2y}{dx^2} + (1 + \sin x)\frac{dy}{dx} + (\cos x)y = 2x \quad \ldots\ldots\ldots \qquad (1)$$

is obtained by differentiating the equation

$$x\frac{dy}{dx} + (\sin x)y = x^2 + 5 \ldots\ldots\ldots\ldots\ldots \qquad (2)$$

So , according to our definition , eequation (1) is exact .

Solution of exact differential equation

Let $P_0 y^{(n)} + P_1 y^{(n-1)} + P_2 y^{(n-2)} + \cdots \ldots P_n y = \phi(x)$ ……….(A)

be a differential equation of order n , where $y^{(r)} = \frac{d^r y}{dx^r}$

If equation (A) is obtained by differentiating the equation

$P_0 y^{(n-1)} + Q_1 y^{(n-2)} + Q_2 y^{(n-3)} \ldots\ldots\ldots Q_{n-1} y = \phi_1(x) + c_1$ ……….(B)

then the equation (B) is called the **First Integral** of (A)

It is obvious that $\phi_1(x) = \int \phi(x)\, dx$

If further , (B) is exact and is obtained by differentiating the equation
$P_0 y^{(n-2)} + R_1 y^{(n-3)} + R_2 y^{(n-4)} \ldots\ldots\ldots R_{n-2} y = \phi_2(x) + c_2$ ………(C)

then (C) is called the **First Integral** of (B) and the **Second Integral** of (A). etc .

Condition of exactness of a linear differential equation of order n

Let the linear differential equation of order n be

$$P_0 y^{(n)} + P_1 y^{(n-1)} + P_2 y^{(n-2)} + \cdots \ldots P_n y = \phi(x) \quad \ldots\ldots\ldots. \tag{1}$$

where $P_0, P_1, \ldots, P_n$ and ϕ are functions of x .

Let (1) be exact so that it is obtained from an equation of next lower order simply by differentition.

Let $\boldsymbol{P_0 y^{(n-1)} + Q_1 y^{(n-2)} + Q_2 y^{(n-3)} + \cdots \ldots Q_{n-1} y = \int \phi(x) dx + C}$ ………. (2)

be the first integral of (1) .

Differentiating (1) we get

$$P_0 y^{(n)} + (P'_0 + Q_1) y^{(n-1)} + (Q_1' + Q_2) y^{(n-2)} + \cdots$$
$$\ldots \ldots (Q_{n-2}' + Q_{n-1}) y' + Q'_{n-1} y = \phi(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots \tag{3}$$

Equation (1) and (3) are identical , hence the corresponding co-efficients of the derivatives must be equal . Therefore –

$$P_1 = P_0' + Q_1, \quad P_2 = Q_1' + Q_2, \quad P_3 = Q_2' + Q_3, \; \ldots.. P_{n-1} = Q_{n-2}' + Q_{n-1}$$
and $P_n = Q_{n-1}'$ … …. . … (4)

Expressing the Q_i, s in terms of P_i, s from above we have

$Q_1 = P_1 - P_0{}'$,

$Q_2 = P_2 - Q_1{}' = P_2 - \frac{d}{dx}(P_1 - P_0{}') = P_2 - P_1{}' - P_0{}''$,

$Q_3 = P_3 - Q_2{}' = P_3 - \frac{d}{dx}(P_2 - P_1{}' + P_0'') = P_3 - P_2{}' + P_1{}'' - P_0{}'''$,
etc..

$Q_{n-1} = P_{n-1} - P_{n-2}' + P_{n-3}'' - \cdots + (-1)^{n-1} P_0^{(n-1)}$,

$P_n = Q_{n-1}' = \frac{d}{dx}\left[P_{n-1} - P_{n-2}' + P_{n-3}'' - \cdots + (-1)^{n-1} P_0^{(n-1)}\right]$

$$= P'_{n-1} - P''_{n-2} + P'''_{n-3} - \cdots + (-1)^{n-1}P_0^{(n)}$$

$$\Rightarrow \; \boldsymbol{P_n - P'_{n-1} + P''_{n-2} - P'''_{n-3} - \cdots + (-1)^n P_0^{(n)} = 0} \;\ldots\ldots\ldots\ldots \qquad (5)$$

Which is the required condition for equation (1) tobe exact .
Again if we put back the the above values of $Q_1, Q_2, \ldots, Q_{n-1}$ in (1), we get

$$P_0 y^{(n)} + (P_1 - P_0')\mathrm{y}^{(\mathrm{n}-1)} + (P_2 - P_1' + P_0'')\mathrm{y}^{(\mathrm{n}-2)} + \cdots$$
$$\ldots + \left\{P_{n-1} - P'_{n-2} + P''_{n-3} - \cdots + (-1)^{n-1}P_0^{(n-1)}\right\} y = \int \; \phi(x)dx + C.$$

which is always the form of an equation tobe exact .

We list below , for quick use , the equation (5) when $n = 2,3,4$.

(i) The equation $P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = \phi(x)$ is exact if

$P_2 - P_1' + P_0'' = 0$ and the First integral is given by

$P_0 \frac{dy}{dx} + (P_1 - P_0')y = \int \phi(x) + c$

(ii) The equation $P_0 \frac{d^3y}{dx^3} + P_1 \frac{d^2y}{dx^2} + P_2 \frac{dy}{dx} + P_3 y = \phi(x)$ is exact if

$P_3 - P'{}_2 + P_1'' - P_0''' = 0$ and the First integral is given by

$P_0 \frac{d^2y}{dx^2} + (P_1 - P_0')\frac{dy}{dx} + (P_2 - P_1' + P_0'')y = \int \phi(x) + c$

(iii) The equation $P_o \frac{d^4y}{dx^4} + P_1 \frac{d^3y}{dx^3} + P_2 \frac{d^2y}{dx^2} + P_3 \frac{dy}{dx} + P_4 y = \phi(x)$ is exact if

$P_4 - P_3' + P_2'' - P_1''' + P_4'''' = 0$ and the First integral is given by

$P_0 \frac{d^3y}{dx^3} + (P_1 - P_0')\frac{d^2y}{dx^2} + (P_2 - P_1' + P_0'')\frac{dy}{dx} + (\,P_3 - P_2' + P_1'' - P_0''')y =$
$\int \phi(x) + c$

Example 1. Show that the equation $(1 + x^2)y'' + 4xy' + 2y = \sec^2 x$
is exact and solve it .

Sol. Given equation: $(1 + x^2)y'' + 4xy' + 2y = \sec^2 x$ …….. (1)

Comparing (1) with $P_0 y'' + P_1 y' + P_2 y = \phi(x)$

We have $P_0 = 2 + x^2, \quad P_1 = 4x, \quad P_2 = 2, \quad \phi(x) = \sec^2 x$

Since $P_2 - P_1{}' + P_0'' = 2 - 4 + 2 = 0,$
The given equation is exact.
The first integral of (1) is given by

$$P_0 \frac{dy}{dx} + (P_1 - P_0')y = \int \sec^2 x dx + c_1$$

or $(1 + x^2)\frac{dy}{dx} + 2xy = \tan x + c_1$ …………….. (2)

Comparing (2) with $P_0 \frac{dy}{dx} + P_1 y = \phi(x)$

we have $P_0 = (1 + x^2)$, $P_1 = 2x$. Since $P_1 - P_0' = 2x - 2x = 0$
therefore (2) is also exact and its first integral which is also the general solution of (1) is given by

$$P_0 y = \int \phi(x) dx + c_2$$

or $(1 + x^2)y = \int(\tan x + c_1)dx + c_2$

or $(1 + x^2)y = \log \sec x + c_1 x + c_2$

Example 2: Solve $\cos x \left(\frac{d^2y}{dx^2}\right) + \sin x \left(\frac{dy}{dx}\right) + 2y \cos x = 0.$ …….. (1)

Sol. Comparing (1) with $P_0 y'' + P_1 y' + P_2 y = \phi(x)$,

We have $P_0 = \cos x$, $P_1 = \sin x$, $P_2 = 2\cos x$

$P_2 - P_1' + P_0'' = 2\cos x - \cos x - \cos x = 0.$
Hence the given equation is exact and its first integral is

$$P_0 \left(\frac{dy}{dx}\right) + (P_1 - P_0{}')y = c_1.$$

or $\cos x \left(\frac{dy}{dx}\right) - (2\sin x)y = c_1,$ ………… (2)
which is not exact. Rewritting equation (2) as

$$\left(\frac{dy}{dx}\right) - (2\tan x)y = c_1 \sec x$$

which is linear and its integrating factor (I.F.) is given by
I.F. $= e^{\int (-2\tan x)dx} = e^{2\log \cos x} = \cos^2 x$

∴ The solution of (2) (which is the general solution of (1)) is
$y\cos^2 x = c_1 \int \sec x \cos^2 x \, dx + c_2 = c_1 \int \cos x \, dx + c_2$
or $y\cos^2 x = c_1 \sin x + c_2$

Example 3: Solve the equation $(1 + x + x^2)\frac{d^3y}{dx^3} + (3 + 6x)\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 0$

Sol: Comparing the given with the standard equation we have
$P_0 = (1 + x + x^2)$, $P_1 = (3 + 6x)$, $P_2 = 6$, $P_3 = 0$
Clearly , $P_3 - P_2' + P_1'' - P_0''' = 0$

so the given equation is exact and its first integral is given by

$$(1 + x + x^2)\frac{d^2y}{dx^2} + (4x + 2)\frac{dy}{dx} + 2y = c_1 \quad \ldots\ldots\ldots\ldots \quad (1)$$

Again comparing (1) with the standard equation we have
$P_0 = (1 + x + x^2)$, $P_1 = (4x + 2)$, $P_2 = 2$

$P_2 - P_1' + P_0'' = 2 - 4 + 2 = 0$

Therefore (1) is also exact and its first integral is

$$(1 + x + x^2)\frac{dy}{dx} + (2x + 1)y = c_1 x + c_2 \quad \ldots\ldots \quad (2)$$

Comparing (2) with the standard equation we have

$P_0 = (1 + x + x^2)$, $P_1 = (2x + 1)$
$P_1 - P_0' = (2x + 1) - (2x + 1) = 0$
Therefore (2) is exact . Its first integral is given by

$(1 + x + x^2)y = \frac{c_1 x^2}{2} + c_2 x + c_3$
which is the general solution of the given equation .

Example 4: Show that the differential equation

$$(1 + x^2)\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0$$ is exact and hence solve it.

Sol: Comparing the given equation with the standard equation , we have

$P_0 = (1 + x^2)$, $P_1 = 3x$, $P_2 = 1$
Also $P_2 - P_1' + P_0'' = 1 - 3 + 2 = 0$

Therefore the given equation is exact and its first integral is given by
$(1+x^2)\frac{dy}{dx}+x\,y=c_1$ which is not exact but can be written as
$\frac{dy}{dx}+\left(\frac{x}{(1+x^2)}\right)y=\frac{c_1}{1+x^2}$ ….. (1)

which is linear and its integrating factor is

$$\mu(x)=\exp\int\frac{x}{1+x^2}dx=\exp\int\frac{2x}{2(1+x^2)}dx=\exp\frac{1}{2}\log(1+x^2)$$
$$=\exp\log\sqrt{1+x^2}=\sqrt{1+x^2}$$

Therefore the solution of (1) is given by

$y\sqrt{1+x^2}=\int\frac{c_1\sqrt{1+x^2}}{1+x^2}dx+c_2$
or $y\sqrt{1+x^2}=\int\frac{c_1}{\sqrt{1+x^2}}dx+c_2=c_1\log|x+\sqrt{1+x^2}|+c_2$

which is the general solution of the given equation .

Example 5: Test for exactness and solve $(1+x^2)y''+4xy'+2y=\sec^2 x$ given that $y=0,\ y'=1$ when $x=0$.

Sol: Comparing the given equation with the standard equation , we have
$P_0=(1+x^2)$, $P_1=4x$, $P_2=2$

Now , $P_2-P_1'+P_0''=2-4+2=0$
Therefore the given equation is exact . Its first integral is given by
$P_0\frac{dy}{dx}+(P_1-P_0')y=\int 0\ dx$

or $(1+x^2)\frac{dy}{dx}+2x\,y=c_1$ ……… (1)

Again Comparing the last equation with the standard equation , we have
$P_0=(1+x^2)$, $P_1=2x$ so that $P_1-P_0'=0$

Therefore equation (1) is also exact and its first integral is
$(1+x^2)y=c_1x+c_2$ ………. (2)

which is the general solution of the given equation .
Differentiating (2) w.r.t x we get

Using $y=0,\ y'=1$ when $x=0$ in (1) and (2) we get $c_1=1$, $c_2=0$
$\Rightarrow (1+x^2)y=x$ is the solution with the given initial values of x, y, y' .

Example 6: Solve $(x^3-4x)\frac{d^3y}{dx^3}+(9x^2-12)\frac{d^2y}{dx^2}+18x\frac{dy}{dx}+6y=2.$

Sol: Comparing the given equation with the standard equation

$$P_0y'''+P_1y''+P_2y'+P_3y=\phi(x)$$

we have $P_0=x^3-4x,\quad P_1=9x^2-12,\quad P_2=18x,\quad P_3=6.$
Then

$$P_3-P_2'+P_1''-P_0'''=6-18+18-6=0.$$

Therefore, given equation is exact. Its First Integral is given by

$$P_0\frac{d^2y}{dx^2}+(P_1-P_0')\frac{dy}{dx}+(P_2-P_1'+P_0'')y=\int\phi(x)\,dx+c_1$$

Or $(x^3-4x)\frac{d^2y}{dx^2}+(6x^2-8)\frac{dy}{dx}+6xy=2x+c_1$ …….. (1)

Again Comparing equation (1) with the standard equation

$$P_0y''+P_1y'+P_2y=\phi(x)$$

We have , $P_0=(x^3-4x)$, $P_1=(6x^2-8)$, $P_2=6x$
Also $P_2-P_1'+P_0''=6x-12x+6x=0$.

Therefore equation (1) is also exact and its first integral is given by

$$P_0\frac{dy}{dx}+(P_1-P_0')y=\int\phi(x)\,dx+c_2$$

or $(x^3-4x)\frac{dy}{dx}+(3x^2-4)y=2x^2+c_1x+c_2$ ……….. (2)

Again , comparing (2) with $P_0\frac{dy}{dx}+P_1y=\phi(x)$
we have $P_0=(x^3-4x)$, $P_1=3x^2-4$ and $P_1-P_0'=0$
Therefore (2) is exact , its integral is

$$(x^3-4x)y=\frac{2}{3}x^3+\frac{1}{2}c_1x^2+c_2x+c_3$$

and is the general solution of the given equation .

Exercises

1. Show that the equation
   $\sin x\frac{d^2y}{dx^2}+(\sin x+\cos x)\frac{dy}{dx}+(\cos x)y=e^x$ *is* exact and solve it .

***Ans*:** $y \sin x = e^x + c_1 x + c_2$

2. **Solve :** $\sin^2 x \frac{d^2y}{dx^2} - 2y = 0$

3. Show that equation $x^2(1+x)y'' + 2x(2+3x)y' + 2(1+3x)y = 0$ is exact and solve it . Also find the particular solution given that $x = 1, y = 1, \frac{dy}{dx} = 0$.

   ***Ans* :** $yx^2(x+1) = c_1 x + c_2$ and $yx^2(x+1) = 5x - 3$,

4. Solve the equation $(2x^2 + 3x)\frac{d^2y}{dx^2} + (6x+3)\frac{dy}{dx} + 2y = (x+1)e^x$.

   Ans. $y(3 + 2x) = e^x + c_1 \log x + c_2$

5. Show $(1 + x^2)y'' + 4xy' + 2y = \sec^2 x$ is exact and solve ,

   given that $y = 0, y' = 1$ when $x = 0$.

6. **Solve** $(1 + x^2)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y + \frac{x}{(1+x^2)^2} = 0$

   ***Ans* :** $y = \frac{\tan^{-1} x}{1+x^2} + c_1 \frac{x}{1+x^2} + \frac{c_2}{1+x^2}$

7. Solve $x\frac{d^3y}{dx^3} + (x^2 + x + 3)\frac{d^2y}{dx^2} + (4x+2)\frac{dy}{dx} + 2y = 0$.

   ***Ans*:** $y = \frac{1}{x} e^{-\left(x+\frac{x^2}{2}\right)} \int (c_1 x + c_2) e^{x+\frac{x^2}{2}} dx + \frac{c_3}{x} e^{-\left(x+\frac{x^2}{2}\right)}$

8. **Solve:** $\sin x \frac{d^2y}{dx^2} - \cos x \frac{dy}{dx} + 2\sin x y = 0$.

   Ans. $y = c_1 \sin^2 x + c_2 \cos x - c_2 \sin^2 x \log \tan \frac{x}{2}$

9. **Solve :** $(1 + x + x^2)\frac{d^3y}{dx^3} + (3 + 6x)\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = x^2$.

   Ans. $(1 + x + x^2)y = \frac{x^5}{60} + c_1 x^2 + c_2 x + c_3$

Chapter-7

Linear Differential Equations of Second Order

Th general linear equations of second order take the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \ldots\ldots\ldots\ldots \qquad (1)$$

where P , Q , R are constants or functions of x .

The equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ …………………. (2)

is called the homogeneous part of equation (1) .

The solutions of equation (2) are called the Complimentary Function of (1) .

Some standard solutions of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$:

(A) Taking $y = x^n$, we have

$$\frac{dy}{dx} = nx^{n-1} \quad , \quad \frac{d^2y}{dx^2} = n(n-1)x^{n-2}$$

Therefore $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \Rightarrow n(n-1)x^{n-2} + Pnx^{n-1} + Qx^n = 0$

$$\Rightarrow n(n-1) + Pnx + Qx^2 = 0$$

(B) Taking $y = x$, i.e taking $n = 1$ in (A) we have

$$\frac{dy}{dx} = 1 \quad , \quad \frac{d^2y}{dx^2} = 0$$

Therefore $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \Rightarrow 0 + P + Qx = 0$

$$\Rightarrow P + Qx = 0$$

(C) Taking $y = x^2$, i.e taking $n = 2$ in (A) we have

$$\frac{dy}{dx} = 2x \quad , \quad \frac{d^2y}{dx^2} = 2$$

Therefore $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \Rightarrow 2 + 2Px + Qx^2 = 0$

(D) Taking $y = e^{ax}$, we have

$\frac{dy}{dx} = ae^{ax}$, $\frac{d^2y}{dx^2} = a^2e^{ax}$

Therefore $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \Rightarrow a^2e^{ax} + aPe^{ax} + Qe^{ax} = 0$

$\Rightarrow a^2 + aP + Q = 0$

(E) Taking $y = e^x$, i.e $a = 1$ in (D) we have

$\frac{dy}{dx} = e^x$, $\frac{d^2y}{dx^2} = e^x$

Therefore $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \Rightarrow e^x + Pe^x + Qe^x = 0$

$\Rightarrow 1 + P + Q = 0$

(F) Taking $y = e^{-x}$, i.e $a = -1$ in (D) we have

$\frac{dy}{dx} = -e^x$, $\frac{d^2y}{dx^2} = e^x$

Therefore $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \Rightarrow e^x - Pe^x + Qe^x = 0$

$\Rightarrow 1 - P + Q = 0$

Summary: We take a list below the above six complimentary functions of the general equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ depending on the relation of P and Q .

	Condition	**integral of C.F**
1	$n(n-1) + Pnx + Qx^2 = 0$	$y = x^n$
2	$P + Qx = 0$	$y = x$
3	$2 - Px + Qx^2 = 0$	$y = x^{-1}$
4	$2 + 2Px + Qx^2 = 0$	$y = x^2$
5	$a^2 + aP + Q = 0$	$y = e^{ax}$
6	$1 + P + Q = 0$	$y = e^x$
7	$1 - P + Q = 0$	$y = e^{-x}$

General Solution when one integral of the complementary function is known

Let $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ ………….. (1)

be the given equation and $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ ……….. (2)

be the homogeneous part .

Let $y = u$ be a known part of the complementary function.
Thus $y = u$ is a solution of (2)

$\Rightarrow \frac{d^2u}{dx} + \mathrm{P}\frac{du}{dx} + Qu = 0.$ ………….. (3)

Now let the complete solution of (1) be

$$y = uv$$

where v is a function of x.

Differentiating we get $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

and $\frac{d^2y}{dx^2} = v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}$

Therefore (1) reduces to

$$\left(v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx}\right) + P\left(v\frac{du}{dx} + u\frac{dv}{dx}\right) + Quv = R$$

or $v\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) + u\left(\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right) + 2\frac{du}{dx}\frac{dv}{dx} = R$

or $u\left(\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right) + 2\frac{du}{dx}\frac{dv}{dx} = R,$ using (3)

or $\boldsymbol{\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}}$ ……… ….. (4)

Now putting $\frac{dv}{dx} = q$ so that $\frac{d^2v}{dx^2} = \frac{dq}{dx}$,

Equation (5) will reduce to $\boldsymbol{\frac{dq}{dx} + \left(P + \frac{2}{u}\frac{du}{dx}\right)q = \frac{R}{u}}$
which is a linear equation that can be solved for q and ultimately solve for v

Example 1 : Solve $\frac{d^2y}{dx^2} - \frac{x}{(x-1)}\frac{dy}{dx} + \frac{y}{(x-1)} = x - 1.$

Sol. Comparing the given equation with the standard equation

$$\frac{d^2y}{dx^2} + P\,\frac{dy}{dx} + Qy = R$$

We have $P = -\frac{x}{(x-1)}$, $Q = \frac{1}{(x-1)}$, $R = x - 1$

Now , $P + Qx = -\frac{x}{(1-x)} + \frac{1}{(x-1)}x = 0$

$\therefore u = x$ is a part of the Complimentary Function
Let $y = vx$ be the general solution of of the given equation .
*Differentiating $y = vx$ twice we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{and} \quad \frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

Putting these in the given equation we get ,

$$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - \frac{x}{x-1}\left(v + x\frac{dv}{dx}\right) + \frac{vx}{x-1} = (x-1)$$

** or $x\frac{d^2v}{dx^2} + \left(2 - \frac{x^2}{x-1}\right)\frac{dv}{dx} = (x-1)$

or $\frac{d^2v}{dx^2} + \left(\frac{2}{x} - \frac{x}{(x-1)}\right)\frac{dv}{dx} = \frac{x-1}{x}$ ………….(1)

Let $\frac{dv}{dx} = z \Rightarrow \frac{d^2v}{dx^2} = \frac{dz}{dx}$, then (1) becomes

$$\frac{dz}{dx} + \left(\frac{2}{x} - \frac{x}{(x-1)}\right)z = \frac{(x-1)}{x} \quad \text{………..(2)}$$

which is a linear equation in z .

$$I.F = e^{\int\left(\frac{2}{x}-1-\frac{1}{x-1}\right)dx} = e^{2\log x - x - \log(x-1)} = e^{\log\left(\frac{x^2}{x-1}\right)-x} = \frac{x^2e^{-x}}{x-1}$$

The solution of (1) is :
$$z\frac{x^2e^{-x}}{x-1} = \int \frac{(x-1)}{x}\cdot\frac{x^2}{x-1}e^{-x}dx + c_1$$

or $\quad z\frac{x^2e^{-x}}{x-1} = \int\; xe^{-x}dx + c_1$

or $\; z\frac{x^2e^{-x}}{x-1} = -e^{-x}(x+1) + c_1$

or $\; z = \frac{-(x^2-1)}{x^2} + c_1\frac{(x-1)}{x^2}e^x \;\; \Rightarrow \;\; \frac{dv}{dx} = \frac{-(x^2-1)}{x^2} + c_1\frac{(x-1)}{x^2}e^x$

$\Rightarrow dv = \left[-1 + \frac{1}{x^2} + c_1e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)\right]dx$

Integrating we get
$v = \int\; \left[-1 + \frac{1}{x^2} + c_1e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)\right]dx = -x - \frac{1}{x} + \frac{c_1e^x}{x} \; + c_2$

Hence the general solution of the given equation is :

$y = vx = (-x - \frac{1}{x} + \frac{c_1e^x}{x} \; + c_2\;)x$

or $\; y = -x^2 - 1 + c_1e^x + c_2x$

(Note : readers can skip line $*$ to $**$, instead can remember equation (4) in the discussion and write the reduced equation only)

Example 2: Solve $\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \frac{2}{x^2}y = x\,log\,x.$

Sol: Given $\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \frac{2}{x^2}y = x\,log\,x.$ ……… (1)

Comparing the given equation with the standard equation
$\frac{d^2y}{dx^2} + P\,\frac{dy}{dx} + Qy = R$, we have $P = -\frac{2}{x},\;\; Q = \frac{2}{x^2},\;\; R = x\log\,x$
Clearly $P + Qx = 0$

$\therefore u = x$ is a part of the C.F of (1)
Let $y = vx$ be the general solution
Then v is fround from the equation :

$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$ (where $u = x$)
$\frac{d^2v}{dx^2} + \left(-\frac{2}{x} + \frac{2}{x}\cdot 1\right)\frac{dv}{dx} = \frac{x\cdot\log\,x}{x}$

$\frac{d^2v}{dx^2} = \log\ x$ ………. (2)

$\Rightarrow \frac{d}{dx}\left(\frac{dv}{dx}\right) = \log x \ \Rightarrow \quad d\ \left(\frac{dv}{dx}\right) = \ \log x\ dx$

$\Rightarrow \frac{dv}{dx} = \int \log x\ dx = \ (\log x) \int 1\ dx - \int (\frac{d}{dx} \log x\ \int 1 dx\ )\ dx$

$= x \log x - x + c_1$

$\Rightarrow v = \int (\ x \log x - x + c_1)\ dx = \ \frac{x^2}{2} \log\ x - \frac{x^2}{4} - \frac{x^2}{2} + c_1 x + c_2$

Hence the complete solution of (1) is
$y = vx\ = \left(\frac{x^2}{2} \log x - \frac{x^2}{4} - \frac{x^2}{2} + c_1 x + c_2\right) x$

Example 3: Solve $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = x^3$

Sol: Given : $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = x$ ……….. (1)

Comparing the given equation with the standard equation
$\frac{d^2y}{dx^2} + P\ \frac{dy}{dx} + Qy = R$, we have

$P = -x^2$, $Q = x$, $R = x$
Since $P + Qx = 0$ therefore
$u = x$ is a part of the C.F . of (1)

Let $\boldsymbol{y = uv = vx}$ be the complete solution of (1)
Then v is given by the equation

$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\right) \frac{dv}{dx} = \frac{R}{u}$

or $\frac{d^2v}{dx^2} + \left(\ -x^2 + \frac{2}{x}\right) \frac{dv}{dx} = 1$ ………… (2)

Let $\frac{dv}{dx} = q \quad so\ that\ \frac{d^2v}{dx^2} = \frac{dq}{dx}$

Then (2) becomes $\frac{dq}{dx} + \left(\frac{2}{x} - x^2\right) q = 1$ ……… (3)

which is linear and first order .

$I.F = e^{\int\left(\frac{2}{x}-x^2\right)dx} = e^{\log x^2-\frac{x^3}{3}} = e^{\log x^2}e^{-\frac{x^3}{3}} = x^2e^{-\frac{x^3}{3}}$

Therefore , solution of (3) is :

$$qx^2e^{-\frac{x^3}{3}} = \int x^2e^{-\frac{x^3}{3}}dx = -e^{-\frac{x^3}{3}} + c_1$$

$\Rightarrow \; q = \frac{dv}{dx} = -\frac{1}{x^2} + \frac{c_1e^{\frac{x^3}{3}}}{x^2}$

$\Rightarrow v = \int\left(-\frac{1}{x^2} + \frac{c_1e^{\frac{x^3}{3}}}{x^2}\right)dx \; + c_2$

The general solution is then given by $y = vx$ where v is given by the last equation .

Example 4: Solve $(x+1)\frac{d^2y}{dx^2} - 2(x+3)\frac{dy}{dx} + (x+5)y = e^x$

Sol. The given equation can be written as :

$$\frac{d^2y}{dx^2} - \frac{2(x+3)}{x+1}\frac{dy}{dx} + \frac{x+5}{x+1}y = \frac{e^x}{x+1}. \quad \ldots\ldots\ldots\ldots\ldots \qquad (1)$$

Comparing (1) with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, we get

$P = -\frac{2(x+3)}{x+1},\; Q = \frac{x+5}{x+1},\; R = \frac{e^x}{x+1}$

Here $\quad 1 + P + Q = 1 - \frac{2x+6}{x+1} + \frac{x+5}{x+1} = \frac{x+1-(2x+6)+x+5}{x+1} = 0$.

Therefore $\quad u = e^x$

is a part of the Complimentary Function of (1).

Let the general solution of (1) be $\quad y = uv$.

Then v is given by the equation :

$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$

or $\quad \frac{d^2v}{dx^2} + \left(-\frac{2x+6}{x+1} + \frac{2}{e^x}\frac{de^x}{dx}\right)\frac{dv}{dx} = \frac{e^x}{e^x(x+1)}$

or $\frac{d^2v}{dx^2} + \left(2 - \frac{2x+6}{x+1}\right)\frac{dv}{dx} = \frac{1}{x+1}$

or $\frac{d^2v}{dx^2} - \left(\frac{4}{x+1}\right)\frac{dv}{dx} = \frac{1}{x+1}$ ………….. (2)

Let $\frac{dv}{dx} = q$ so that $\frac{d^2v}{dx^2} = \frac{dq}{dx}$

Then (2) becomes

$\frac{dq}{dx} - \frac{4}{x+1}q = \frac{1}{x+1}$ ………… (3)
which is linear in q and x.

Its integrating factor I.F. $= e^{-\int [\frac{4}{x+1}]dx} = e^{-4\log(x+1)} = \mathrm{e}^{\log(x+1)^{-4}} = (x+1)^{-4}$. and solution of (3) is

$$q(x+1)^{-4} = \int \frac{1}{x+1} \cdot (x+1)^{-4}dx + C_1 = \int (x+1)^{-5}dx + c_1$$

or $\frac{dv}{dx} = -\frac{1}{4} + c_1(x+1)^4$ or $dv = \left[-\frac{1}{4} + c_1(x+1)^4\right]dx.$

Integrating , we get $v = -\frac{x}{4} + \frac{c_1}{5}(x+1)^5 + c_2$.
Hence , the complete solution of (1) is given by

$y = uv = e^x\left(-\frac{x}{4} + \frac{c_1}{5}(x+1)^5 + c_2\right)$

Example 5: Sole the equation $\frac{d^2y}{dx^2} + (1-\cot x)\frac{dy}{dx} - (\cot x)y = \sin^2 x$.

Sol: Given $\frac{d^2y}{dx^2} + (1-\cot x)\frac{dy}{dx} - (\cot x)y = \sin^2 x$ …… (1)

Comparing the given equation with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q\,y = R$
we have $P = (1-\cot x)$, $Q = (-\cot x)$, $R = \sin^2 x$
Since $(1 - P + Q) = 0$, therefore ,
$u = e^{-x}$ is part of the $C.F$ of (1)
Let $y = uv$ be the complete solution of (1) .
Then v is given by
$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$

or $\frac{d^2v}{dx^2} - (1+\cot x)\frac{dv}{dx} = e^x\sin^2 x$ ……… (2)

Let $q = \frac{dv}{dx}$ so that $\frac{dq}{dx} = \frac{d^2v}{dx^2}$

Equation (2) becomes
$\frac{dq}{dx} - (1 + \cot x)q = e^x \sin^2 x$
which is a linear equation of first order , Its I.F is given by

$$I.F = e^{\int -(1+\cot x)dx} = \frac{e^{-x}}{\sin x}$$

Therefore , its solution is given by

$$q\frac{e^{-x}}{\sin x} = \int \frac{e^{-x}}{\sin x}\ e^x \sin^2 x\ dx = \int \sin x\, dx = -\cos x + c_1$$

$$\Rightarrow\ q = e^x(-\sin x\cos x + c_1 \sin x) = e^x\left(-\frac{1}{2}\sin 2x + c_1 \sin x\right)$$

$$\Rightarrow \frac{dv}{dx} = c_1 e^x \sin x - \frac{1}{2}e^x \sin 2x$$

$$\Rightarrow v = \int c_1 e^x \sin x\, dx - \frac{1}{2}\int e^x \sin 2x\, dx$$

$$= c_1\frac{e^x}{2}(\sin x - \cos x) - \frac{1}{2}\ \frac{e^x}{5}(\sin 2x - 2\cos 2x) + C_2$$

$$= C_1 e^x(\sin x - \cos x) - \frac{e^x}{10}(\sin 2x - 2\cos 2x) + C_2 \quad : \text{ where } C_1 = \frac{c_1}{2}$$

Hence , the complete solution of (1) is given by

$$y = uv = \left(C_1 e^x(\sin x - \cos x) - \frac{e^x}{10}(\sin 2x - 2\cos 2x) + C_2\right)e^{-x}$$

$$= C_1(\sin x - \cos x) - \frac{1}{10}(\sin 2x - 2\cos 2x) + C_2 e^{-x}$$

Changing the dependent Variable / Reducting to Normal form / removal of first derivative

Let $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ ………….. (1)

Let the general solution of (1) be

$y = uv$ ……… (2)

where u and v are functions of x.

Differentiating (2) twice we get

$\frac{dy}{dx} = \frac{du}{dx} v + u\frac{dv}{dx}$ and $\frac{d^2y}{dx^2} = \frac{d^2u}{dx^2} v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}$

Putting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from above in (1) we get

$$\frac{d^2u}{dx^2} v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2} + P\left(\frac{du}{dx} v + u\frac{dv}{dx}\right) + Quv = R$$

or $u\frac{d^2v}{dx^2} + \left(Pu + 2\frac{du}{dx}\right)\frac{dv}{dx} + v\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) = R$

or $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} + \frac{1}{u}\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) v = \frac{R}{u}$ ………. (3)

If we take $P + \frac{2}{u}\frac{du}{dx} = 0$, the first order derivative in (3) will be removed

and reduce to $\frac{d^2v}{dx^2} + \frac{1}{u}\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) v = \frac{R}{u}$ ………. (4)

Now $P + \frac{2}{u}\frac{du}{dx} = 0 \Rightarrow \frac{du}{u} = -\frac{1}{2}P\,dx \Rightarrow \log u = -\frac{1}{2}\int P\,dx$

or $\boldsymbol{u = e^{-\frac{1}{2}\int Pdx}}$ …………… (5)

$\Rightarrow \frac{du}{dx} = e^{-\frac{1}{2}\int Pdx}\left(-\frac{1}{2}P\right) = -\frac{Pu}{2}$ $\left(\frac{d}{dx}\int P\,dx = P\right)$

$$\Rightarrow \frac{d^2u}{dx^2} = -\frac{1}{2}P\frac{du}{dx} - \frac{1}{2}\frac{dP}{dx}u$$

$\Rightarrow \frac{d^2u}{dx^2} = -\frac{1}{2}P\left(-\frac{1}{2}Pu\right) - \frac{1}{2}\frac{dP}{dx}u,$ putting $\frac{du}{dx} = \frac{Pu}{2}$

$$\Rightarrow \frac{1}{u}\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) = \frac{1}{u}\left(\frac{1}{4}P^2u - \frac{1}{2}\frac{dP}{dx}u - \frac{1}{2}P^2u + Qu\right)$$

$= Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = I$ (say)

Therefore , equation (4) reduces to

$\boldsymbol{\frac{d^2v}{dx^2}} + Iv = S$ ………. (6)

where $\boldsymbol{S = \frac{R}{u}}$, $\boldsymbol{I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}}$

Equation (6) is called the **normal form** of equation of (1)

Equation (6) can be solved easily if I is constant or $I = \frac{k}{x^2}$ for some constant k ……………………..

Example: Solve the equation $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y = \frac{1}{x}\sin 2x$

Sol: Comparing the given equation with the standard equation we have

$P = \frac{2}{x}$, $Q = 1$, $R = \frac{1}{x}\sin 2x$

Let $u = e^{-\frac{1}{2}\int P dx} = e^{-\int \frac{1}{x} dx} = e^{\log \frac{1}{x}} = \frac{1}{x}$

Let $y = uv$ be the general solution . Then the given equation reduces to

$\frac{d^2v}{dx^2} + \left(Q - \frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx}\right) v = \frac{R}{u}$

or $\frac{d^2v}{dx^2} + \left(1 - \frac{1}{x^2} + \frac{1}{x^2}\right) v = \sin 2x$

or $(D^2 + 1)v = \sin 2x$ …………….. (1) where $D = \frac{d}{dx}$

The auxiliary equation of (1) is $m^2 + 1 = 0 \Rightarrow m = \pm i$

Therefore $C.F = a\cos x + b\sin x$

$P.I = \frac{1}{D^2+1}\sin 2x = \frac{1}{-4+1}\sin 2x = -\frac{1}{3}\sin 2x$

$\Rightarrow v = a\cos x + b\sin x - \frac{1}{3}\sin 2x$

Thus , the general solution of the given equation is $y = uv$

or $y = \frac{1}{x}\left(a\cos x + b\sin x - \frac{1}{3}\sin 2x\right)$

Example 6: Solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2}\sin x$

Sol: Given $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2}\sin x$ ………. (1)

Comparing with the standard equation $\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}y = \mathrm{R}$

We have $P = -4x$, $Q = 4x^2 - 1$, $R = -3e^{x^2}\sin x$

Let $u = e^{-\frac{1}{2}\int Pdx} = e^{-\frac{1}{2}\int (-4x)dx} = e^{2\cdot\frac{x^2}{2}} = e^{x^2}$

let $y = uv$ be the complete solution of (1)
With this substitution , equation (1) reduces to normal form
$\frac{d^2v}{dx^2} + Iv = S$ ………… (2)

where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = 4x^2 - 1 - 4x^2 + 2 = 1$

and $S = \frac{R}{u} = \frac{-3e^{x^2}\sin x}{e^{x^2}} = -3\sin x$
Therefore (2) becomes $\frac{d^2v}{dx^2} + v = -3\sin x$ ………. (3)

The characteristic equation of (3) is : $m^2 + 1 = 0$
$\Rightarrow m = \pm i$ and $C.F = c_1\cos x + c_2\sin x$

$\text{P.I} = \frac{1}{D^2+1}(-3\sin x) = (-3)\frac{1}{D^2+1}\sin x = (-3)\left(-\frac{x}{2}\cos x\right)$

The complete primitive of the given equation is
$y = uv = e^{x^2}\left[c_1\cos x + c_2\sin x + \frac{3}{2}x\cos x\right]$

Example 7: Solve $\frac{d^2y}{dx^2} + (4\operatorname{cosec}2x)\frac{dy}{dx} + (2\tan^2 x)y = e^x\cot x$ reducing to normal form .

Sol. Given $\frac{d^2y}{dx^2} + (4\operatorname{cosec}2x)\frac{dy}{dx} + (2\tan^2 x)y = e^x\cot x$ ……….. (1)

Comparing with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

We have $P = 4\operatorname{cosec}2x$, $Q = 2\tan^2 x$ and $R = e^x\cot x$

Let $u = e^{-\frac{1}{2}\int Pdx} = e^{\int(-2\operatorname{cosec}2x)dx} = e^{\int -\frac{2}{\sin 2x}dx} = e^{-\int\frac{1}{\sin x\cos x}dx}$
$= e^{-\int\frac{\sec^2 x}{\tan x}dx} = e^{-\log\tan x} = \cot x$

Let the complete solution of (1) be
$y = uv$

Then v is given by :

$\frac{d^2v}{dx^2} + Iv = S$ ……………… (2)

where

$I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = 2\tan^2 x - \frac{1}{4}(16\,\text{cosec}^2\, 2x) - \frac{1}{2}(-8\,\text{cosec}\, 2x \cot 2x)$

$= 2\tan^2 x - 4\,\text{cosec}^2\, 2x + 4\,\text{cosec}\, 2x \cot 2x$

$= 2\tan^2 x - 4\, cosec^2 2x + 4\, cosec^2 2x \cos 2x$

$= 2\tan^2 x - 4\, cosec^2 2x\ (\ 1 - \cos 2x)$

$= 2\tan^2 x - 4\ \ cosec^2 2x\ (\ 2\sin^2 x)$

$= 2\tan^2 x - 8\ \frac{\sin^2 x}{(\sin 2x)^2} = 2\tan^2 x - \frac{2}{\cos^2 x} = \ 2\tan^2 x - \frac{2(\sin^2 x + \cos^2 x)}{\cos^2 x} = \ -2$

and $\quad S = \frac{R}{\text{u}} = \frac{(e^x \cot x)}{\cot x} = e^x$

Then (2) becomes

$\frac{d^2v}{dx^2} - 2v = e^x$

or $(D^2 - 2)v = e^x$ ……….. (3)

The auxiliary equation of (3) is

$m^2 - 2 = 0 \Rightarrow$ m = $\pm\sqrt{2}$

∴ C.F. of (3) $= c_1 e^{x\sqrt{2}} + c_2 e^{-x\sqrt{2}}$

P.I. $= \frac{1}{D^2-2} e^x = \frac{1}{(1-2)} e^x = -e^x$

Hence , $v = C.F + P.I = c_1 e^{x\sqrt{2}} + c_2 e^{-x\sqrt{2}} - e^x$

The general solution of (1) is given by

$y = uv$

or $y = \cot\ x\left(c_1 e^{x\sqrt{2}} + c_2 e^{-x\sqrt{2}} - e^x\right)$

Changing the independent variable

Let $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ ………….. (1)

be the given equation .
Let the independent variable be changed from x to z by some relation .

Therefore $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\frac{dz}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\right)\frac{dz}{dx} + \frac{dy}{dz}\frac{d^2z}{dx^2}$

$= \frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}\frac{dz}{dx} + \frac{dy}{dz}\frac{d^2z}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}$

replacing these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, equation (1) reduces to

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2} + P\frac{dy}{dz}\frac{dz}{dx} + Qy = R$$

or $\boldsymbol{\frac{d^2y}{dz^2} + \frac{\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}\frac{dy}{dz} + \frac{Q}{\left(\frac{dz}{dx}\right)^2}y = \frac{R}{\left(\frac{dz}{dx}\right)^2}}$

Or $\boldsymbol{\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1}$ ……… (2)

where $P_1 = \frac{\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}$, $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$ and $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$ ………….. (3)

Equation (2) is of the same form as equation (1) and as such will be useful if P_1 , Q_1 are constants .
If $Q = \pm k\, f(x)$ then choosing z such that $\left(\frac{dz}{dx}\right)^2 = f(x)$ will reduce Q_1 to a constant .

If further , P_1 also becomes constant , then (2) is a linear equation of order two with constant coefficient and can be solved by previous methods .
Also in equation (3) , if we make $P_1 = 0$
Then $\boldsymbol{\frac{d^2z}{dx^2} + P\frac{dz}{dx} = 0 \Rightarrow \frac{d^2z}{dx^2} = -P\frac{dz}{dx}}$ **or** $\boldsymbol{\frac{1}{\frac{dz}{dx}}\frac{d}{dx}\left(\frac{dz}{dx}\right) = -P}$

$\Rightarrow \frac{1}{\frac{dx}{dx}} d\left(\frac{dz}{dz}\right) = -Pdx$

$\Rightarrow \log\left(\frac{dz}{dx}\right) = -\int Pdx$

$\Rightarrow \frac{dz}{dx} = e^{-\int Pdx} \quad \Rightarrow \quad z = \int\left(e^{-\int Pdx}\right)dx$ ………….. (4)

If with this substitution , Q_1 becomes constant , then (2)

will be of the form $\boldsymbol{\frac{d^2y}{dz^2} + Q_1 y = R_1}$ ………………. (5)

where $\boldsymbol{z = \int\left(e^{-\int Pdx}\right)dx}$, $\boldsymbol{Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}}$, $\boldsymbol{R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}}$ …………….. (6)

We shall stress to remember relation (2) , (3) or (5) and (6) for direct use in the examples and exercises .

We give an example below where we started by making $P_1 = 0$ and hope that Q_1 becomes constant .

Example 8: Solve $\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \frac{9}{x^6}y = \frac{1}{x^{10}}$

Sol: Given $\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \frac{9}{x^6}y = \frac{1}{x^{10}}$ ………….. (1)

Comparing with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

We have $P = \frac{3}{x}, Q = \frac{9}{x^6}$ and $R = \frac{1}{x^{10}}$

Let $z = \int e^{-\int Pdx} = \int \frac{1}{x^3}dx = -\frac{1}{2x^2}$. ……….. (2)

Then $\frac{dz}{dx} = \frac{1}{x^3} \Rightarrow \left(\frac{dz}{dx}\right)^2 = \frac{1}{x^6}$.

(It is clear that $\frac{Q}{\left(\frac{dz}{dx}\right)^2}$ is constant)

With the substitution as given in (2) , equation (1)
(using the relation 5 , 6 in the discussion) becomes

$$\frac{d^2y}{dz^2} + Q_1 y = R_1 \text{ ....} \qquad (3)$$

where $Q' = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 9$ and $R' = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{1}{x^4} = 4\text{z}^2$

Equation (3) becomes

$(D^2 + 9)y = 4\text{z}^2$ ....(4) $\left(D = \frac{d}{dz} \right)$

Whose $C.F = c_1 \cos 3z + c_2 \sin 3z$

$P.I = \frac{1}{D^2+9} 4z^2 = \frac{1}{9}\left(1 + \frac{D^2}{9}\right)^{-1} 4z^2 = \frac{1}{9}\left(1 - \frac{D^2}{9} + \frac{D^4}{81} \ldots\ldots\right) 4z^2$
$= \frac{4z^2}{9} - \frac{8}{81}$

Therefore $y = (c_1 \cos 3z + c_2 \sin 3z) + \frac{4z^2}{9} - \frac{8}{81}$

or $y = \left(c_1 \cos\left(3\left(-\frac{1}{2x^2}\right)\right) + c_2 \sin\left(3\left(-\frac{1}{2x^2}\right)\right)\right) + \frac{1}{9x^4} - \frac{8}{81}$

$= c_1 \cos\left(\frac{3}{2x^2}\right) - c_2 \sin\left(\frac{3}{2x^2}\right) + \frac{1}{9x^4} - \frac{8}{81}$

Example 9: Solve

$$\frac{d^2y}{dx^2} + \tan x \frac{\text{dy}}{\text{dx}} - (2\cos^2 x)y = 2\cos^4 x.$$

Sol. Given : $\frac{d^2y}{dx^2} + \tan x \frac{\text{dy}}{\text{dx}} - (2\cos^2 x)y = 2\cos^4 x$ .... (1)

Comparing (1) with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, we have

$P = \tan x$, $Q = -2\cos^2 x$ and $R = 2\cos^4 x$

Let $z = \int e^{-\int P dx} = \int e^{-\int \tan x dx} dx = \int e^{\log \cos x} dx = \int \cos x \, dx = \sin x$

Therefore $\frac{dz}{dx} = \cos x \Rightarrow \left(\frac{dx}{dx}\right)^2 = \cos^2 x$

Changing x to z with the substitution above , Equation (1) becomes

$\frac{d^2y}{dz^2} + \frac{Q}{\left(\frac{dz}{dz}\right)^2} y = \frac{R}{\left(\frac{dz}{dx}\right)^2}$

or $\frac{d^2y}{dz^2} - 2y = 2\cos^2 x = 2(1-z^2)$

Taking $D = \frac{d}{dz}$, we have

$(D^2-2)y = 2(1-z^2)$ ……………….. (2)

Auxiliary equation of (2) is $m^2 - 2 = 0$ giving $m = \pm\sqrt{2}$.
∴ C.F. of (2) $= c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{z}}$

and $P.I = \frac{1}{(D^2-2)} 2(1-z^2) = -\frac{1}{1-\frac{D^2}{2}}(1-z^2) = \left(1-\frac{D^2}{2}\right)^{-1}(1-z^2)$

$= \left(1+\frac{D^2}{2}+\cdots\right)(1-z^2) = 1+z^2-1 = z^2$

Hence the required solution is
$y =$ C.F. + P.I
or $y = c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{z}} + z^2$
or $y = c_1 e^{\sqrt{2}\sin x} + c_2 e^{-\sqrt{2}\sin x} + \sin^2 x$

In the example below , we shall start by makinga substitution in such a way that Q_1 becomes constant and hope that P_1 also becomes constant .

Example 10: Solve $\frac{d^2y}{dx^2} - \frac{\mathrm{y}}{\mathrm{x}} + 4x^2y = x^4$

Sol. Given : $\frac{d^2y}{dx^2} - \frac{\mathrm{y}}{\mathrm{x}} + 4x^2y = x^4$ ……………… (1)

Comparing with the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$,

We have $P = -\frac{1}{x}$, $Q = 4x^2$, $R = x^4$.

Choose z such that $\left(\frac{dz}{dx}\right)^2 = 4x^2$ or $\frac{dz}{dx} = 2x$ so that $z = x^2$ ……….. (2)

Then, (1) reduces to

$\frac{d^2y}{dz^2} + P_1\frac{\mathrm{dy}}{\mathrm{dz}} + Q_1 y = R_1$ …………… (3)

where $P_1 = \frac{\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2} = \frac{2-\frac{2x}{x}}{4x^2} = 0$, $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4x^2}{4x^2} = 1$

and $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{x^4}{4x^2} = \frac{x^2}{4} = \frac{z}{4}$

Therefore equation (3) becomes

$(D^2 + 1)y = \frac{z}{4}$ …………….. (4)

C.F. of (4) $= c_1 \cos z + c_2 \sin z$ ………….. (5)

and P.I. $= \frac{1}{D^2+4}\left(\frac{z}{4}\right) = \frac{1}{4}(1 + D^2)^{-1}z = \frac{1}{4}(1 - D^2 + D^4 - \cdots.)z = \frac{z}{4}$.

$\therefore\ y = \text{CF} + \text{PI} = c_1 \cos z + c_2 \sin z + \frac{z}{4} = c_1 \cos x^2 + c_2 \sin x^2 + \frac{x^2}{4}$

which is the general solution of (1)

Method of Variation of Parameters

Let $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$…………………….. (1)

Suppose $y = u\ and\ y = v$ be independent solutions of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ (i.e $u\ and\ v$ are part of the complimentary solution of (1))

Therefore $u_2 + Pu_1 + Qu = 0$ $, v_2 + Pv_1 + Qv = 0$…….(*)

Then $y = au + bv$ --------------------- (2)

is also the general solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

Let $y = Au + Bv$ -------------------------- (2)

be the general solution of equation (1) , where A, B are functions of x.
Differentiating we get

$y_1 = Au_1 + Bv_1 + (uA_1 + vB_1)$ ------------ (3)

where $A_1 = \frac{dA}{dx}$.

In order to simplify the process , we take one extra condition called the Auxilliary

Condition namely –

$uA_1 + vB_1 = 0$ ------------------ (4)

so that (3) becomes $y_1 = Au_1 + Bv_1$----------- (5)

$\Rightarrow y_2 = Au_2 + Bv_2 + A_1u_1 + B_1v_1$ ------------ (6)

Putting the value of y , y_1 and y_2 from (2) , (5) and (6) in (1)

we get $(Au_2 + Bv_2 + A_1u_1 + B_1v_1) + P(Au_1 + Bv_1) + Q(Au + Bv) = R$
or $A(u_2 + Pu_1 + Qu) + B(v_2 + Pv_1 + Qv) + A_1u_1 + B_1v_1) = R$

$\Rightarrow A_1u_1 + B_1v_1 = R$ ---------------------- (7)
using (*)

Now , A_1 and B_1 can be solved from (4) and (7) to get
$A_1 = \frac{dA}{dx} = h(x)$, $B_1 = \frac{dB}{dx} = g(x)$ say

So that A and B can be obtained by integration .

A quick method to find A and B

Solving (4) and (7) we get get

$$A_1 = \frac{dA}{dx} = -\frac{vR}{uv_1 - u_1v} = -\frac{vR}{W} \quad , \quad B_1 = \frac{dB}{dx} = \frac{uR}{uv_1 - u_1v} = \frac{uR}{W}$$

(Where $W = (uv_1 - u_1v) = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix}$ called the Wronskian of u and v)

from which we shall get

$$A = -\int \frac{vR}{W}\, dx + C_1 \quad , \quad B = \int \frac{uR}{W} dx + C_2$$

The general solution is then given by $y = Au + Bv$

or $\boldsymbol{y = -u \int \frac{vR}{W}\, dx + v \int \frac{uR}{W} dx}$

Example 11: Solve $(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \mathrm{y} = (x-1)^2$ by the method of variation of parameters.

Sol. The given equation can be written as

$\frac{d^2y}{dx^2} - \frac{x}{(x-1)}\frac{dy}{dx} + \frac{y}{(x-1)} = (x-1)$ ……… (1)

Comparing with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

we have P $= \frac{-x}{(x-1)}$, $Q = \frac{1}{(x-1)}$, $R = (x-1)$

Now $1 + P + Q = 1 - \frac{x}{x-1} + \frac{1}{x-1} = \frac{x-1-x+1}{x-1} = 0$

$\Rightarrow u = e^x$ is a part of the $C.F$ of (1)

Also $P + Qx = \frac{-x}{x-1} + \frac{x}{x+1} = 0$

$\Rightarrow v = x$ is a part of the $C.F$ of (1)

The wroskian of u and v is given by

$W = (uv_1 - u_1v) = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix} = e^x(1-x)$

Let $y = Au + Bv = Ae^x + Bx$

be the complete solution of (1)
Then $A = -\int \frac{vR}{W}\,dx = -\int \frac{x(x-1)}{e^x(1-x)}\,dx = \int xe^{-x}\,dx = -e^{-x}(1+x) + c_1$,

$B = \int \frac{uR}{W}dx = \int \frac{e^x(x-1)}{e^x(1-x)}\,dx = \int -1dx = -x + c_2$

Therefore the complete solution is given by
$y = Ae^x + Bx = (-e^{-x}(1+x) + c_1)e^x + (-x + c_2)x$

Example 12: Solve $\frac{d^2y}{dx^2} + m^2y = \sec\, mx$ by the method of variation of parameters.

Sol. Given equation is $\frac{d^2y}{dx^2} + a^2y = \sec\, ax$ ……… (1)

The auxiliary equation is $m^2 + a^2 = 0 \Rightarrow m = \pm ia$
Therefore $C.F$ of (1) is $y = C_1\cos ax + C_2\sin ax$

We have $u = \cos ax$, $v = \sin ax$ are parts of the complimentary function .
The wroskian of u and v is given by

$$W = (uv_1 - u_1v) = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a$$

Let $y = A\cos\ ax + B\sin\ ax$ be the complete solution of (1)
Then

$$A = -\int \frac{vR}{W}\,dx = -\int \frac{\sin ax \sec ax}{a}\,dx = \frac{1}{a^2}\int \frac{-a\sin ax}{\cos ax}dx = \frac{1}{a^2}\log\cos ax + c_1\ ,$$

$$B = \int \frac{uR}{W}dx = \int \frac{(\cos ax \sec ax)}{a}\,dx = \frac{x}{a} + c_2$$

Therefore the complete solution is given by

$$y = \left(\frac{1}{a^2}\log\cos ax + c_1\right)\cos ax + \left(\frac{x}{a} + c_2\right)\sin ax$$

Example 13: Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin\ x$ by the method of variation of parameters.

Sol. Writing $D = \frac{d}{dx}$, the given equation is $(D^2 - 2D)y = e^x \sin x \ldots..$ (1)

The auxiliary equation of (1) is :

$m^2 - 2m = 0 \Rightarrow\ m = 0\,, 2$
Therefore $C.F = C_1e^{0x} + C_2e^{2x} = C_1 + C_2e^{2x}$
It can be seen that $u = 1,\ v = e^{2x}$ are part of the $C.F$.

The wroskian of $1\ and\ e^{2x}$ is given by
$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x}$$

Let $y = Au + Bv = A + Be^{2x}$ be the complete solution of (1)

Then $A = -\int \frac{vR}{W}\,dx = -\int \frac{(e^{2x}e^x \sin x)}{2e^{2x}}\,dx = -\frac{1}{2}\int e^x \sin x\,dx$

$$= -\frac{e^x}{4}(\sin x - \cos x) + c_1\ ,$$

$$B = \int \frac{uR}{W} dx = \int \frac{(e^x \sin x)}{2e^{2x}} \, dx = \frac{1}{2} \int e^{-x} \sin x \, dx = \frac{e^{-x}}{4} (-\sin x - \cos x) + c_2$$

Therefore , the complete solution of (1) is given by

$$y = A + Be^{2x} = -\frac{e^x}{4} (\sin x - \cos x) + c_1 + \left(\frac{e^{-x}}{4} (-\sin x - \cos x) + c_2\right) e^{2x}$$

$$= -\frac{e^x}{4} (\sin x - \cos x) + c_1 + \frac{e^x}{4} (-\sin x - \cos x) + c_2 e^{2x}$$

$$= -\frac{e^x}{2} \sin x + c_2 e^{2x} + c_1$$

Example 14: Solve the equation $\frac{d^2y}{dx^2} + 4y = 4\tan\ 2x$

Sol. Given equation is $\frac{d^2y}{dx^2} + 4y = 4\tan\ 2x$ …………….. (1)

which can be written as $D^2 + 4 = 4\tan\ 2x$ ………….. (2)

The auxiliary equation is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$
Therefore $C.F = C_1 \cos 2x + C_2 sin 2x$

We have $u = \cos x$, $v = \sin x$ *are parts* of the $C.F.$

Let $y = AU + Bv = A \cos x + B \sin x$ be the complete solution of (1) .
Now , the wroskian of u and v is given by

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin x & 2\cos 2x \end{vmatrix} = 2$$

Therefore , $A = \int \frac{-vRdx}{W} = -\int \frac{\sin 2x \cdot 4 \tan 2x}{2} dx = -2 \int \frac{\sin^2 2x}{\cos 2x} dx$

$$= -2 \int \frac{1 - \cos^2 2x}{\cos 2x} dx = -2 \int \left[\frac{1}{\cos 2x} - \cos 2x\right] dx$$

$$= -2[\int \sec 2x dx - \int \cos 2x dx = -2 \left[\frac{\log|\sec 2x + \tan 2x|}{2} - \frac{\sin 2x}{2}\right] + c_1$$

$$= -[\log|\sec 2x + \tan 2x|] + \sin 2x + c_1$$

$$B = \int \frac{uRdx}{W} = \int \frac{\cos 2x \cdot 4 \tan 2x}{2} dx = 2 \int \cos 2x \frac{\sin 2x}{\cos 2x} dx$$

$$= 2\int \sin 2x dx = -\cos 2x + c_2$$

the complete solution of the given equation is given by

$$y = Au + Bv$$

or $y = ((-[\log|\sec 2x + \tan 2x|] + \sin 2x + c_1)\cos x + (-\cos 2x + c_2)\sin x$

Example 15: Solve the differential equation: $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^2\log x$ using method of variation of parameters.

Sol: Let $x = e^t$ $\therefore \log x = t$ so that $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$

$$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dt} \Rightarrow x\frac{dy}{dx} = Dy \quad \text{where } D = \frac{d}{dt}$$

Similarly $\frac{d^2y}{dx^2} = D(D-1)y$

$\therefore$ Given differential becomes

$(D(D-1) - 4D + 6)y = te^{2t}$
$\Rightarrow (D^2 - 5D + 6)y = te^{2t}$ ………… (1)

Auxiliary equation of (1) is: $(m^2 - 5m + 6) = 0$
or $(m-2)(m-3) = 0$ $\Rightarrow m = 2,3$
C.F. $= C_1e^{2t} + C_2e^{3t}$

Therefore $u = e^{2t}$ and $v = e^{3t}$ are part of the $C.F$
The wroskian of u and v is given by

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t}$$

Let $y = Au + Bv$ be the complete solution of (1)

Then $A = -\int \frac{e^{3t}te^{2t}}{e^{5t}}dt = -\int tdt = -\frac{t^2}{2} + c_1$

$$B = \int \frac{e^{2t}te^{2t}}{e^{5t}}dt = \int te^{-t}dt = -te^{-t} + \int e^{-t}dt$$

$$= -te^{-t} - e^{-t} + c_2$$

Therefore the required solution is

$$y = Au + Bv = \left(-\frac{t^2}{2} + c_1\right)e^{2t} + (-te^{-t} - e^{-t} + c_2)e^{3t}$$

$$= \left(-\frac{(\log x)^2}{2} + c_1\right)x^2 - (\log x + 1)x^2 + c_2x^2$$

Example 16: By the method of variation of parameters, solve $x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(x+1)y = x^3$, given that $u = xe^{2x}$ is a solution of

$L.H.S = 0$

Sol: Given equation is

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(x+1)}{x^2}y = x$$

Comparing with the standard equation , we have

$$P = -\frac{2(1+x)}{x}, \qquad Q = \frac{2(x+1)}{x^2} \quad , R = x$$

Since $\mathrm{P} + \mathrm{Q}x = 0$
$\therefore$ $v = x$ is a part of C.F
Also , $u = xe^{2x}$ is part of $C.F$ (given)
The wroskian of u and v is given by

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} xe^{2x} & x \\ e^{2x}(1+2x) & 1 \end{vmatrix} = -2x^2e^{2x}$$

Let $y = Au + Bv = Ax + Bxe^{2x}$

Then $A = -\int \frac{vR}{W}dx = \int \frac{x^2}{2x^2e^{2x}}dx = \frac{1}{2}\int e^{-2x}dx = -\frac{1}{4}e^{-2x} + c_1$
$B = \int \frac{uR}{W}\,dx = \int \frac{x^2e^{2x}}{-2x^2e^{2x}}dx = \int -\frac{1}{2}dx = \left(-\frac{x}{2} + c_2\right)$

The complete solution of the given equation is given by

$$y = Au + Bv = \left(-\frac{1}{4}e^{-2x} + c_1\right)xe^{2x} + \left(-\frac{x}{2} + c_2\right)x$$

Solution by factorizing the Operator

Let $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$…………………….. (1)

be the given equation .

Let $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$

Suppose that $L.H.S$ of (1) can be factorized into linear factors of D of the form

$f_1(D,x)f_2(D,x)y$ (order sensitive) , where f_1 and f_2 are functions of D and x .

Then (1) can be written as

$f_1(D,x)f_2(D,x)y = R$ ……… (2)

Taking $f_2(D,x)y = v$, (2) will reduce to

$f_1(D,x)v = R$, which is a linear equation of first order and can be solved for v .
The general solution will be found by solving v from the equation
$f_2(D,x)y = v$.

Example 17: Solve $x\frac{d^2y}{dx^2} + (x-2)\frac{dy}{dx} - 2y = x^3$.

Sol: Let $D = \frac{d}{dx}$. The given equation can be written as
$[xD^2 + (x-2)D - 2]y = x^3$

$or\ (xD-2)(D+1)y = x^3$ …………… (1)

Put $(D+1)y = v$
Then Equation (1) b
ecomes $(xD-2)v = x^3$

or $x\frac{dv}{dx} - 2v = x^3$

or $\frac{dv}{dx} - \frac{2}{x} v = x^2$ ………………….. (2)

which is linear and first order .

I.F. of (2) $= e^{\int\left(-\frac{2}{x}\right)dx} = e^{\log x^{-2}} = \frac{1}{x^2}$.

Therefore , solution of (2) is given by

$\frac{v}{x^2} = \int \frac{x^2}{x^2}\, dx = \int 1\, dx = (x + c_1)$

$\Rightarrow v = x^3 + c_1 x^2$

$\Rightarrow (D+1)y = x^3 + c_1 x^2$

or $\frac{dy}{dx} + y = (\, x^3 + c_1 x^2)$ …………… (3)

I.F. of (3) $= e^{\int 1 dx} = e^x$

.

∴ The solution of (3) (which is also solution to the given differential equation) is given by

$ye^x = \int e^x(\, x^3 + c_1 x^2)dx \, = \int \, e^x x^3 dx + c_1 \int \, x^2 e^x dx$

$= x^3 e^x - \int 3x^2 e^x dx + c_1\left(x^2 e^x - \int 2xe^x dx\right)$

$= x^3 e^x + c_1 x^2 e^x - \left(\, 3x^2 e^x - \int 6xe^x dx\right) - c_1\left(2xe^x - \int 2e^x dx\right)$

$= x^3 e^x + (c_1 - 3)x^2 e^x + 6xe^x - 6e^x - 2c_1 xe^x + 2c_1 e^x + c_2$

$= e^x x^3 + (c_1 - 3)(x^2 - 2x + 2)e^x + c_2$

$\Rightarrow . \, y = x^3 + (c_1 - 3)(x^2 - 2x + 2) + c_2 e^{-x}$

Example 18: Solve $x\frac{d^2y}{dx^2} + (x-2)\frac{dy}{dx} - 2y = x^3$.

Sol. Let $D = \frac{d}{dx}$, then the given equation can be written as

$[xD^2 + (x-2)D - 2]y = x^2$. …………. (1)

We have , $xD^2 + (x-2)D - 2 = xD^2 + xD - 2D - 2$

$$= xD(D+1) - 2(D+1) = (xD-2)(D+1)$$

Hence (1) becomes

$(xD-2)(D+1)y = x^3$ ……………… (2)

Let $(D+1)y = v$ ……………………….. (3)

From (2) $(xD-2)v = x^3$.

or $x\frac{dv}{dx} - 2v = x^3$

or $\frac{dv}{dx} - \frac{2}{x}v = x^2$ …………………. (4)

which is linear and first order . Its. I.F. = $e^{\int -\frac{2}{x}dx} = e^{-2\log x} = x^{-2} = 1/x^2$
Therefore solution of (4) is

$\frac{v}{x^2} = \int \frac{x^2}{x^2}\,dx + c_1$ or $\frac{v}{x^2} = x + c_1$ or $v = x^3 + c_1x^2$

Putting the value of v in (3) , we get

$$\frac{dy}{dx} + y = x^3 + c_1x^2$$

which is again a linear and first order equation . Its I.F. = $e^{\int dx} = e^x$.
Therefore $ye^x = \int e^x(x^3 + c_1x^2)dx$

or $ye^x = (x^3 + c_1x^2)e^x - \int(3x^2 + 2c_1x)e^x dx$

$= (x^3 + c_1x^2)e^x - (3x^2 + 2c_1x)e^x + \int (6x + 2c_1)e^x\,dx$

$= (x^3 + c_1x^2)e^x - (3x^2 + 2c_1x)e^x + (6x + 2c_1)e^x - \int 6e^x dx$

$= (x^3 + c_1x^2)e^x - (3x^2 + 2c_1x)e^x + (6x + 2c_1)e^x - 6e^x + c_2$

Exercises

Solve the following differential Equations ($y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$

1. $\frac{d^2y}{dx^2} - \cot x\frac{dy}{dx} - (1 - \cot x)y = e^x \sin x$.

   Ans **:** $y = c_1e^x + c_2e^{-x}(\cos x + 2\sin x) - \frac{1}{2}e^x\cos x$

2. $x^2y'' - 2x(1+x)y' + 2(1+x)y = x^2.$

   ***Ans* :** $2y = x(c_1e^{2x} + c_2 - x)$

3. $x^3y'' + 2xy' - 2y = 1.$

   ***Ans*:** $y = c_1x + c_2xe^{2/x} - \frac{1}{2}$

4. $(x+2)y'' - (4x+9)y' + (3x+7)y = 0.$

   Ans. $y = c_1e^x + c_2e^{3x}(2x+3)$

5. $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 4x^2y = x$

6. $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 3)y = e^{x^2}$

7. $xy'' - y' - 4x^3y = x^5.$

   ***Ans* :** $y = c_1e^{x^2} + c_2e^{-x^2} - \frac{x^2}{4}$

8. $xy'' - y' + 4x^3y = 8x^3\sin\ x^2.$

   Ans. $y = c_1\cos\ x^2 + c_2\sin\ x^2 - x^2\cos\ x^2$

9. $y'' - 4y' + 4y = e^x\sin\ x.$

   ***Ans* :** $y = (c_1 + c_2x)e^{2x} + \frac{1}{2}e^x\cos\ x$

10. $xy'' - (2x-1)y' + (x-1)y = e^x$

11. Solve $\frac{d^2y}{dx^2} - x^2 \cdot \frac{dy}{dx} + xy = x$

Given $y = x$ is a solution of $x^2y'' + xy' - y = 0.$

Chapter-8

Symmetric Simultaneous and Total Differential Equations

Pre-requisites

Let $\frac{a}{b} = \frac{c}{d} = \frac{p}{q} \;=\; r\ \ (say\,)$

$\Rightarrow a = br\ \ , c = dr\ \ , \qquad p = qr$

If $l\,, m\,, n$ are three numbers not all zero then

$\frac{la+mc+np}{lb+md+nq} = \frac{lbr+mdr+nqr}{lb+md+nq} = r = \frac{a}{b} = \frac{c}{d} = \frac{p}{q}$

Simultaneous Differential equations (symmetrical form)

Equations of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ ……………… (1)

Suppose from (1) we can find two relations of the form

$u(x, y, z) = c_1$ *……………(2)*

and $v(x, y, z) = c_2$ *………(3)*

Then the general solution of equation (1) is of the form $f(c_1\,, c_2\,) = 0$ *or* $c_1 = \phi(c_2)$ *for some function* $f\,, \phi$ *. We can also simply take the relations (2) and (3) together as the complete solution of (1) .*

Rule I for solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

By equating two of the three fractions of (1), we may be able to get an equation in only two variables. On integrating the differential equation in only two variables we shall obtain one of the relations in the general solution of (1).
This method may be repeated to give another relation with help of two other fractions of (1).

Example 1: Solve the equation $\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{z}$

Sol: Given $\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{z}$ …………….. (1)

From the first two fractions of (1) we have
$\frac{dx}{x} = \frac{dy}{y}$ or $\frac{dx}{x} - \frac{dy}{y} = 0$

Integrating we get $\log x - \log y = \log c_1$
or $\frac{x}{y} = c_1$ …………(3)

from the last two fractions of (1) , we have
$\frac{dy}{y} = dz$. Integrating we get , $\log y = z + C$
$\Rightarrow y = e^{z+C} = e^z e^C = e^z c_2$
$\Rightarrow ye^{-z} = c_2$ …………….(3)

Hence , the general solution of (1) is given by $c_1 = f(c_2)$
or $\frac{x}{y} = f(ye^{-z})$ for some function f .

Example 2: Solve $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$

Sol: Taking the first two fractions

we get $\frac{dx}{yz} = \frac{dy}{zx}$ or $xdx - ydy = 0$ or $2xdx - 2ydy = 0$
$\Rightarrow d(x^2 - y^2) = 0$

Integrating, we get $x^2 - y^2 = c_1$

Taking the last two fractions ,

we get $\frac{dy}{z} = \frac{dz}{y}$ or $ydz - zdy = 0$ or $2ydy - 2zdz = 0$
$\Rightarrow d(y^2 - z^2) = 0$

Integrating we get , $y^2 - z^2 = c_2$
The general solution is of the form $f(c_1, c_2) = 0$
or $f(x^2 - y^2 , y^2 - z^2) = 0$ for some function f .

Example 3: Solve $dx = \frac{dy}{y^2} = dz$

Sol: Given $dx = \frac{dy}{y^2} = dz$ …………….. (1)
Taking the first two fractions

we get $\mathrm{dx} = \frac{\mathrm{dy}}{\mathrm{y}^2} \quad \Rightarrow dx - \frac{dy}{y^2} = 0$

Integrating, we get $x + \frac{1}{y} = c_1$

Taking the first and last fractions ,

we get $dx = dz$ or $dx - dz = 0$

Integrating we get , $x - z = c_2$

The general solution is of the form $f(c_1, c_2) = 0$
or $f\left(x + \frac{1}{y}, x - z\right) = 0$ for some function f .

Rule II for solving $\boldsymbol{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$

Suppose only one relation $u(x, y, z) = c_1$ can be found by using rule I. Then, sometimes we try to use this relation in expressing one variable in terms of the others.

This may help us to obtain an equation in two variables. The solution of this equation will give a second relation of the form $v(x, y, z) = c_2$.

Example 4: Solve $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{zxy - 2x^2}$.

Sol. Given $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{zxy - 2x^2}$ ……………… (1)

Taking the first two fractions we have $\frac{dx}{x} = \frac{dy}{y} = 0$

$\Rightarrow \frac{dx}{x} - \frac{dy}{y} = 0$. Integrating we get , $\log x - \log y = \log c_1$
or $\frac{x}{y} = c_1$ ………………. (2)

$\Rightarrow x = c_1 y$ ……………. (3)

Taking the second and third fractions and suing (3) , we have

$$\frac{dy}{y^2} = \frac{dz}{c_1 z y^2 - 2c_1^2 y^2}$$

or $\quad c_1 dy = \frac{dz}{z-2c_1^2}$.

Integrating we get , $\quad c_1 y - \log(z - 2c^2{}_1) = c_2$
using (3) we get

$x - \log(z - 2x^2/y^2) = c_2$. ………….. (4)

The complete solution is given by $f(c_1, c_2)$ for some functions f .
where c_1 and c_2 are given by (2) and (4) .

Example 5: Solve $\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2}$.

Sol : Given $\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2}$ …………….. (1)

Taking the first two fractions , we have
$dx = -dy \Rightarrow \quad dx + dy = 0$
Intgerating we get $x + y = c_1$ …………….. (2)

Taking the last two fractions and using (2) we get
$\frac{zdz}{z^2+(x+y)^2} + dy = 0$ or $\frac{zdz}{z^2+c_1^2} + dy = 0$

or $\frac{2zdz}{z^2+c_1^2} + 2dy = 0$

Integrating we get $\log(z^2 + c_1^2) + 2y = c_2$
or $\log(z^2 + (x+y)^2) + 2y = c_2$ ………… (3)

The complete solution is of the form
$c_1 = \phi(c_2)$ where c_1, c_2 are given by (2) and (3) .

Rule III for solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ ……… (1)

The use of Lagrange's Multipliers

Let m_1, m_2, m_3 be three numbers or funtions of x, y, z

Then $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{m_1 dx + m_2 dy + m_3 dz}{m_1 P + m_2 Q + m_3 R}$

If $m_1 P + m_2 Q + m_3 R = 0$ then $m_1 dx + m_2 dy + m_3 dz = 0$

Choosing m_1, m_2, m_3 in such a way that $m_1 dx + m_2 dy + m_3 dz$ is integrable, one solution $u(x, y, z) = c_1$ can be found .

The method can be repeated with a different set of m_1, m_2, m_3 to get another relation of the form $v(x, y, z) = c_2$

(Here m_1, m_2, m_3 are known as Lagrange's multipliers)

Example 6: Solve the simultaneous equations $\frac{xdx}{z^2-2yz-y^2} = \frac{dy}{y+z} = \frac{dz}{y-z}$.

Sol: The given equations are $\frac{xdx}{z^2-2yz-y^2} = \frac{dy}{y+z} = \frac{dz}{y-z}$ …………… (1)

Choosing x, y, z as multipliers, each fraction of (1) are equal to

$$\frac{xdx + ydy + zdz}{z^2 - 2yz - y^2 + y(y+z) + z(y-z)} = \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow xdx + ydy + zdz = 0$

Integrating, we get $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C$

,

or $x^2 + y^2 + z^2 = c_1$

From last two fractions of Equation (1), we have $\frac{dy}{y+z} = \frac{dz}{y-z}$

$\Rightarrow (y - z)dy = (y + z)dz$

$\Rightarrow ydy - (zdy + ydz) - zdz = 0$

$\Rightarrow ydy - d(yz) - zdz = 0$

Integrating, we get $\frac{y^2}{2} - yz - \frac{z^2}{2} = C_1$

or $y^2 - 2yz - z^2 = c_2$

Therefore , the general solution of the given equation is of the form
$c_1 = f(c_2)$

or $x^2 + y^2 + z^2 = f(y^2 - 2yz - z^2)$ for some function f .

Example 7: Solve $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$.

Sol: Given $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$ …………………….. (1)

Choose multipliers $1, -1, 0$; $0, 1, -1$; $1,1,1$

each fractions of (1) are equal to $\frac{dx-dy}{y-x}$, $\frac{dy-dz}{z-y}$ and $\frac{dx+dy+dz}{2(x+y+z)}$

Therefore $\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} = \frac{dx+dy+dz}{2(x+y+z)}$ ………… (2)

From first two fractions , we have $\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$

$\Rightarrow \frac{d(x-y)}{y-x} = \frac{d(y-z)}{z-y}$ or $\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$

On integrating, we get $\log(x-y) = \log(y-z) + \log c_1$

$\Rightarrow \frac{(x-y)}{y-z} = c_1$ ………….. (3)

Again choosing the first and the last fractions of (2) ,

we have, $\frac{dx-dy}{y-x} = \frac{dx+dy+dz}{2(x+y+z)}$ or $\frac{d(x-y)}{(x-y)} = -\frac{d(x+y+z)}{2(x+y+z)}$

Integrating we get , $\log(x-y) = -\frac{1}{2}\log(x+y+z) + \log C$

$= -\log(x+y+z)^{\frac{1}{2}} + \log C$

$\Rightarrow \log(x-y) + \log(x+y+z)^{\frac{1}{2}} = \log C$

$\Rightarrow (x-y)(x+y+z)^{\frac{1}{2}} = C$

or $(x-y)^2(x+y+z) = c_2$ ………………. (4)

Thus , the general solution of (1) is of the form
$f(c_1, c_2) = 0$ for some function f . where c_1 , c_2 are given by (3) and (4)

Example 8: Solve $\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}$

Sol: Taking multipliers 1 , 1 , 1 each fraction are equal to

$$\frac{dx+dy+dz}{z-y+x-z+y-x} = \frac{dx+dy+dz}{0}$$

$$\Rightarrow dx + dy + dz = 0$$

or $d(x + y + z) = 0$

Integrating we get $x + y + z = c_1$

Taking multipliers x , y , z each fractions are equal to

$$\frac{xdx+ydy+zdz}{xz-xy+yx-yz+zy-zx} = \frac{xdx+ydy+zdz}{0}$$

$$\Rightarrow xdx + ydy + zdz = 0$$

or $d(x^2 + y^2 + z^2) = 0$
Integrating we get ,

$$x^2 + y^2 + z^2 = c_2$$

The general solution is of the form
$f(c_1, c_2) = 0$

or $f(x + y + z \ , x^2 + y^2 + z^2) = 0$ for some function f .

Example 9: Solve $\frac{dx}{y(x+y)+3z} = \frac{dy}{x(x+y)-3z} = \frac{dz}{z(x+y)}$.

Sol: Given $\frac{dx}{y(x+y)+3z} = \frac{dy}{x(x+y)-3z} = \frac{dz}{z(x+y)}$ ……………. (1)

Taking multipliers 1 ,1 , 0 each fractions of (1) are equal to $\frac{dx+dy}{(x+y)^2}$.

Therefore $\frac{dx+dy}{(x+y)^2} = \frac{dz}{z(x+y)} \Rightarrow \frac{d(x+y)}{x+y} = \frac{dz}{z}$

$$\Rightarrow \frac{d(x+y)}{x+y} - \frac{dz}{z} = 0$$

Integrating we get , $\log(x + y) - \log z = \log c_1$
or $\log\frac{x+y}{z} = \log c_1 \Rightarrow \frac{x+y}{z} = c_1$

Again , taking multipliers $x\,, -y\,, 0$, each fractions of (1) are equal to $\frac{xdx-ydy}{3z(x+y)}$

Therefore $\frac{xdx-ydy}{3z(x+y)} = \frac{dz}{z(x+y)}$ or $xdx - ydy - 3dz = 0$

Integrating we get , $\frac{x^2}{2} - \frac{y^2}{2} - 3z = C$

or $x^2 - y^2 - 6z = c_2$

Therefore , the general solution is given by

$f\left(\frac{x+y}{z}\,, x^2 - y^2 - 6z\right) = 0$ for some function f .

Example 10: Solve $\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$.

Sol: Given $\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$ ……………. (1)

Choosing x, y, z as multipliers, each fraction of (1) are equal to

$$\frac{xdx + ydy + zdz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)} = \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow\ xdx + ydy + zdz = 0$ or $2xdx + 2ydy + 2zdz = 0.$

Integrating we get , $x^2 + y^2 + z^2 = c_1$ ………… (2)

Again choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, each fraction of (1) are equal to

$$\frac{dx/x + dy/y + dz/z}{(y^2 - z^2) + (z^2 - x^2) + (x^2 - y^2)} = \frac{dx/x + dy/y + dz/z}{0}$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating we get , $\log\, x + \log\, y + \log\, z = \log\, c_2$
or $\log\, xyz = \log\, c_2$
$\Rightarrow\ xyz = c_2$ ………….. (3)

The complete solution is given $f(c_1, c_2) = 0$ where c_1, c_2 are given by the relations (2) and (3).

Example 11. Solve $\frac{dx}{y-zx} = \frac{dy}{x+yz} = \frac{dz}{x^2+y^2}$.

Sol. Given equation : $\frac{dx}{y-zx} = \frac{dy}{x+yz} = \frac{dz}{x^2+y^2}$ ………… (1)

Choosing $x, -y, z$ as multipliers, each fraction of (1) are equal to

$$= \frac{xdx - ydy + zdz}{x(y - zx) - y(x + yz) + z(x^2 + y^2)} = \frac{xdx - ydy + zdz}{0}$$

$\Rightarrow$ $xdx - ydy + zdz = 0$ or $2xdx - 2ydy + 2zdz = 0$.

Integrating we get , $x^2 - y^2 + z^2 = c_1$ ………… (2)
Again, choosing $y, x, -1$ as multipliers, each fraction of (1)

$$= \frac{ydx + xdy - dz}{y(y - zx) + x(x + yz) - (x^2 + y^2)} = \frac{ydx + xdy - dz}{0}$$

$\Rightarrow$ $ydx + xdy - dz = 0$ or $d(xy) - dz = 0$

Integrating we get , $xy - z = c_2$ ……………… (3)
The general solution is given by the relations (2) and (3).

Example 12. Solve $\frac{dx}{x-y} = \frac{dy}{x+y} = \frac{dz}{2xz}$.

Sol. Given equation : $\frac{dx}{x-y} = \frac{dy}{x+y} = \frac{dz}{2xz}$ …………….. (1)

Taking the first two fractions of (1) , we have

$$\frac{dy}{dx} = \frac{x+y}{x-y} \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

Let $\frac{y}{x} = v$ or $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Then (2) becomes

$v + x\frac{dv}{dx} = \frac{1+v}{1-v}$ or $x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v-v(1-v)}{1-v} = \frac{1+v^2}{1-v}$

or $\frac{1-v}{1+v^2}dv = \frac{dx}{x}$ or $\left(\frac{2}{1+v^2} - \frac{2v}{1+v^2}\right)dv = \frac{2dx}{x}$

or $\frac{2}{x}dx + \frac{2v}{1+v^2}dv = \frac{2}{1+v^2}\ dv$

Integrating we get , $\log x^2 + \log(1+v^2) = 2\tan^{-1} v + C$

or $x^2(1+v^2) = e^C e^{2(\tan^{-1} v)}$ or $x^2(1+v^2) = c_2 e^{2\tan^{-1} v}$

or $(x^2+y^2)e^{-2\tan^{-1}(y/x)} = c_1$ …………………… (3)

Choosing $1,1,-\frac{1}{z}$ as multipliers, each fraction of (1)

$$= \frac{dx+dy-(1/z)dz}{(x-y)+(x+y)-(1/z)\times(2xz)} = \frac{dx+dy-(1/z)dz}{0}$$

$$\Rightarrow\ dx + dy - \frac{1}{z}dz = 0$$

Integrating we get , $x + y - \log z = c_2$ ……… (4)

The realtions (2) and (3) together constitutes the general solution of the given equation .

Example 13. Solve $\frac{dx}{(x^2+y^2)} = \frac{dy}{2xy} = \frac{dz}{z(x+y)}$

Sol. Given $\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{(x+y)z}$ ………………. (1)

Choosing 1,1,0 as multipliers, each fraction of (1) are equal to

$$\frac{dx+dy}{x^2+y^2+2xy} = \frac{dx+dy}{(x+y)^2}\ \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Choosing $1,-1,0$ as multipliers, each fraction of (1) are equal to

$$\frac{dx-dy}{x^2+y^2-2xy} = \frac{dx-dy}{(x-y)^2}\ \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Equating (2) and (3) we have , $\frac{dx+dy}{(x+y)^2} = \frac{dx-dy}{(x-y)^2}$

Integrating, $-\frac{1}{x+y} = -\frac{1}{x-y} + C$ or $\frac{1}{x+y} = \frac{1}{x-y} + c_1$

$$\Rightarrow \frac{x-y}{x+y} = c_1\ \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Equating the last fraction of (1) and (2), we have

$\frac{dx+dy}{(x+y)^2} = \frac{dz}{(x+y)z}$ or $\frac{d(x+y)}{x+y} - \frac{dz}{z} = 0$

Integrating, $\log(x+y) - \log z = \log c_2$

or $(x+y)/z = c_2$ ………….. (5)

Relations (4) and (5) together constitute the general solution .

Total Differential equations

A total differential equation involves the total derivative of a function with respect to all its variables . In three dimensional space, total differential equations take the form

$Pdx + Qdy + Rdz = 0$ ……………. (1)

where P, Q, R are functions of x, y, z .

If there exists a function u of x, y, z such that its total derivative du is equal to the $L.H.S$ of (1) or its multiple i.e $du = Pdx + Qdy + Rdz$ or $du = \lambda(Pdx + Qdy + Rdz)$ then $\boldsymbol{u(x,y,z) = c}$ obtained directly by integration is a solution of (1) . In most cases however , equation (1) cannot be so easily solved or may not be integrable at all . We discuss below one theorem that guarantee the integrability of equation (1) called the Necessary and sufficient conditions for integrability of total differential equation.

Necessary and sufficient conditions for integrability

The Necessary and sufficient condition for the integrability of the equation : $Pdx + Qdy + Rdz = 0.$ **is**

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0.$$

***Proof* : The condition is necessary**

Let $Pdx + Qdy + Rdz = 0$ ………….(1)

be the given equation where P, Q, R are functions of x, y, z.

Let (1) be integrable and its integral be $u(x,y,z) = c$

Then the total differential du must be equal to a multiple of $Pdx + Qdy + Rdz$.

i.e $du = \lambda(Pdx + Qdy + Rdz)$ (λ need not be a constant) and since $du = \left(\frac{\partial u}{\partial x}\right) dx + \left(\frac{\partial u}{\partial y}\right) dy + \left(\frac{\partial u}{\partial z}\right) dz.$

$$\lambda P = \frac{\partial u}{\partial x'}, \quad \lambda Q = \frac{\partial u}{\partial y'} \text{ and } \quad \lambda R = \frac{\partial u}{\partial z} \;\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2)$$

From the first two equations of (2), we get

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial \mathrm{y}}(\lambda P) = \lambda\frac{\partial P}{\partial y} + P\frac{\partial \lambda}{\partial y}$$

and $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(\lambda Q) = \lambda\frac{\partial Q}{\partial x} + Q\frac{\partial \lambda}{\partial x}$ and as $\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x \partial y}$

We have $\lambda\frac{\partial P}{\partial y} + P\frac{\partial \lambda}{\partial y} = \lambda\frac{\partial Q}{\partial x} + Q\frac{\partial \lambda}{\partial x}$

$$\Rightarrow \quad \lambda\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\partial \lambda}{\partial x} - P\frac{\partial \lambda}{\partial y} \;\ldots\ldots\ldots\ldots \qquad (3)$$

Similarly , $\lambda\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = R\frac{\partial \lambda}{\partial y} - Q\frac{\partial \lambda}{\partial z}$ ………….. (4)

$$\text{and} \quad \lambda\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = P\frac{\partial \lambda}{\partial z} - R\frac{\partial \lambda}{\partial x} \;\ldots\ldots\ldots\ldots\ldots \qquad (5)$$

Multiplying (3) , (4) and (5) by R, P and Q respectively and adding, we get
$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0.$
This proves the condition is necessary .

Conversely , The Condition is Sufficient

Let the coefficients P, Q and R satisfy the condition

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0. \;\ldots\ldots\ldots\ldots \qquad (6)$$

Consider the equation $\mathrm{P}dx + \mathrm{Q}dy = 0$ ………….. (7)

We can assume that (7) is exact for otherwise we can always multiply by its integrating factor to make it exact .
Since (7) is exact , we have $\mathrm{P}dx + \mathrm{Q}dy = d\,\mathrm{V}$ for some function V .

$$\Rightarrow \; \mathrm{P}dx + \mathrm{Q}dy = \frac{\partial \mathrm{V}}{\partial x}dx + \frac{\partial \mathrm{V}}{\partial y}dy$$

$\therefore\ \mathrm{P} = \frac{\partial \mathrm{V}}{\partial x}$ and $\mathrm{Q} = \frac{\partial \mathrm{V}}{\partial y}$ ……… (8)

$$\Rightarrow \quad \frac{\partial \mathrm{P}}{\partial z} = \frac{\partial^2 \mathrm{V}}{\partial z\, \partial x} \text{ and } \frac{\partial \mathrm{Q}}{\partial z} = \frac{\partial^2 \mathrm{V}}{\partial z\, \partial y}$$

Also, $\frac{\partial \mathrm{P}}{\partial y} = \frac{\partial^2 \mathrm{V}}{\partial y\, \partial x}$ and $\frac{\partial \mathrm{Q}}{\partial x} = \frac{\partial^2 \mathrm{V}}{\partial x\, \partial y} = \frac{\partial^2 V}{\partial y \partial x}$

$$\therefore\ \frac{\partial \mathrm{P}}{\partial y} = \frac{\partial \mathrm{Q}}{\partial x}$$

Substituting the above values in Equation (6) , we get

$$\frac{\partial \mathrm{V}}{\partial x}\left(\frac{\partial^2 \mathrm{V}}{\partial z\, \partial y} - \frac{\partial \mathrm{R}}{\partial y}\right) + \frac{\partial \mathrm{V}}{\partial y}\left(\frac{\partial \mathrm{R}}{\partial x} - \frac{\partial^2 \mathrm{V}}{\partial z\, \partial x}\right) + \mathrm{R}\left(\frac{\partial \mathrm{Q}}{\partial x} - \frac{\partial \mathrm{Q}}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial \mathrm{V}}{\partial x}\left(\frac{\partial^2 \mathrm{V}}{\partial z\, \partial y} - \frac{\partial \mathrm{R}}{\partial y}\right) + \frac{\partial \mathrm{V}}{\partial y}\left(\frac{\partial \mathrm{R}}{\partial x} - \frac{\partial^2 \mathrm{V}}{\partial z\, \partial x}\right) = 0$$

$$\Rightarrow \frac{\partial \mathrm{V}}{\partial x} \cdot \frac{\partial}{\partial y}\left(\frac{\partial \mathrm{V}}{\partial z} - \mathrm{R}\right) - \frac{\partial \mathrm{V}}{\partial y} \cdot \frac{\partial}{\partial x}\left(\frac{\partial \mathrm{V}}{\partial z} - \mathrm{R}\right) = 0$$

$$\Rightarrow \begin{vmatrix} \frac{\partial V}{\partial x} & \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial z} - R\right) \\ \frac{\partial V}{\partial y} & \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z} - R\right) \end{vmatrix} = 0.$$

The last equation shows that the Jacobian of V and $\left(\frac{\partial V}{\partial z} - R\right)$ is zero . Hence V and $\left(\frac{\partial V}{\partial z} - R\right)$ are functionally related (*see theory of jacobian*) .

which implies that there exists a relation between V and $\left(\frac{\partial \mathrm{V}}{\partial z} - \mathrm{R}\right)$ independent of x and y.

Finally we express $\left(\frac{\partial V}{\partial z} - R\right)$ as a function of z and V .

$\frac{\partial V}{\partial z} - R = \phi(z, V)$. ………..(8)

Now, $Pdx + Qdy + Rdz = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \left(\frac{\partial V}{\partial z} - \phi\right)dz$ (using (8) and (9))

$$= \left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) - \phi dz = dV - \phi dz$$

Thus (1) may be written as

$dV - \phi dz = 0$
which is an equation in two variables which will have solution of the form $g(V, z) = 0.$ for some function g .

This proves the condition is sufficient .

……………………

The conditions for exactness of $\boldsymbol{Pdx + Qdy + Rdz = 0}$

The necessary and sufficient condition for the equation $Pdx + Qdy + Rdz = 0$ tobe exact is

$$.\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \text{ and } \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

(Note that when the above conditions are satisfied, the condition

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \text{ is also satisfied)}$$

We shall leave the proof as it is easily available .

Method I- Solution by Inspection

When the condition of integrability is satisfied, by rearranging the terms in the given equation, we may be able easily integrate to get the general solution .

Example 14: Solve $z(1 - z^2)dx + zdy - (x + y + xz^2)dz = 0.$

Sol: Equation can be written as

$z(dx + dy) - z^2(zdx + xdz) - (x + y)dz = 0$

or $zd(x + y) - z^2d(xz) - (x + y)dz = 0$

or $\frac{zd(x+y) - (x+y)dz}{z^2} - d(xz) = 0$

or $d\left(\frac{x+y}{z}\right) - d(xz) = 0$

Integrating we get

$\frac{x+y}{z} - xz = c$ which is the required solution .

Example 15. Find $f(y)$ such that the total differential equation

$\left(\frac{yz+z}{x}\right)dx - zdy + f(y)dz = 0$ is integrable and solve it.

Sol. The given equation can be written as

$$(yz+z)dx - xzdy + xf(y)dz = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots \tag{1}$$

Comparing (1) with $Pdx + Qdy + Rdz = 0$, we have

$$P = yz + z, \quad Q = -xz \quad \text{and} \quad R = xf(y).$$

$\Rightarrow \frac{\partial P}{\partial y} = z\,, \frac{\partial P}{\partial z} = (y+1)\,, \frac{\partial Q}{\partial x} = -z\,, \frac{\partial Q}{\partial z} = -x\,, \frac{\partial R}{\partial x} = f(y)\,, \frac{\partial R}{\partial y} = xf'(y)$

Suppose that (1) is integrable , then

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0.$$

Using (2) and denoting df/dy by f', (3) gives

$(yz+z)\big(-x - xf'(y)\big) - xz\big(f(y) - (y+1)\big) + xf(y)\big(z - (-z)\big) = 0.$
or $\quad xz(1+y)f'(y) = xzf(y)$

$\Rightarrow \frac{df(y)}{f(y)} = \frac{dy}{y+1}$

Integrating we get , $\log f(y) = \log(y+1) + \log k$
or $f(y) = k(y+1)$, (k is constant)
Putting this value $f(y)$ in (1), we get

$z(y+1)dx - xzdy + xk(y+1)dz = 0.$

$\frac{dx}{x} - \frac{dy}{y+1} + \frac{\mathrm{k}dz}{z} = 0.$

Integrating, $\log x - \log(y+1) + k\log z = \log c.$

or $\log\frac{xz^k}{y+1} = \log c \quad \Rightarrow \quad \frac{xz^k}{y+1} = c$ which is the required solution .

Example 16: Find $f(z)$ such that $\left(\frac{(y^2+z^2-x^2)}{2x}\right)dx - ydy + f(z)dz = 0$ is integrable and solve it.

Sol. The given equation can be written as

$(y^2 + z^2 - x^2)dx - 2xydy + 2xf(z)dz = 0.$ ……….. (1)

Comparing (1) with $Pdx + Qdy + Rdz = 0$, we have

$P = y^2 + z^2 - x^2,\ Q = -2xy,\ R = 2xf(z).$

$\Rightarrow \frac{\partial P}{\partial y} = 2y\ ,\ \frac{\partial P}{\partial z} = 2z\ ,\ \frac{\partial Q}{\partial x} = -2y\ ,\ \frac{\partial Q}{\partial z} = 0\ ,\ \frac{\partial R}{\partial x} = 2f(z)\ ,\frac{\partial R}{\partial y} = 0$

If (1) is integrable then

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0.$$

$$\Rightarrow (y^2 + z^2 - x^2)(0 - 0) - 2xy(2f(z) - 2z) + 2xf(z)[2y - (-2y)] = 0$$

or $\Rightarrow -4xy(f(z) - z) + 8xyf(z) = 0$
or $f(z) = -z.$

Putting $f(z) = -z$ in (1),
We have $(y^2 + z^2 - x^2)dx - 2xydy - 2xzdz = 0.$

or $(x^2 + y^2 + z^2)dx - 2x^2dx - 2xydy - 2xzdz = 0$

or $(x^2 + y^2 + z^2)dx - x(2xdx + 2ydy + 2zdz) = 0$

$(x^2 + y^2 + z^2)dx = x(2xdx + 2ydy + 2zdz)$

or $\frac{dx}{x} = \frac{\mathrm{d}(\mathrm{x}^2+y^2+z^2)}{x^2+y^2+z^2}$
Integrating we get ,
$\log x = \log(x^2 + y^2 + z^2) + \log c$ or $x = \mathrm{c}(x^2 + y^2 + z^2).$

Method II - Coefficients are homogeneous of same degree

Let $Pdx + Qdy + Rdz = 0$ ………….. (1)

be the given equation and we assume it is integrable

i.e $P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0.$

If P, Q, R are homogeneous of same degree , then :

Case I : General Method for Homogeneous coefficients

Taking $x=uz$, $y=vz$. Then $dx=udz+zdu$ and $dy=zdv+vdz$.
The given equation will reduce to the form

$$\frac{f_1(u,v)du+f_2(u,v)dv}{f(u,v)}+\frac{dz}{z}=0 \quad \ldots\ldots\ldots\ldots \qquad (3)$$

If (3) can be integrated directly , then the solution follows .
Otherwise , the numerator of the first fraction of (3) can be express as $df \pm g(u,v,du,dv)$ and express the fraction as the sum of partial fractions

to get $\frac{df(u,v)}{f(u,v)} \pm \frac{g(u,v,du,dv)}{f(u,v)}+\frac{dz}{z}=0$ .... (4)

Upon solving (4) , The general solution is found putting $u=\frac{x}{z}$, $v=\frac{y}{z}$.

Case II : If $Px+Qy+Rz \neq 0$

Let $\phi(x,y,z)=Px+Qy+Rz$.

Then $d\phi=Pdx+Qdy+Rdz+(xdP+ydQ+zdR)$

$$\Rightarrow Pdx+Qdy+Rdz=d\phi-(xdP+ydQ+zdR)$$

The given equation on dividing by ϕ becomes $\frac{Pdx+Qdy+Rdz}{\phi}=0$

or $\frac{d\phi-(xdP+ydQ+zdR)}{\phi}=0$ or $\frac{d\phi}{\phi}-\frac{xdP+ydQ+zdR}{\phi}=0$ ....(2)

The general solution is found by integrating equation (2) .

Example 17: Solve $y(y+z)dx+z(x+z)dy+y(y-x)dz=0.$

Sol: Given $y(y+z)dx+z(x+z)dy+y(y-x)dz=0$ .... (1)
Comparing with $Pdx+Qdy+Rdz=0$

we have $P = y(y+z)\,, Q = z(x+z)\,,\ R = y(y-z)$

$\Rightarrow \frac{\partial Q}{\partial z} = x + 2z\,, \frac{\partial R}{\partial y} = 2y - z\ ,\ \frac{\partial R}{\partial x} = 0\,, \frac{\partial P}{\partial z} = y\,, \frac{\partial P}{\partial y} = 2y + z\,, \frac{\partial Q}{\partial x} = z$

Now , $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= y(y+z)(x-2y+3z) + z(x+z)(-y) + y(y-z)(2y)$

$= 0$ (verify)

Therefore , the given equation is integrable .

Let $x = uz\ ,\ y = vz$ …………….. (2)

Then $dx = udz + zdu\ ,\ dy = vdz + zdv$
Substituting these in (1) we get

$vz(vz+z)(udz+zdu) + z(uz+z)(vdz+zdv) + vz(vz-uz)dz = 0$
or $z^3 v(v+1)du + z^3(u+1)dv +$

$\left(z^2uv(v+1) + z^2v(u+1) + z^2v(v-u)\right)dz = 0$
or $z^3v(v+1)du + z^3(u+1)dv + z^2v(v+1)(u+1)dz = 0$

or $\frac{du}{u+1} + \frac{dv}{v(v+1)} + \frac{dz}{z} = 0$

or $\frac{du}{u+1} + \frac{dv}{v} - \frac{dv}{v+1} + \frac{dz}{z} = 0$

Integrating we get ,

$\log \frac{(u+1)vz}{v+1} = \log c$

or $\frac{(u+1)vz}{v+1} = c$ or $\frac{y(x+z)}{y+z} = c$.

Example 18: Solve $(yz + z^2)dx - xzdy + xydz = 0$

Sol: Given $(yz + z^2)dx - xzdy + xydz = 0$ ………….. (1)

Comparing with $Pdx + Qdy + Rdz = 0$ we have
$P = yz + z^2, \quad Q = -xz\ , R = xy$

$\frac{\partial Q}{\partial z} = -x \;\; , \frac{\partial R}{\partial y} = x \,, \frac{\partial R}{\partial x} = y \;\; , \frac{\partial P}{\partial z} = y + 2z \,, \frac{\partial P}{\partial y} = z \;\; , \frac{\partial Q}{\partial x} = -z$

Now , $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= (yz + z^2)(-2x) - xz(-2z) + xy(2z) = 0$

Therefore the given equation is integrable .

Let $\phi(x, y, z) = Px + Qy + Rz = xyz + xz^2 - xyz + xyz$
i.e $\phi(x, y, z) = xyz + xz^2 \neq 0$ …………… (2)

(We can use case II of the above method .)

Then $d\phi = (yz + z^2)dx + xzdy + (xy + 2xz)dz$ ………… (3)

The given equation can be written as :

$\frac{(yz+z^2)dx - xzdy + xydz}{\phi} = 0$

or $\frac{(yz+z^2)dx + xzdy + (xy+2xz)dz - 2xydy - 2xzdz}{\phi} = 0$

or $\frac{(yz+z^2)dx + xzdy + (xy+2xz)dz - 2xz(dy+dz)}{\phi} = 0$

or $\frac{d\phi}{\phi} - \frac{2xz(dy+dz)}{\phi} = 0 \quad \Rightarrow \frac{d\phi}{\phi} - \frac{2(dy+dz)}{y+z} = 0$

Integrating we get ,

$\text{Log}\,\phi - \log(y + z)^2 = \log c$

$\Rightarrow \frac{\phi}{(y+z)^2} = c$ or $\frac{xz(y+z)}{(y+z)^2} = c$ or $xz = c(y + z)$

Method III - Use of auxiliary equations

Let $Pdx + Qdy + Rdz = 0$ ………….. (1)

be the given equation which is **not exact** . If (1) is integrable then
$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0.$ …………….. (2)

If the previous method are not suitable to solve equation (1) , then Comparing The coeficients of P, Q, R in (1) and (2), we obtain a simultaneous equations, (called the auxiliary equations of (1)) as

$$\frac{dx}{\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

(note that the denominators above are not zero as the equation is not exact)
Equations (3) can be solved by methods previous chapter.

Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ …………… (4)

be two solutions of (3) .
If (4) constitutes the general solution of (1) then (1) is identical to an equation of the form : $Adu + Bdv = 0.$ ….. (5)

Comparing (1) and (5) will get the values of A and B and upon solving (5), we shall get the general solution .

Example 19: Solve the equation $z(z-y)dx + z(z+x)dy + x(x+y)dz = 0$

Sol: Given : $z(z-y)dx + z(z+x)dy + x(x+y)dz = 0$ …… (1)

Comparing the given equation with $Pdx + Qdy + Rdz = 0$,
we get $P = z(z-y),\ Q = z(z+x),\ R = x(x+y)$.

Therefore $\frac{\partial P}{\partial y} = -z\,, \frac{\partial P}{\partial z} = 2z - y\,, \frac{\partial Q}{\partial x} = z\,,\ \frac{\partial Q}{\partial z} = 2z + x\,, \frac{\partial R}{\partial x} = 2x + y\,, \frac{\partial R}{\partial y} = x$

Now $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$$= z(z-y)(2z + x - x) + z(z+x)(2x + y - 2z + y) + x(x+y)(-z-z)$$
$$= 0$$

Therefore , the given equation is integrable.
The auxiliary equations of the given equation are :

$$\frac{dx}{\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}} = \frac{dy}{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}}$$

$$\Rightarrow \frac{dx}{z} = \frac{dy}{x+y-z} = \frac{dz}{-z} \quad \ldots\ldots\ldots\ldots \quad (2)$$

From first and third fractions of (2) , we get $dx + dz = 0$.
Integrating we get , $x + z = c_1$ or $u = c_1$ where $u = x + z$.

Using Multipliers 1,1,0 , each fractions of (2) are equal to $\frac{dx+dy}{x+y}$.

Equating this with the thirs fraction of (2) we get
$\frac{dx+dy}{x+y} = \frac{dz}{-z}$ or $\frac{d(x+y)}{x+y} + \frac{dz}{z} = 0$

Integrating we get , $\log(x + y) + \log z = \log c_2$

$\Rightarrow (x + y)z = c_2$ or $v = c_2$ where $v = (x + y)z$
Let (1) be identical with

$Adu + Bdv = 0$ ………… (3)

Now $Adu + Bdv = 0 \Rightarrow Ad(x + y) + Bd(xz + yz) = 0$
$\Rightarrow (A + zB)dx + zBdy + (A + x + y)dz = 0$ ……………. (4)
Comparing (1) and (4)

we get $A + zB = z(z - y)$; $zB = z(z + x)$ and $A + x + y = x(x + y)$,
From the second relation above , we get

$B = (x + z) = u$
and $A = z(z - y) - zB = z(z - y) - z(z + x) = -v$.
Therefore (3) becomes

$-vdu + udv = 0 \Rightarrow \frac{du}{u} - \frac{dv}{v} = 0.$

On integrating we get, $\log u - \log v = \log c \Rightarrow \frac{u}{v} = c$
$\Rightarrow u = cv$ or $x + z = cz(x + y)$ is the required solution of the given equation.

Method IV - General Method

Let $Pdx + Qdy + Rdz = 0$ ……………. (1)
be the given equation . Assuming that it is integrable , we proceed as below to find its solution .

Step 1. We treat one of the variables, say z as a constant so that $dz = 0$.
Then (1) becomes $Pdx + Qdy = 0$ ………… (2)

Step 2. Let the solution of (2) be $u(x,y)=f(z)$, where $f(z)$ is an arbitrary function of z tobe determined .

Step 3. Differentiate $u(x,y)=f(z)$ totally and then compare the result with the given equation $Pdx+Qdy+Rdz=0$. After comparing we shall get an equation in two variables f and z. If the coefficient of f or z involve functions of x,y, it will always be possible to remove them with the help of $u(x,y)=f(z)$.

Step 4. Solving the equation obtained in step 3 to obtain f. Putting this value of f in $u(x,y)=f(z)$, we shall get the required solution of the required equation.

Example 20: Solve $yzdx+2zxdy-3xydz=0$

Sol: Comparing the given equation with $Pdx+Qdy+Rdz=0$

we have $P=yz\ \ ;Q=2zx\ \ ;R=-3xy$

$\Rightarrow \frac{\partial P}{\partial y}=z\,,\frac{\partial P}{\partial z}=y\ \ ,\frac{\partial Q}{\partial x}=2z\ ,\frac{\partial Q}{\partial z}=2x\ ,\frac{\partial R}{\partial x}=-3y\ ,\frac{\partial R}{\partial y}=\ -3x$

Now $P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$

$=yz(2x+3x)+2zx(-3y-y)-3xy(z-2z)$

$=5xyz-8xyz+3xyz=0$

Therefore, the given equation is integrable.
Assuming $x=$ Constant so that $dx=0$
The given equation reduces to
$2zxdy-3xydz=0\Rightarrow\ \ 2zdy-3ydz=0$
$\Rightarrow\ 2\frac{dy}{y}-3\frac{dz}{z}=0$

Integrating we get , $2\log y-3\log z=f(x)$ ……….. (1)
Differentiating we get

$f'(x)dx=\frac{2}{y}dy-\frac{3}{z}dz$
or $-yz\,f'(x)dx+2z\,dy-3y\,dz\ =0$
or $-xyz\,f'(x)dx+2xz\,dy-3xy\,dz=0$

Comparing the above equation with the given equation we have

$-xyzf'(x) = yz \quad \Rightarrow f'(x) = -\frac{1}{x} \Rightarrow \quad d\big(f(x)\big) = -\frac{1}{x}dx$

integrating we get $f(x) = -\log x + \log c = \log \frac{c}{x}$

Putting this value of $f(x)$ in (1) we get ,

$2\log y - 3\log z = \log \frac{c}{x}$ or $\log \frac{y^2}{z^3} = \log \frac{c}{x}$

$\Rightarrow \frac{y^2}{z^3} = \frac{c}{x}$ or $\frac{xy^2}{z^3} = c$ is the required solution .

Example 21. Solve $zydx + (x^2y - zx)dy + (x^2z - xy)dz = 0.$

Sol. Given : $zydx + (x^2y - zx)dy + (x^2z - xy)dz = 0.$ ……. (1)

Comparing with $Pdx + Qdy + Rdz = 0$

we have $P = zy$, $Q = x^2y - zx$, $R = x^2z - xy$

$$\Rightarrow \frac{\partial Q}{\partial z} = -x , \frac{\partial R}{\partial y} = -x , \frac{\partial R}{\partial x} = 2xz - y , \frac{\partial P}{\partial z} = y , \frac{\partial P}{\partial y} = z , \frac{\partial Q}{\partial x} = 2xy - z$$

Now , $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= zy(0) + (x^2y - zx)(2xz - 2y) + (x^2z - xy)(2z - 2xy) = 0$

Therefore , the given equation is integrable .
Treating x as constant so that $dx = 0$, (1) reduces to

$(x^2y - zx)dy + (x^2z - xy)dz = 0$ or $x^2(ydy + zdz) - x(zdy + ydz) = 0$
or $x^2(ydy + zdz) - xd(yz) = 0$

Integrating , we get

$\frac{x^2(y^2+z^2)}{2} - xyz = f(x)$
or $x^2(y^2 + z^2) - 2xyz = 2f(x)$ ………………… (2)

where f is function tobe determined .

Differentiating (2), we have

$$2x(y^2+z^2)dx + x^2\,(2ydy+2zdz) - 2xydz - 2yzdx - 2zxdy = 2f'(x)dx$$

$$[x(y^2+z^2) - yz - f'(x)]dx + (x^2y - xz)dy + (x^2z - xy)dz = 0$$

Comparing the above equation with (1), we have

$$x(y^2+z^2) - yz - f'(x) = yz$$

or $x(y^2+z^2) - 2yz = f'(x)$ or $\frac{x^2(y^2+z^2)}{2} - xyz = \frac{x}{2}f'(x)$

$f(x) = \frac{x}{2}f'(x) \quad \Rightarrow \frac{f'(x)}{f(x)} = \frac{2}{x} \quad \Rightarrow \frac{1}{f(x)}\,df(x) = \frac{2}{x}dx$

Integrating we get $\log f(x) = \log x^2 + \log c$

$\Rightarrow \; f(x) = cx^2.$

$\frac{x^2(y^2+z^2)}{2} - xyz = cx^2$ or $x^2(y^2+z^2-2c) = 2xyz$

Example 22: Solve the equation

$$2yzdx + zxdy - xy(1+z)dz = 0. \;\ldots\ldots\ldots\ldots \quad (1)$$

Sol: Comparing Equation (1) with Pdx + Qdy + R$dz = 0$, we get

P $= 2yz$, Q $= zx$ and R $= -xy(1+z)$

$$\Rightarrow \frac{\partial P}{\partial y} = 2z, \frac{\partial P}{\partial z} = 2y, \quad \frac{\partial Q}{\partial x} = z, \frac{\partial Q}{\partial z} = x, \frac{\partial R}{\partial y} = -x - xz, \frac{\partial R}{\partial x} = -y - yz$$

Now $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$$= 2yz(x + x + xz) + zx(-y - yz - 2y) - xy(1+z)(2z - z)$$

$$= 2yz(2x + xz) + zx(-3y - yz) - (xy + xyz)z$$

$$= 4xyz + 2xyz^2 - 3xyz - xyz^2 - xyz - xyz^2 = 0$$

∴ the given equation is integrable .

Taking z as constant so that $dz = 0$ Equation (1) can be written as

$2yzdx + zxdy = 0 \quad \Rightarrow 2ydx + xdy = 0 \;\Rightarrow\; 2\frac{dx}{x} + \frac{dy}{y} = 0$

Integrating, we get
$\log\, x^2 + \log\, y = \log\, f(z)$
$\Rightarrow \log x^2 y = \log f(z) \quad \Rightarrow x^2 y = f(z)$ …………….. (2)

Differentiating totally , we get

$2xydx + x^2dy = f'(z)dz \quad \Rightarrow 2xydx + x^2dy - f'(z)dz = 0$
$\Rightarrow\; 2yzdx + xzdy - \frac{z}{x}\phi'(z)dz = 0$ ……………. (3)

Comparing Equations (1) and (3), we get

$$\frac{z}{x}f'(z) = xy(1+z)$$

$\Rightarrow f'(z) = x^2y\left(\frac{1+z}{z}\right) \Rightarrow f'(z) = f(z)\left(\frac{1+z}{z}\right)$

$\Rightarrow \quad \frac{df(z)}{dz} = f(z)\left(\frac{1+z}{z}\right) \quad or \quad \frac{df(z)}{f(z)} = \left(\frac{1+z}{z}\right)\, dz$

Integrating, we get $\quad \log\, f(z) = \log\, z + z + \log\, c$
or $\quad \log\frac{f(z)}{cz} = z \;\Rightarrow f(z) = cze^z$ or $x^2y = cze^z$

which is the solution of the equation.

Example 23: Solve $(x^2 + y^2 + z^2)dx - 2xydy - 2xzdz = 0.$

Sol : Given $(x^2 + y^2 + z^2)dx - 2xydy - 2xzdz = 0$ …………. (1)
Comparing the given equation with $Pdx + Qdy + Rdz = 0$ we have

$P = x^2 + y^2 + z^2 \;,\; Q = -2xy \;,\; R = -2xz$

$\Rightarrow \frac{\partial P}{\partial y} = 2y \;, \frac{\partial P}{\partial z} = 2z \;, \frac{\partial Q}{\partial x} = -2y \;, \frac{\partial Q}{\partial z} = 0 \;, \frac{\partial R}{\partial x} = -2z \;, \frac{\partial R}{\partial y} = 0$

Now $\mathrm{P}\left(\frac{\partial \mathrm{Q}}{\partial z} - \frac{\partial \mathrm{R}}{\partial y}\right) + \mathrm{Q}\left(\frac{\partial \mathrm{R}}{\partial x} - \frac{\partial \mathrm{P}}{\partial z}\right) + \mathrm{R}\left(\frac{\partial \mathrm{P}}{\partial y} - \frac{\partial \mathrm{Q}}{\partial x}\right)$

$= (x^2 + y^2 + z^2)(\,0) - 2xy(-2z - 2z) - 2xz(2y + 2y) = 0$
Therefore , the given equation is integrable .

Let x be treated as constant, so that $dx = 0$. Then (1) becomes
$-2xydy - 2xzdz = 0$ or $2ydy + 2zdz = 0$

Integrating we get , $y^2 + z^2 = f(x)$ (say) …………… (2)

Differentiating (2) we get ,

$2ydy + 2zdz = f'(x)dx$
or $xf'(x)dx - 2xydy - 2xzdz = 0$ ... ….. (3)

Comparing (3) with (1) we get
$xf'(x) = x^2 + y^2 + z^2$ or $xf'(x) = x^2 + f(x)$

or $\frac{df(x)}{dx} - \frac{1}{x} f(x) = x$, which is a differential linear equation

I.F. $= e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

$\frac{1}{x} f(x) = \int \frac{x}{x} dx + c = x + c$

or $f(x) = x^2 + cx$

or $y^2 + z^2 = x^2 + cx$

which is the general solution of (1)

Exercises

Solve the following equations

1. $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$

   Ans. $x + y + z = c_1$, $xyz = c_2$

2. $\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$

   Ans : $x^2 + y^2 + z^2 = c_1, xyz = c_2$

3. $\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$

Ans : $\frac{x-y}{y-z} = c_1, xy + yz + zx = c_2$

4. $\frac{\mathrm{d}x}{x^2-yz} = \frac{\mathrm{d}y}{y^2-zx} = \frac{\mathrm{d}z}{z^2-xy}$.

5. $\frac{\mathrm{d}x}{y+zx} = \frac{\mathrm{d}y}{-x-yz} = \frac{\mathrm{d}z}{x^2-y^2}$.

6. $\frac{dx}{1+y} = \frac{dy}{1+x} = \frac{dz}{z}$

Ans : $z(x-y) = c_1, 2 + x + y = c_2 z$

7. $\frac{dx}{y^3x-2x^4} = \frac{dy}{2y^4-x^3y} = \frac{dz}{9z(x^3-y^3)}$.

Ans : $x^3y^3z = c_1$, $\frac{y}{x^2} + \frac{x}{y^2} = c_2$

8. $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{zxy-2x^2}$.

Ans : $\frac{x}{y} = c_1$, $e^{-x}\left(z - 2\frac{x}{y}\right) = c_2$

9. $\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2y^2z^2}$.

Ans : $x^3 - y^3 = c_1, x^3 + \frac{3}{z} = c_2$

10. $\frac{xdx}{y^2z} = \frac{dy}{zx} = \frac{dz}{y^2}$

Ans. $x^3 - y^3 = c_1, x^2 - z^2 = c_2$

11. Solve $3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0$,

Ans. $x^2z^2 - 2y = cz^2$

12. Solve $(y^2 + z^2 - x^2)dx - 2xydy - 2xzdz = 0$,

Ans. $x^3 + y^3 = e^{2z} + ce^z$

13. Solve $(y^2 + z^2 + x^2)dx - 2xydy - 2xzdz = 0$,

Ans. $y^2 + z^2 = x^2 + cx$

14.Solve $zydx + (x^2y - xz)dy + (x^2y - xy)dy + (x^2z - xy)dz = 0$

Ans. $x\log z = cy$, $x^2(y^2 + z^2 - 2c) = 2xyz$

15. Solve $2xzdx + zdy - dz = 0.$

Ans. $x^2 + y - \log z = c.$

16. Solve $z^2dx + (z^2 - 2yz)dy + (2y^2 - yz - zx)dz = 0.$

Ans. $xz + yz - y^2 = cz^2$

17.Solve $(y^2 + yz)dx + (z^2 + zx)dy + (y^2 - xy)dz = 0.$

Ans. $y(x + z) = c(y + z)$

18.Solve $z(x^2 - yz - z^2)dx + (x + z)xzdy + x(z^2 - x^2 - xy)dz = 0.$

Ans. $x(y^2 + z) = z(x + y)(1 - cy)$

19.Solve $3ydx - 3xdy - y^2dz = 0.$

Ans. $y = x(c - z^2)$

20.Solve $y^2dx - 2x^2dy + (xy - zy^3)dz = 0.$

Ans. $x\log z = cy$

Chapter-9

Introduction to Partial Differential Equation

Introduction

Partial differential equations arise from a situation when the number of independent variables in the problem is two or more. Under such a situation, any dependent variable will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables.

Origin of Partial Differential Equation

Elimination of arbitrary constants/functions

Consider a relation

$F(x, y, z, a, b) = 0$ .... (1)

between x, y, z where a and b are arbitrary constants and z is a function of two variables x and y and where F is a known relation . Differentiating this equation with respect to x and y partially and using the chain rule . we shall obtain two relations

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

or $\dfrac{\partial F}{\partial x} + p\dfrac{\partial F}{\partial z} = 0$ and $\dfrac{\partial F}{\partial y} + q\dfrac{\partial F}{\partial z} = 0$ .... (2)

where $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$

The two arbitrary constants a and b, can be eliminated from the three relations given by (1) and (2) to obtain another relation of the form :

$f(x, y, z, p, q) = 0$ .... (3)

which is the partial differential equation arises from the system of surface (1).

Example 1: Eliminate the arbitrary constants from the relation

$ax^2 + by^2 + z^2 = 1$ ....(1)

Sol: Differentiating partially w.r.t x and y in turn

$$2ax + 2zp = 0 \Rightarrow ax^2 = -xzp \quad ;$$

$$2by + 2zq = 0 \Rightarrow by^2 = -y\,zq$$

Adding the last two equations and using (1)

$$(ax^2 + by^2) = -z(xp + yq)$$

or $(1 - z^2) = -z(xp + yq)$ or $px + qy = z - \frac{1}{z}$

Example 2: Eliminate the constants a, b from

$$2z = (ax + y)^2 + b$$

Sol: $2p = 2a(ax + y)$, $2q = 2(ax + y)$

$\Rightarrow p = a(ax + y)$, $q = (ax + y)$

$\Rightarrow pq = a(ax + y)^2 = \frac{(q-y)q^2}{x}$

Example 3: Eliminate the constants a, b from

$$z = (x + a)(y + b)$$

Sol: $p = (y + b)$, $q = (x + a)$

$\Rightarrow pq = (x + a)(y + b) = z$

Example 4: Find the differential equation of all planes having equal intercepts on the X and Y axis .

Solution: Equation of the plane having equal intercepts with the X and Y-axis is : $\frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1.$ ………… (1)

Differentiating partially wr.t ' x ' we get

$\frac{1}{a} + 0 + \frac{1}{c}\frac{\partial z}{\partial x} = 0$ or $\frac{1}{a} + \frac{1}{c}p = 0$ or $\frac{1}{a} = -\frac{1}{c}p$ ….. (2)

Differentiang (1) partially w.r.t 'y' we get $0 + \frac{1}{a} + \frac{1}{c}\frac{\partial z}{\partial y} = 0$

$\frac{1}{a} + \frac{1}{c}q = 0$ or $\frac{1}{a} = -\frac{1}{c}q$ …….. (3)

From (2) and (3) $\Rightarrow -\frac{1}{c}p = -\frac{1}{c}q \Rightarrow$ p = q which is the required equation.

Suppose on the other hand , we have two known functions u and v of three variables x , y , z connected by an arbitrary function F of the form :

$$F(u, v) = 0 \quad \text{…………………} \quad (4)$$

Then, differentiating equation (4) totally with respect to x and y, we shall obtain two more equations

$$\frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right\} + \frac{\partial F}{\partial v}\left\{\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right\} = 0 \quad \text{………} \quad (5)$$

$$\frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right\} + \frac{\partial F}{\partial v}\left\{\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right\} = 0 \quad \text{………} \quad (6)$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from equations (5) and (6) , we obtain an equation

$$p\frac{\partial(u,v)}{\partial(y,z)} + q\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)} \quad \text{………………} \quad (7)$$

where the jacobian $\frac{\partial(f\ ,g)}{\partial(r\ ,s)}$ is given by $\begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial g}{\partial r} \\ \frac{\partial f}{\partial s} & \frac{\partial g}{\partial s} \end{vmatrix}$

Since u and v are known functions of x , y , z , then so are $u_x, u_y , u_z, v_x, v_y , v_z$ Thus, equation (7) is a partial differential equation of the same form as (3) namely :

$$f(x, y, z, p, q) = 0$$

Example 5: Eliminate the arbitrary function f from the equations:

$$z = xy + f(x^2 + y^2)$$

Sol: We have , $p = y + 2xf'(x^2 + y^2)$

$\Rightarrow f'(x^2 + y^2) = \frac{p-y}{2x}$

and $q = x + 2y\, f'(x^2 + y^2)$

or $q = x + 2y\left(\frac{p-y}{2x}\right)$ or $qx - py = x^2 - y^2$

Example 6: Eliminate the arbitrary function f from : $z = x + y + f(xy)$

Sol: $\mathbf{p} = 1 + yf'(xy) \Rightarrow \quad f'(xy) = \frac{p-1}{y}$

and $q = 1 + xf'(xy) = 1 + \frac{x(p-1)}{y}$ or $px - qy = x - y$

Exercise

1. Eliminate arbitrary constants from $ax^2 + by^2 + cz^2 = 1$.

2. Eliminate ' a ' and ' b ' from $z = ax^3 + by^3$.

3. Form the partial differential equation by eliminating arbitrary function $f \; and \; g$ from the following relations

   (a) $f(xyz, x^2 + y^2 + z^2) = 0$
   (b) $z = f(xy) + g\left(\frac{x}{y}\right)$.
   (c) $z = e^{ax} f(x + y)$.
   (d) $xyz = f(x^2 + y^2 + z^2)$
   (e) $f(x^2 + y^2, z - xy) = 0$
   (f) $z = f(\frac{y}{x})$

Order and Degree

Order and degree of a partial differential equation are defined in the same way as those of ordinary differential equations .

Linear Partial differential equations of first order

An equation of the form $f(x, y, z, p, q) = 0$ where the highest degree of p and q is 1 and there is no term containing the product pq . The most common of these equations are of the form

$Pp + Qq = R$ ………….. (1)

known as **Lagrange's equation** or **quasi-linear** , where $P, Q,$ and R are given functions of $x, y,$ and z (which do not involve p or q). A relation of the type
$F(x, y, z, a, b) = 0$ ………… (2)

containing two arbitrary constants a and b and which is a solution of a partial differential equation $f(x, y, z, p, q) = 0$ of the first order is called a **complete solution** or a **complete integral** of that equation. A relation of the type

$F(u, v) = 0$ ……….. (3)

involving an arbitrary function F connecting two known functions u and v of x, y, and z and providing a solution of a first-order partial differential equation $f(x, y, z, p, q) = 0$ is called a **general solution** or a **general integral**.

Theorem: The general solution of the linear partial differential equation
$Pp + Qq = R$ ………. (1)

is of the form $f(u, v) = 0$ where f is an arbitrary function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two independent solutions of the equations
$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. ………… (2)

Proof: Since $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are independent solutions of (2) therefore u and v must satisfy the equations

$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0$ …………… (3)

and $P\frac{\partial v}{\partial x} + Q\frac{\partial v}{\partial y} + R\frac{\partial v}{\partial z} = 0$ …………… (4)

From (3) and (4) we have

$$\frac{P}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}} = \frac{Q}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}} = \frac{R}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}$$

or $\frac{P}{\frac{\partial(u,v)}{\partial(y,z)}} = \frac{P}{\frac{\partial(u,v)}{\partial(z,x)}} = \frac{P}{\frac{\partial(u.v)}{\partial(x,y)}}$ ……….. (5)

or $\frac{\frac{\partial(u,v)}{\partial(y,z)}}{P} = \frac{\frac{\partial(u,v)}{\partial(z,x)}}{Q} = \frac{\frac{\partial(u.v)}{\partial(x,y)}}{R}$

$\Rightarrow \frac{\partial(u,v)}{\partial(y,z)} = kP \; , \frac{\partial(u,v)}{\partial(z,x)} = kQ \; , \frac{\partial(u.v)}{\partial(x,y)} = kR$. ……….. (6)

Now , differentiating $F(u,v) = 0$ with respect to x and y, respectively, we have

$$\frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right\} + \frac{\partial F}{\partial v}\left\{\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right\} = 0$$

$$\frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right\} + \frac{\partial F}{\partial v}\left\{\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right\} = 0$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from these equations, we obtain

$$\frac{\partial z}{\partial x}\frac{\partial(u,v)}{\partial(y,z)} + \frac{\partial z}{\partial y}\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)} \quad \ldots\ldots\ldots\ldots.. \quad (7)$$

$\Rightarrow kPp + kQq = kR$ using (6)
or $Pp + Qq = R$

Hence, we have seen that $f(u,v)$ is a solution of the (1) if only if $u(x,y,z) = c_1$ and
$v(x,y,z) = c_2$ are the solutions of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

Note**:** The equation (2) is called the characteristic equation or Lagrange's auxiliary equation of (1) .

Summary: To solve the equation Pp + Qq = R

1. Form the auxiliary equations $\frac{dx}{\mathrm{P}} = \frac{dy}{\mathrm{Q}} = \frac{dz}{\mathrm{R}}$
2. Solve the auxiliary equations by the method of grouping or by the method of multipliers or both to get two independent solutions $u = c_1$ and $v = c_2$
3. Then f$(u,v) = 0$ or $u = f(v)$ is the general solution .

Example 7: Find the general integral of the equation

$$z(x+y)p + z(x-y)q = x^2 + y^2.$$

Sol. The characteristic equations are

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2} \quad \ldots\ldots\ldots. \quad (1)$$

Using multipliers $x, -y, -z$ each fractions of (1) are equal to

$$\frac{xdx - ydy - zdz}{0}$$

$\Rightarrow xdx - ydy - zdz = 0$ or $2xdx - 2ydy - 2zdz = 0$

or $d(x^2 - y^2 - z^2) = 0$ and integrating we get

$(x^2 - y^2 - z^2) = c_1$ ………… (2)

Using multipliers y, x, $-z$ each fractions of (1) are equal to $\frac{ydx+xdy-zdz}{0}$

$ydx + xdy - zdz = 0$ or $d\left(xy - \frac{z^2}{2}\right) = 0$

Integrating we habe $2xy - z^2 = c_2$

Thus, the general solution is$F(c_1, c_2) = 0$

or $F(x^2 - y^2 - z^2, 2xy - z^2) = 0$,

where F is an arbitrary function.

Example 8: Solve $z(xp - yq) = y^2 - x^2$.

Sol: The Lagrange's auxiliary equations are

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ or $\frac{dx}{zx} = \frac{dy}{-zy} = \frac{dz}{y^2-x^2}$ ……… (1)

From the first two fractions we get $\frac{dx}{zx} = \frac{dy}{-zy}$ or $\frac{dx}{x} + \frac{dy}{y} = 0$

Integrating , $\log x + \log y = \log c_1 \Rightarrow xy = c_1$

Taking multipliers 1,1 ,0 each fractions of (1) are equal to $\frac{dx+dy}{zx-zy}$.

Therefore $\frac{dx+dy}{zx-zy} = \frac{dz}{y^2-x^2}$ or $\frac{dx+dy}{2z} = \frac{dz}{2(x+y)}$

or $2(x + y)d(x + y) - 2zdz = 0$

Integrating , $(x + y)^2 - z^2 = c_2$

The general solution is $f(xy, \ (x+y)^2 - z^2) = 0$ for some functions f .

Example 9. Solve $y^2p - xyq = x(z-2y)$.

Sol. Given $y^2p - xyq = x(z-2y)$
The auxiliary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} \quad \ldots\ldots\ldots\ldots \quad (1)$$

From the first two fractions of (1) we have

$\frac{dx}{y} = \frac{dy}{-x}$ or $2xdx + 2ydy = 0$

Intregrating we get $x^2 + y^2 = c_1$ ……….. (2)

From last two fractions of (1) we have

$-\frac{dy}{y} = \frac{dz}{z-2y}$
or $-zdy + 2ydy = ydz$ or $2ydy = ydz + zdy = d(yz)$

On integration, we get $y^2 = yz + c_2$
or $y^2 - yz = c_2$ ………. (3)

$The\ general\ solution\ is\ \ given\ by\ \ c_1 = f(c_2)$

or $(x^2 + y^2 = f(y^2 - yz)$

Example 10. Solve $(x^2 - y^2 - z^2)p + 2xyq = 2xz$.

Sol. Given $(x^2 - y^2 - z^2)p + 2xyq = 2xz$
The auxiliary equations are

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

From the last two fractions of (1) we have

$\frac{dy}{y} = \frac{dz}{z}$ or $\frac{dy}{y} - \frac{dz}{z} = 0$
Integrating we get $\log y - \log z = \log c_1$ or $\log \frac{y}{z} = \log\ c_1$

$\Rightarrow \quad \frac{y}{z} = c_1 \quad \ldots\ldots\ldots\ldots\ldots$ (2)

Using multipliers x, y, z , each fractions of (1) are equal to

$$\frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)}$$

Equating the above fraction with the last fraction of (1) we have

$$\frac{2xdx+2ydy+2zdz}{(x^2+y^2+z^2)} = \frac{dz}{z}$$

Integrating we get : $\log(x^2+y^2+z^2) = \log z + \log c_2$

or $\frac{x^2+y^2+z^2}{z} = c_2$

Hence , the general solution is given by :

$$f\left(\frac{y}{z}\right) = \frac{(x^2+y^2+z^2)}{z}$$

Example 11. Find the general solution of the equation $xzp + yzq = -(x^2+y^2)$.

Sol: The Lagrange's auxiliary equations are :

$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{-(x^2+y^2)}. \quad \ldots\ldots\ldots\ldots$ (1)

Taking the first two integrals , we have

$$\frac{dx}{xz} = \frac{dy}{yz} \quad \Rightarrow \frac{dx}{x} = \frac{dy}{y} \Rightarrow \log x - \log y = \log c_1$$

$$\Rightarrow u(x,y,z) = \frac{x}{y} = c_1.$$

Using multipliers $x, y, 0$, each fractions of (1) are equal to $\frac{xdx+ydy}{z(x^2+y^2)}$.

Equating with the third fraction of (1) , we have

$$\frac{xdx+ydy}{z(x^2+y^2)} = \frac{dz}{-(x^2+y^2)}$$

$\Rightarrow xdx + ydy + zdz = 0$
Integrating we get $x^2 + y^2 + z^2 = c_2$

The general solution of the given equation is

$$f\left(\frac{x}{y}, x^2 + y^2 + z^2\right) = 0$$

where f is arbitrary function.

Example 12: Find the general integral of the quasi-linear equation

$$\boldsymbol{px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)}.$$

Sol: The given equation can be written as

$$x(z - 2y^2)p + y(z - y^2 - 2x^3)q = z(z - y^2 - 2x^3).$$

Lagrange's auxiliary equations are

$$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (1)$$

The last two fractions of (1) give $\frac{dy}{y} = \frac{dz}{z}$

$\Rightarrow y = c_1 z$ Thus $\Rightarrow u(x, y, z) = \frac{y}{z} = c_1$ .... (2)

Taking multipliers $0, 2y, -1$, each fraction of (1) are equal to

$$\frac{2ydy - dz}{2y^2(z-y^2-2x^3) - z(z-y^2-2x^3)} = \frac{d(y^2-z)}{(z-y^2-2x^3)(2y^2-z)}$$

Equating with the first fraction , we have

$$\frac{dx}{x(z-2y^2)} = \frac{d(y^2-z)}{(z-y^2-2x^3)(2y^2-z)}$$

or $\frac{dx}{x} = \frac{d(y^2-z)}{y^2-z+2x^3} = \frac{dr}{r+2x^3}$, where $r = y^2 - z$

or $2x^3dx + rdx - xdr = 0$

or $\quad 2xdx - \frac{xdr - rdx}{x^2} = 0 \Rightarrow \quad 2xdx - d\left(\frac{r}{x}\right) = 0$

Integrating, we have $x^2 - \frac{r}{x} = c_2$,

Hence $\quad v(x, y, z) = x^2 + \frac{z}{x} - \frac{y^2}{x^2} = c_2$ ……………. (3)

The general solution is given by
$c_1 = f(c_2)$ for some function f , where c_1 and c_2 are from (2) and (3) .

Example 13: Solve $\quad y^2p - xyq = x(z - 2y)$

Sol. Given equation is $\quad y^2p - xyq = x(z - 2y)$
Langrange's auxiliary equations are

$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$ ……………. (1)

Using 1st and 2nd fraction of (1) we have

$\frac{dx}{y^2} = \frac{dy}{-xy} \Rightarrow xdx = -ydy \quad \Rightarrow xdx + ydy = 0$

Integrating we get $\frac{x^2}{2} + \frac{y^2}{2} = c$ or $\quad x^2 + y^2 = c_1$ ……….. (2)

Again , taking 2nd and 3rd fraction of (1) we get

$\frac{dy}{-xy} = \frac{dz}{x(z-2y)} \quad \Rightarrow \frac{dz}{dy} = \frac{-z+2y}{y} \Rightarrow \frac{dz}{dy} + \frac{z}{y} = 2$ ……… (3)

Which is a linear differential equation .

$\text{IF} = e^{\int p(y)dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y.$

Hence the solution of (3) is : $zy = \int \; 2ydy + c_2$

$\Rightarrow zy = y^2 + c_2 \quad$ or $\quad yz - y^2 = c_2$ ……. (4)

From (2) and (4), the complete solution is

$\phi(x^2 + y^2, zy - y^2) = 0$ where ϕ is an arbitrary function .

Example 14: Solve $(y + zx)p - (x + yz)q = x^2 - y^2$

Sol: Given, $(y + zx)p - (x + yz)q = x^2 - y^2$

The Lagrange's Auxiliary equations are

$$\frac{dx}{y+zx} = \frac{dy}{-x-yz} = \frac{dz}{x^2-y^2} \quad \ldots\ldots\ldots.. \qquad (1)$$

Choosing $y, x, 1$ as multipliers, we get

$$\frac{ydx + xdy + dz}{y^2 + xyz - x^2 - xyz + x^2 - y^2} = \frac{ydx + xdy + dz}{0}$$

Thus, $ydx + xdy + dz = 0 \Rightarrow xy + z = c_1$ ……….. (2)

Choosing $x, y, -z$ an multipliers , we get

$$\frac{xdx+ydy-zdz}{xy+x^2z-xy-y^2z-z(x^2-y^2)} = \frac{xdx+ydy-zdz}{0}$$

$\Rightarrow\ xdx + ydy - zdz = 0 \Rightarrow x^2 + y^2 - z^2 = c_2$

Therefore the complete solution is $f(x^2 + y^2 - z^2, xy + z) = 0$ where f is an arbitrary function .

Example 15: Solve $px(x + y) = qy(x + y) - (x - y)(2x + 2y + z)$

Sol: The given equation can be written as

$$px(x + y) - qy(x + y) = -(x - y)(2x + 2y + z)$$

The Lagrange's auxiliary equations are

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{-(x-y)(2x+2y+z)} \ldots\ldots\ldots\ldots\ldots\ldots \qquad (1)$$

Now from first two fractions of equation (1), we have

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} \quad \Rightarrow \frac{dx}{x} + \frac{dy}{y} = 0$$

On integration, we get $\log x + \log y = \log c_1 \Rightarrow xy = c_1$ ………. (2)

Choosing multipliers $2, 2, 1$ each fractions of (1) are equal to $\frac{2dx+2dy+dz}{-(x-y)z}$

Equating this with the last fraction of (1) we have

$$\frac{dz}{2x+2y+z} = \frac{2dx+2dy+dz}{z} \text{ or } (2x+2y+z)d(2x+2y+z) - zdz = 0$$

On integration, we get $(2x+2y+z)^2 - z^2 = c_2$

The complete solution is given by
$c_2 = f(c_1)$ or $(2x+2y+z)^2 - z^2 = f(xy)$ for some function f .

Exercises

Find the general integrals of the linear partial differential equations:

1. $x(x^2+3y^2)p - y(3x^2+y^2)q = 2z(y^2-x^2)$
2. $x^2p + y^2\mathrm{q} = (x+y)z.$
3. $(y+x)p + (x-y)q = \frac{x^2+y^2}{z}$
4. $p + zq = 6x$ satisfying the condition $z(0,y) = 3y.$
5. $(2xy-1)p + (z-2x^2)q = 2(x-yz)$

Integral Surfaces Passing through a Given Curve

Let the curve c be given in parametric equations as

$$x = x(t),\ y = y(t),\ z = z(t) \ \ldots\ldots\ldots.. \qquad (1)$$

where t is a parameter
Let $u(x,y,z) = c_1,\ v(x,y,z) = c_2$ ………… (2)

be the particular solutions of the equation $Pp + Qq = R$, where

$u(x,y,z) = c_1,\ v(x,y,z) = c_2$ are as described in the last section .

In order to find the integral surface which passes through the curve c ,
the particular solution $u = c_1$, $v = c_2$ must satisfy the conditions

$$u\{x(t), y(t), z(t)\} = c_1,\ v\{x(t), y(t), z(t)\} = c_2 \ \ldots\ldots.. \qquad (3)$$

If we eliminating $'t'$ from (3) , we shall obtain an equation of the form
$F(c_1, c_2) = 0$

The required integral surface will then be given by $F(\, u, v) = 0$.

Example 16: Find the equation of the integral surface of the differential equation
$2y(z-3)\, p + (2x-z)q = y(2x-3)$ which passes through the circle
$z = 0, x^2 + y^2 = 2x.$

Sol: The Lagrange's auxiliary equations are

$$\frac{dx}{2y(z-3)} = \frac{dy}{(2x-z)} = \frac{dz}{y(2x-3)} \quad \ldots\ldots\ldots\ldots\ldots\ldots.. \qquad (1)$$

From the first and last fractions, we get

$$\frac{dx}{z-3} = \frac{2dz}{2x-3} \Rightarrow (2x-3)dx - 2(z-3)dz = 0$$

$$\Rightarrow u(x, y, z) = x^2 - 3x - z^2 + 6z = c_1 \quad \ldots\ldots\ldots\ldots\ldots \qquad (2)$$

Using multipliers $1\,, 2y\,, -2$, each fractions of (1) are equal to

$$\frac{dx + 2ydy - 2dz}{2y(z-3) + 2y(2x-z) - 2y(2x-3)} = \frac{dx + 2ydy - 2dz}{0}$$

$$\Rightarrow dx + 2(ydy - dz) = 0$$

Integrating we get , $v(x, y, z) = x + y^2 - 2z = c_2$

Hence the general solution of the given equation is $\phi(u, v) = 0$

or $\phi(x^2 - 3x - z^2 + 6z, x + y^2 - 2z) = 0.$

The parametric equations of the given curve are

$$x = 1 + \cos t\,, \qquad y = \sin t\,, \ z = 0$$

Therefore $(1 + \cos\, t)^2 - 3(1 + \cos\, t) = c_1$

$$\Rightarrow \cos^2\, t - \cos\, t = 2 + c_1$$

$$\text{and } 1 + \cos\ t + \sin^2\ t = c_2$$
$$\Rightarrow\ \cos\ t - \cos^2\ t = c_2$$

so that $c_1 + c_2 + 2 = 0$

Thus the required equation of the integral surface is

$$(x^2 - 3x - z^2 + 6z) + (x + y^2 - 2z) + 2 = 0$$

or $x^2 + y^2 - z^2 - 2x + 4z + 2 = 0$

Example 17: Find the integral surface of the equation

$$x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z,$$

which contains the straight line $x + y = 0, z = 1$.

Sol: Auxiliary equations are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1)$$

By Choosing multipliers $x, y, -1$, we have

$$\frac{xdx + ydy - dz}{x^2y^2 + x^2z - x^2y^2 - y^2z - x^2z + y^2z} = \frac{xdx + ydy - dz}{0}$$

$$\Rightarrow\ xdx + ydy - dz = 0$$

Integrating we get , $x^2 + y^2 - 2z = c_1$ ………… (2)

Choosing multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, we get

$$\frac{\frac{dx}{x} + \frac{dx}{x} + \frac{dz}{z}}{y^2 + z - x^2 - z + x^2 - y^2} = \frac{\frac{dx}{x} + \frac{dx}{x} + \frac{dz}{z}}{0}$$

$$\Rightarrow \frac{dx}{x} + \frac{dx}{x} + \frac{dz}{z} = 0$$

Integrating we get , $\log xyz = \log c_2 \Rightarrow\ xyz = c_2$ …………….. (3)

The Parametric equations of straight line are

$$x = t, y = -t, z = 1$$

Substitute in (3) we have $t(-t)(1) = -t^2 = c_2$

and from (2) we have

$t^2 + (-t)^2 - 2(1) = c_1 \Rightarrow \quad 2t^2 - 2 = c_1$
Eliminate t we have $-2c_2 - 2 = c_1$
Or $c_1 + 2c_2 + 2 = 0$

Hence, the integral surface, which contains the straight line

$$x^2 + y^2 - 2z + 2xyz + 2 = 0$$

Example 18. Find the integral surface of the linear partial differential equation

$$x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$$

which contains the straight line $x + y = 0, z = 1$.

Sol: The auxiliary equations have integrals

$$\frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{(x^2 - y^2)z}$$

$$xyz = c_1,\ x^2 + y^2 - 2z = c_2$$

For the curve in question we have the freedom equations

$$x = t,\ y = -t,\ z = 1$$

Substituting these values in the pair of equations (4), we have the pair

$$-t^2 = c_1,\ 2t^2 - 2 = c_2$$

and eliminating t from them, we find the relation

$$2c_1 + c_2 + 2 = 0$$

showing that the desired integral surface is

$$x^2 + y^2 + 2xyz - 2z + 2 = 0$$

Example 19: Find the general integral of the partial differential equation $(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$ and also the particular integral which passes through the line $x = 1, y = 0$.

Sol. Given $(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$.

Lagrange's auxiliary equations are $\frac{dx}{2xy-1} = \frac{dy}{z-2x^2} = \frac{dz}{2x-2yz}$ ………… (1)

Taking $z, 1, x$ as multipliers, each fraction of (1) are equal $\frac{zdx+dy+xdz}{0}$

$\Rightarrow zdx + dy + xdz = 0$ or $d(xz) + dy = 0$

Integrating, $xz + y = c_1$. ……… (2)

Taking $x, y, \frac{1}{2}$ as multipliers, each fraction of (1) are equal to $\frac{\left\{xdx+ydy+\frac{1}{2}dz\right\}}{0}$

$\Rightarrow\ xdx + ydy + \frac{1}{2}dz = 0$ or $2xdx + 2ydy + dz = 0$

Integrating, $x^2 + y^2 + z = c_2$. ……. (3)

The particular integral passing through the line $x = 1, y = 0$ is found by putting $x = 1$ and $y = 0$ in (2) and (3) .

Now $z = c_1$ and $1 + z = c_2 \ \Rightarrow\ 1 + c_1 = c_2$

or $1 + xz + y = x^2 + y^2 + z$

or $x^2 + y^2 + z - xz - y = 1.$

Example 20: Find the integral surface of the partial differential equation $(x - y)y^2p + (y - x)x^2q = (x^2 + y^2)z$ passing through the curve $xz = a^3, y = 0.$

Sol. Given $(x - y)y^2p + (y - x)x^2q = (x^2 + y^2)z$

Lagrange's auxiliary equations are $\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{(x^2+y^2)z}$ ………. (1)

Using multipliers $1, -1, 0$. Each fraction of $(3) = \frac{dx-dy}{(x-y)(y^2+x^2)}$

$\therefore\ \frac{dx-dy}{(x-y)(y^2+x^2)} = \frac{dz}{(x^2+y^2)z} \ \Rightarrow\ \frac{d(x-y)}{x-y} - \frac{dz}{z} = 0$

Integrating we get , $\frac{x-y}{z} = c_1$ . . . . (2)

Taking the first two fractions, we have $\frac{dx}{y^2} = \frac{dy}{x^2} \ \Rightarrow\ 3x^2dx + 3y^2dy = 0$

Integrating we get , $x^3 + y^3 = c_2$. . . . . (3)

The parameteric equation of the given curve is $z = t,\ x = \frac{a^3}{t},\ y = 0$
Substituting these values in (2) and (3), we get

$\frac{a^3}{t^2} = c_1 \Rightarrow \quad t^2 = \frac{a^3}{c_1} \Rightarrow t^6 = \frac{a^9}{c_1^3}$

and $\left(\frac{a^3}{t}\right)^3 = c_2 \Rightarrow t^3 = \frac{a^9}{c_2} \Rightarrow \; t^6 = \frac{a^{18}}{c_2^2}$

$\Rightarrow \frac{a^9}{c_1^3} = \frac{a^{18}}{c_2^2}$ or $c_2^2 = c_1^3 a^9 \Rightarrow (x^3 + y^3)^2 = \frac{a^9(x-y)^3}{z^3}$

or $z^3(x^3 + y^3)^2 = a^9(x - y)^3$.

Exercises

1. Find the integral surface of the linear first order partial differential equation $yp + xq = z - 1$ which passes through the curve
   $z = x^2 + y^2 + z, y = 2x$

2. Find the general solution of the equation $2x(y + z^2)p + y(2y + z^2)q = z^3$ and deduce that $yz(z^2 + yz - 2y) = x^2$ is a solution.

3. Find the general integral of the equation $(x - y)p + (y - x - z)q = z$ and the particular solution through the circle $z = 1, x^2 + y^2 = 1$.

4. Find the general solution of the differential equation $x(z + 2a)p + (xz + 2yz + 2ay)q = z(z + a)$ and the integral surfaces which pass through the curves: (a) $y = 0,\ z^2 = 4ax$ (b) $y = 0,\ z^3 + x(z + a)^2 = 0$

5. Find the equation of the integral surface of the differential equation $(x - y)y^2p + (y - x)x^2q = (x^2 + y^2)z$ which passes through the curve $xz = 1, y = 0$.

Surfaces Orthogonal to a Given System of Surfaces

Suppose we are given a one-parameter family of surfaces characterized by the equation

$f(x, y, z) = c$ …………(1)

The surfaces orthogonal to the system (1) are the surfaces generated by the integral curves of the equations

$$\frac{dx}{f_x} = \frac{dy}{f_y} = \frac{dz}{f_z}$$

Example 21: Find the surface which is orthogonal to system $z = cxy\,(x^2 + y^2)$ and passes through the hyperbola $x^2 - y^2 = a^2,\ z = 0$.

Sol: The given one parameter system is $\frac{xy(x^2+y^2)}{z} = \frac{1}{c} = C$

Let $f(x, y, z) = \frac{xy(x^2+y^2)}{z}$

$\Rightarrow\ f_x = \frac{y(x^2+y^2)+2x^2y}{z}\ \ ,\quad f_y = \frac{x(x^2+y^2)+2xy^2}{z},\quad f_z = \frac{-xy(x^2+y^2)}{z^2}$

The auxiliary equations are for the orthogonal surface are :

$$\frac{dx}{\frac{y(x^2+y^2)+2x^2y}{z}} = \frac{dy}{\frac{x(x^2+y^2)+2xy^2}{z}} = \frac{dz}{\frac{-xy(x^2+y^2)}{z^2}}$$

or $\frac{dx}{y(x^2+y^2)+2x^2y} = \frac{dy}{x(x^2+y^2)+2xy^2} = \frac{zdz}{-xy(x^2+y^2)}$ ………… (1)

Using multipliers $(x, y, 1)$, each ratio of (1) are equal to

$$\frac{xdx+ydy+zdz}{3x^3y+xy^3+x^3y+3xy^3-x^3y-xy^3} = \frac{xdx+ydy+zdz}{3xy(x^2+y^2)}$$

Equating this with 3rd fraction of (1), we get

$$\frac{xdx+ydy+zdz}{3xy(x^2+y^2)} = \frac{zdz}{-xy(x^2+y^2)}\ \Rightarrow\ xdx + ydy + zdz = -3zdz$$

$\Rightarrow\ xdx + ydy + 4zdz = 0$

Integrating we get $\frac{x^2}{2} + \frac{y^2}{2} + \frac{4z^2}{2} = \frac{c_1}{2}$

or $x^2 + y^2 + 4z^2 = c_1$ …………….. (2)

Using multipliers $(x, y, 0)$ and $(x, -y, 0)$ in (1) and equating the two fractions we get $\frac{xdx+ydy}{3x^3y+xy^3+x^3y+3xy^3} = \frac{xdx-ydy}{3x^3y+xy^3-x^3y-3xy^3}$

$$\Rightarrow \frac{xdx+ydy}{4x^3y+4xy^3} = \frac{xdx-ydy}{2x^3y-2xy^3}$$

$\Rightarrow \frac{xdx+ydy}{x^2+y^2} = \frac{2(xdx-ydy)}{x^2-y^2}$

Integrating, we get $\log(x^2+y^2) = 2\log(x^2-y^2) + \log c_2$

$\Rightarrow \frac{x^2+y^2}{(x^2-y^2)^2} = c_2$ …………..(3)

Also , given hyperbola : $x^2 - y^2 = a^2,\ z = 0$

Its parametric equations are :

$x = a\sec\theta,\ x = a\tan\theta,\ z = 0$

∴ from (2), $c_1 = a^2\sec^2\theta + a^2\tan^2\theta$

$c_1 = a^2(\sec^2\theta + \tan^2\theta)$

And from (3) we have

$c_2 = \frac{a^2\sec^2\theta + a^2\tan^2\theta}{(a^2\sec^2\theta - a^2\tan^2\theta)^2}$ or $c_2 = \frac{\sec^2\theta+\tan^2\theta}{a^2(\sec^2\theta-\tan^2\theta)^2}$

or $c_2 = \frac{\sec^2\theta+\tan^2\theta}{a^2(1)^2}$ $\Rightarrow c_2 = \frac{\frac{c_1}{a^2}}{a^2(1)^2}$

$c_2 = \frac{c_1}{a^4}$ or $\frac{c_1}{c_2} = a^4$

∴ The required surface orthogonal to the given system is

$\frac{(x^2-y^2)^2(x^2+y^2+4z^2)}{x^2+y^2} = a^4$

Example 22: Find the surface which intersects the surfaces of the system

$$z(x+y) = c(3z+1)$$

orthogonally and which passes through the circle $x^2 + y^2 = 1, z = 1$.

Sol: Let $f = \frac{z(x+y)}{3z+1}$,

Then $f_x = \frac{z}{3z+1}$, $f_y = \frac{z}{3z+1}$, $f_z = \frac{1}{(3z+1)^2}$

The auxiliary equation of the system orthogonal to the given system is

$$\frac{dx}{f_x} = \frac{dy}{f_y} = \frac{dz}{f_z} \quad \text{or} \quad \frac{dx}{z(3z+1)} = \frac{dy}{z(3z+1)} = \frac{dz}{(x+y)} \quad \ldots\ldots\ldots\ldots \quad (1)$$

From the first two fractions of (1) we have $dx - dy = 0$

$$\Rightarrow \; x - y = c_1 \quad \ldots\ldots. \quad (2)$$

Choosing $x\,, y\,,\; (-3z^2 - z)$ as multipliers , each fraction of (1) are equal to
$\frac{xdx+ydy-(3z^2+z)dz}{0}$ Therefore $xdx + ydy - (3z^2 + z)dz = 0$

Integrating we get $\frac{x^2}{2} + \frac{y^2}{2} - z^3 - \frac{z^2}{2} = C$

$$\text{or} \quad x^2 + y^2 - 2z^3 - z^2 = c_2 \;\ldots\ldots\ldots \quad (3)$$

Thus any surface which is orthogonal to the given surfaces has equation of the form

$$x^2 + y^2 - 2z^3 - z^2 = f(x - y)$$

If the above surface passes through the circle $x^2 + y^2 = 1, z = 1$

$f(x - y) = \; x^2 + y^2 - 2z^3 - z^2 = -2$

Hence , the required surface is $x^2 + y^2 - 2z^3 - z^2 = -2$
or $x^2 + y^2 - 2z^3 - z^2 + 2 = 0$

Example 23: Find the equation of the system of surfaces which cut the system

$x^2 + y^2 + z^2 = cxy$ and passes through $x = 0,\; y^1 + z^2 = 1$.

Sol: The given system of surfaces is $f(x, y, z) = \frac{x^2+y^2+z^2}{xy} = c.$

$\Rightarrow f_x = \frac{1}{y} - \frac{y}{x^2} - \frac{z^2}{xy^2}$, $f_y = -\frac{x}{y} + \frac{1}{x} - \frac{z^2}{xy^2}$, $f_z = \frac{2z}{xy}$

The auxiliary equations are $\frac{dx}{f_x} = \frac{dy}{f_y} = \frac{dz}{f_z}$

$$\text{or} \quad \frac{dx}{\frac{1}{y} - \frac{y}{x^2} - \frac{z^2}{xy^2}} = \frac{dy}{-\frac{x}{y} + \frac{1}{x} - \frac{z^2}{xy^2}} = \frac{dz}{\frac{2z}{xy}} \quad \text{or} \quad \frac{xdx}{x^2 - y^2 - z^2} = \frac{ydy}{-x^2 + y^2 - z^2} = \frac{dz}{2z} \;\ldots\ldots \quad (1)$$

Using multipliers 1 ,1 , z we have $xdx + ydy + zdz = 0$

Integrating we get $x^2 + y^2 + z^2 = c_1$ ………… (2)

Using multipliers $1, -1, 0$ and equating with the last fraction of (1) wehave

$\frac{xdx-ydy}{x^2-y^2} = \frac{dz}{z}$ or $\frac{d(x^2-y^2)}{x^2-y^2} - \frac{2dz}{z} = 0$. Integrating we get

$\frac{x^2-y^2}{z^2} = c_2$ ………….. (3)

Thus , The general quation of the required system of orthogonal surfaces is
$x^2 + y^2 + z^2 = f\left(\frac{x^2-y^2}{z^2}\right)$ where f is an arbitrary function .

if the surface contains the circle $x = 0, y^2 + z^2 = 1$

Then $f\left(\frac{x^2-y^2}{z^2}\right) = 1$

Thus , the particular surface is given by $x^2 + y^2 + z^2 = 1$

Nonlinear Partial Differential Equations of the First Order

The partial differential equation $F(x, y, z, p, q) = 0$ …..(1) in which the function F is not necessarily linear in p and q. A solution of $f(x, y, z, a, b) = 0$ of (1) that contains two arbitrary constants is called a **complete solution** .
A solution of $f(x, y, z, a_0, b_0) = 0$ obtained by giving a and b some particular values is called a **particular integral** / **particular solution** .

The relation between x, y, and z obtained by eliminating a and b from
$f(x, y, z, a, b) = 0$, $\frac{\partial f}{\partial a} = 0$, $\frac{\partial f}{\partial b} = 0$ is called a **singular solution** of (1) .

Compatible System of First order Equations :

The two first order partial differential equations
$f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$ ………… (1)

are said tobe compatible if any solution of one is a solution of the other.
Let $J = \frac{\partial(f,g)}{\partial(p,q)} \neq 0$

Then from equation (1) , p and q can be solved as functions of x, y, z

say $\quad p = \phi(x, y, z), \quad q = \psi(x, y, z)$ ……………. (2)

Thus , the two equations in (1) will be compatible if equation (2) is integrable and since $\quad dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = pdx + qdy$,

Therefore , (2) is integrable if $dz = pdx + qdy$ is integrable .
or $\quad \phi\, dx + \psi dy - 1\, dz = 0$ …………. (3)

is integrable .
Also , (3) is of the form ($Pdx + Qdy + Rdz = 0$) is integrable if

$$\phi(\psi_z - 0) + \psi(0 - \phi_z) + (-1)(\phi_y - \psi_x) = 0$$

or $\phi\psi_z - \psi\phi_z - \phi_y + \psi_x = 0$

or $\quad \psi_x + \phi\psi_z = \phi_y + \psi\phi_z$ ………. (4)

Now , differentiating the first equation of (1) w.r.t x and z and using (2)

$$\frac{df}{dx} = f_x + f_p p_x + f_q q_x = f_x + f_p \phi_x + f_q \psi_x = 0 \quad \text{…………….} \qquad (5)$$

and $\quad f_z + f_p\phi_z + f_q\psi_z = 0$ ………..(6)

$(6) \times \phi + (5) \Rightarrow \; f_x + \phi f_z + f_p(\phi_x + \phi\phi_z) + f_q(\psi_x + \phi\psi_z) = 0$

Similarly $\quad g_x + \phi g_z + g_p(\phi_x + \phi\phi_z) + g_q(\psi_x + \phi\psi_z) = 0$

From the last two equations we can solve to get

$$\psi_x + \phi\psi_z = \frac{(f_x+\phi f_z)g_p - (g_x+\phi g_z)f_p}{f_p g_q - f_q g_p} = \frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(x,p)} + \phi\frac{\partial(f,g)}{\partial(z,p)}\right\}. \quad \text{….} \qquad (7)$$

Similarly , $\quad \phi_y + \psi\phi_z = -\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(y,q)} + \psi\frac{\partial(f,g)}{\partial(z,q)}\right\}$ …………… (8)

Thus , Equation (3) is integrable if or $\quad \psi_x + \phi\psi_z = \phi_y + \psi\phi_z$

i.e if $\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(x,p)} + \phi\frac{\partial(f,g)}{\partial(z,p)}\right\} + \frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(y,q)} + \psi\frac{\partial(f,g)}{\partial(z,q)}\right\} = 0$

or $[f, g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0$ …….. (9)

and is the condition for the compatibility of the equations in (1) .

Summary: Given two equations $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$,

verify condition (9) . Solve for p and q . Put p and q in
$dz = pdx + qdy$ and solve .

Example 24: Show that equations $xp = yq$, $z(xp + yq) = 2xy$ are compatible and solve them.

Sol: Let $f = xp - yq = 0$, $g = z(xp + yq) - 2xy = 0$

Then $\frac{\partial(f,g)}{\partial(x,p)} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} = pzx - x(zp - 2y) = 2xy$

$$\frac{\partial(f,g)}{\partial(z,p)} = -px^2 - xyq \quad , \quad \frac{\partial(f,g)}{\partial(y,q)} = -2xy \quad , \quad \frac{\partial(f,g)}{\partial(z,q)} = xyp - qy^2$$

Now $[f, g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$

$$= 2xy - p^2x^2 - xypq - 2xy + xypq - q^2y^2 = 0$$

∴ Equations are compatible.

Solving given equations $p = \frac{y}{x}q$ and $z(\frac{xy}{x}q + yq) = 2xy$
$\Rightarrow q = \frac{x}{z}$ and $p = \frac{y}{z}$

Putting these in $dz = pdx + qdy$ we have

$$dz = \frac{y}{z}dx + \frac{x}{z}dy \quad \text{or} \quad zdz = ydx + xdy$$

$\Rightarrow$ $zdz - d(xy) = 0$
Integrating, $z^2 - 2xy = c$

1. Show that the equations $xp - yq = x$, $x^2p + q = xz$ are compatible and find their solution. Show that the equation $z = px + qy$ is compatible with any equation

2. $f(x, y, z, p, q) = 0$ that is homogeneous in x, y, and z.
   Solve completely the simultaneous equations
   $z = px + qy$, $2xy(p^2 + q^2) = z(yp + xq)$

Charpit's Method

Let the given partial differential equation be
$f(x, y, z, p, q) = 0$ ………….. (1)

We introduce another compatible differentail equation
$g(x, y, z, p, q, a) = 0$ ………… (2)

and such that p and q can be solved to solved to get
$p = p(x, y, z, a),\ q = q(x, y, z, a)$

and that $dz = p(x, y, z, a)dx + q(x, y, z, a)dy$
is integrable. Since f and g are compatible , we must have

$$[g, f] = \frac{\partial(g,f)}{\partial(x,p)} + p\frac{\partial(g,f)}{\partial(z,p)} + \frac{\partial(g,f)}{\partial(y,q)} + q\frac{\partial(g,f)}{\partial(z,q)} = 0$$

or $g_x f_p - g_p f_x + p\big(g_z f_p - g_p f_z\big) + g_y f_q - g_q f_y + q\big(g_z f_q - g_q f_z\big) = 0$

or $f_p \frac{\partial g}{\partial x} + f_q \frac{\partial g}{\partial y} + \big(pf_p + qf_q\big)g_z - (f_x + pf_z)g_p - \big(f_y + qf_z\big)g_q = 0$

and its subsidiary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-\big(f_y + qf_z\big)} \quad \text{…………..} \qquad (3)$$

These equations, which are known as Charpit's equations .

From (5) we can solve for p and q as functions of $x\,, y\,, z\,, a$ and the complete solution is found by solving $dz = p(x, y, z, a)dx + q(x, y, z, a)dy$
It should be noted that not all of Charpit's equations (3) need be used, but that p or q must occur in the solution obtained.

Example 25: Solve: $px + qy = pq$ …………. (1)

Sol: Let $f(x, y, z, p, q) = px + qy - pq = 0$

then $f_x = p, f_y = q, f_2 = 0, f_p = x - q, f_q = y - p.$
Charpit's auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{-dp}{f_x + pf_2} = \frac{dq}{f_y + q \cdot f_z}.$$

$\Rightarrow \ \frac{dx}{x-q} = \frac{dy}{y-p} = \frac{dz}{p(x-q)+q(y-p)} = \frac{-dp}{p} = \frac{-dq}{q}$ ……….. (2)

from (2), we have $\frac{-dp}{b} = -\frac{dq}{q} \ \Rightarrow \frac{dp}{p} = \frac{dq}{q}$

on integration, we have $\log p = \log q + \log a$

$\Rightarrow p = aq$

putting value of p in (1), we have $(ax+y)q = aq^2$

$\Rightarrow q = \frac{ax+y}{a}$

therefore, $p = ax + y.$

The complete solution is given by the solution of

$$dz = pdx + qdy$$

$$\Rightarrow dz = (ax+y)dx + \frac{ax+y}{a} \cdot dy$$

$$\Rightarrow adz = (ax+y)(adx+dy)$$

on integrating, we get $az = \frac{(ax+y)^2}{2} + b'$

or $2az = (ax+y)^2 + 2b' = (ax+y)^2 + b$

Example 26: Find a complete integral of $pxy + pq + qy = yz$

Sol. Given $f(x,y,z,p,q) = pxy + pq + qy - yz = 0$ …… (1)

$\Rightarrow f_x = py$, $f_y = px + q - z$, $f_z = -y$, $f_p = xy + q$, $f_q = p + y$

Charpit's Auxiliary equations are

$$\frac{dx}{\frac{\partial f}{\partial p}} = \frac{dy}{\frac{\partial f}{\partial q}} = \frac{dz}{p\left(\frac{\partial f}{\partial p}\right)+q\left(\frac{\partial f}{\partial q}\right)} = \frac{\mathrm{dp}}{-\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)} = \frac{\mathrm{dp}}{-\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)}$$

or $\frac{dx}{(xy+q)} = \frac{dy}{(p+y)} = \frac{dz}{p(xy+q)+q(p+y)} = \frac{dp}{0} = -\frac{dq}{px+q)+qy}$

The fourth fraction gives $dp = 0$ so that $p = a$

Putting $p = a$ in (1), we have $axy + aq + qy = yz$

or $q(a+y) = y(z-ax)$

or $\quad q = \frac{y(z-ax)}{(a+y)}$

Puting these values of p and q in $\; dz = pdx + qdy,$ we get

$dz = adx + \frac{y(z-ax)}{a+y}dy \quad$ or $\quad dz - adx = \frac{y(z-ax)}{a+y}dy$

or $\; \frac{d\,(z-ax)}{z-ax} = \frac{y}{a+y}dy = dy - \frac{a}{a+y}dy$

Integrating we get , $\; \log\,(z-ax) = y - a\log\,(a+y) + \text{c}$

or $\; \log\,(z-ax) + \log\,(a+y)^a = y + c$

or $\; \log(z-ax)(a+y)^a = y + c$

$(z-ax)(a+y)^a = e^c e^y = be^y$

Example 27: Solve $\; z^2 = pqxy \;$ by Charpit's method .

Sol: We have $\; f(x,y,z,p,q) = pqxy - z^2 = 0$

$f_x = pqy \;, f_y = pqx \;, f_z = -2z \;, f_p = qxy \;, f_q = pxy$

Charpit's Auxiliary equations are :

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{dp}{-(f_z+pf_z)} = \frac{dq}{-(f_y+qf_z)}$$

or $\quad \frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{pqxy+pqxy} = -\frac{dp}{pqy-2zp} = -\frac{dq}{pqx-2zq} \quad$ …...……… (1)

using multipliers $\; 0\,,0\,,0\;,-\frac{1}{p}\;,\frac{1}{q}\;$ each fractions of (1) are equal to

$\frac{\frac{dp}{p}-\frac{dq}{q}}{(qy-\text{px})}$ ……….. (2)

using multipliers $\frac{1}{x}\,,-\frac{1}{y}\;,0\;,0\,,0\;$ each fractions of (1) are equal to

$\frac{\frac{dx}{x}-\frac{dy}{y}}{qy-px}$ ………………. (3)

Equating (2) and (3) we have

$$\frac{dp}{p}-\frac{dq}{q}=\frac{dy}{y}-\frac{dx}{x} \quad \text{or} \quad \frac{dp}{p}+\frac{dx}{x}=\frac{dy}{y}+\frac{dq}{q}$$

Integrating we get $\log\ p+\log\ x=\log\ y+\log\ q+\log\ a$
$\Rightarrow\ px=qya \quad \Rightarrow\ p=\frac{qya}{x}$

Form given equation , we get $z^2=\frac{q^2ay^2x}{x}$

$\Rightarrow\ q=\frac{z}{\sqrt{a}y}=\frac{z}{by}$ where $b=\sqrt{a}$ So that $p=\frac{bz}{x}$

Putting the value of p and q in $dz=pdx+qdy$, we have

$dz=\frac{bz}{x}dx+\frac{z}{by}dy$ or $\frac{dz}{z}=\frac{b}{x}dx+\frac{1}{by}dy$

Integrating we get $\log z=b\log x+\frac{1}{b}\log y+\log c=\log x^b y^{\frac{1}{b}}c$
or $z=cx^b y^{\frac{1}{b}}$ which is required complete integral.

Example 28: Find complete integral of the equation $xp+3yq=2(z-x^2q^2)$.

Sol: Given $xp+3yq=2(z-x^2q^2)$ ……… (1)

Let $f(x,y,z,p,q)=xp+3yq-2(z-x^2q^2)$

Then $f_x=p+4xq^2$, $f_y=\ 3q$, $f_z=-2$, $f_p=x$, $f_q=3y+4x^2q$

Charpit's Subsidiary equations are

$$\frac{dx}{x}=\frac{dy}{3y+4x^2q}=\frac{dz}{px+3qy+4x^2q^2}=\frac{dp}{-p-4xq^2+2p}=\frac{dq}{-q}$$

First and last fractions give $\frac{dx}{x}=\frac{dq}{-q}\Rightarrow\ \frac{dq}{q}+\frac{dq}{x}=0$

integrating gives , $\log qx=\log a\Rightarrow q=\frac{a}{x}$

Substituting in Eq. (1) gives $p=\frac{2(z-a^2)}{x}-\frac{3ya}{x^2}$

Putting the value of p and q in $dz=pdx+qdy$ we have

$$dz = \left[\frac{2(z-a^2)}{x} - \frac{3ya}{x^2}\right] dx + \frac{a}{x} dy$$

or $x^2 dz = 2x(\text{z} - a^2)dx - 3aydx + axdy$

or $x^2 dz - 2x(\text{z} - a^2)dx = -3aydx + axdy$

$$\Rightarrow \ x^4 d\left(\frac{z-a^2}{x^2}\right) = -3aydx + axdy$$

$$\Rightarrow \ d\left(\frac{z-a^2}{x^2}\right) = \frac{ax}{x^3} dy - \frac{3ay}{x^4} dx$$

$$\Rightarrow \ d\left(\frac{z-a^2}{x^2}\right) = d\left(\frac{ay}{x^3}\right)$$

Integrating, $\frac{z-a^2}{x^2} = \frac{ay}{x^3} + b \quad \Rightarrow \ z = a\left(a + \frac{y}{x}\right) + bx^2$

Example 29. Find a complete integral of the equation
$p^2x + q^2y = z$ ………….. (1)

Sol: Charpit's auxiliary equations are

$$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{2(p^2x+q^2y)} = \frac{dp}{p-p^2} = \frac{dq}{q-q^2}$$

$$\Rightarrow \ \frac{p^2dx+2pxdp}{p^2x} = \frac{q^2dy+2qydq}{q^2y}$$

$\Rightarrow \frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}$. Integrating gives $p^2x = aq^2y$ ………. (2)

where a is a constant. Solving equations (1) and (2) for p, q, we have
$p = \left\{\frac{az}{(1+a)x}\right\}^{\frac{1}{2}}, \ q = \left\{\frac{z}{(1+a)y}\right\}^{\frac{1}{2}}$

Putting these values off p and q we have

$$\left(\frac{1+a}{z}\right)^{\frac{1}{2}} dz = \left(\frac{a}{x}\right)^{\frac{1}{2}} dx + \left(\frac{1}{y}\right)^{\frac{1}{2}} dy$$

$$\Rightarrow \{(1+a)z\}^{\frac{1}{2}} = (ax)^{\frac{1}{2}} + y^{\frac{1}{2}} + b$$

Exercises

Find the complete integrals of the equations:

1. $(p^2 + q^2)y = qz$

2. $p = (z + qy)^2$

3. $px^5 - 4q^3x^2 + 6x^2z - 2 = 0$

4. $2(y + zq) = q(xp + yq)$

5. $2(z + xp + yq) = yp^2$

Special Types of First-order Equations

In this section we shall consider some special types of first-order partial differential equations whose solutions may be obtained easily by Charpit's method.

(I) Equations Involving Only p and q.

For equations of the type $f(p, q) = 0$ ……… (1)

Charpit's equations reduce to $\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0}$

An obvious solution of these equations is $p = a$

the corresponding value of q being obtained from (1) in the form
$f(a, q) = 0$

so that $q = Q(a)$ a constant. The solution of the equation is then
$z = ax + Q(a)y + b$

where b is a constant.

We have taken here $dp = 0$. Sometimes it is easier to take $dq = 0$
and proceed in a similar way .

Example 29. Find a complete integral of the equation $pq = 1$.

In this case $Q(a) = 1/a$, so that we see, from equation (4), that a complete integral is
$z = ax + \frac{y}{a} + b$
which is equivalent to

$a^2x + y - az = c$

where a, c are arbitrary constants.

(II) Equations Not Involving the Independent Variables.

If the partial differential equation is of the type

$f(z, p, q) = 0$ ……… (3)

Charpit's equations take the forms

$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-pf_z} = \frac{dq}{-qf_z}$

the last of which leads to the relation $p = aq$ ……… (4)

Solving (3) and (4), we obtain expressions for p, q from which a complete integral of $dz = pdx + qdy$ can be found .

Example 30. Find a complete integral of the equation $p^2z^2 + q^2 = 1$. Putting $p = aq$, we find that

$q^2(1 + a^2z^2) = 1,\ q = (1 + a^2z^2)^{-\frac{1}{2}},\ p = a(1 + a^2z^2)^{-\frac{1}{2}}$
Hence

$(1 + a^2z^2)^k dz = adx + dy$

which leads to the complete integral
$az(1 + a^2z)^{\frac{1}{2}} - \log\left[az + (1 + a^2z^2)^{\frac{t}{t}}\right] = 2a(ax + y + b)$

(III) Separable Equations.

If the partial differential equation can be written in the form
$f(x, p) = g(y, q)$
For such an equation Charpit's equations become

$\frac{dx}{f_p} = \frac{dy}{-g_q} = \frac{dz}{pf_p - qg_q} = \frac{dp}{-f_x} = \frac{dq}{-g_y}$

so that we have an ordinary differential equation

$\frac{dp}{dx} + \frac{f_x}{f_p} = 0$

in x and p which may be solved to give p as a function of x and an arbitrary constant a .

Hence we can determine $p, \; q$ from the relations
$f(x, p) = a, \; g(y, q) = a$

and solve the equation $dz = pdx + qdy$.

Example 31: Solve the equaiton $q = xyp^2$

Sol: The given equation can be written as
$xp^2 = \frac{q}{y}$

Let $xp^2 = a$ and $\frac{q}{y} = a$

Then $p = \sqrt{\frac{a}{x}}$ and $q = ay$

Using the values of p and q in $dz == pdx + qdy$

We have , $dz = \sqrt{\frac{a}{x}} dx + aydy$

Integrating, we get $z = 2\sqrt{a}(x)^{\frac{1}{2}} + a\frac{y^2}{2} + b$

or $2z = \; 4\sqrt{ax} + ay^2 + 2b$ or

$16ax - (2z - ay^2 - 2b)^2 = 0$

which is the complete integral.

Example 32: .Find a complete integral of $p^2y(1 + x^2) = qx^2$.

Sol. The given equation can be written as :

$\frac{p^2(1+x^2)}{x^2} = \frac{q}{y}$

Let $\frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = a^2$

$\Rightarrow p = \frac{ax}{\sqrt{1+x^2}},$ and $q = a^2y.$

Putting p and q in $dz = pdx + qdy$ yields

$dz = \frac{ax}{\sqrt{1+x^2}}dx + a^2ydy$

$z = a\sqrt{1+x^2} + \frac{a^2y^2}{2} + b$

where a and b are arbitrary constant .

Example 33: Solve $p^3 + q^3 = x + y$. $p^3 + q^3 = x + y$

Sol: Let $p^3 - x = y - q^3 = a$

$p^3 - x = a \Rightarrow p^3 = x + a \Rightarrow p = (x+a)^{\frac{1}{3}}$

and $y - q^3 = a \Rightarrow q^3 = y - a \Rightarrow q = (y-a)^{\frac{1}{3}}$

Now, $dz = pdx + qdy$
$\Rightarrow dz = (x+a)^{\frac{1}{3}}dx + (y-a)^{\frac{1}{3}}dy$

Integrating we get
$\Rightarrow z = \frac{3}{4}(x+a)^{\frac{4}{3}} + \frac{3}{4}(y-a)^{\frac{4}{3}} + b$

(IV) Clairaut Equations.

A first-order partial differential equation is said to be of Clairaut type if it can be written in the form $z = px + qy + f(p,q)$

The corresponding Charpit equations becomes:
$\frac{dx}{x+f_p} = \frac{dy}{y+f_q} = \frac{dz}{px+qy+pf_p+qf_q} = \frac{dp}{0} = \frac{dq}{0}$

The last two fractions give $p = a$, $q = b$ so that

$z = ax + by + f(a, b)$ ………..(5) becomes the complete solution.

The singular solution of the given equation is found by eliminating a , b from $\frac{\partial z}{\partial a} = 0$, $\frac{\partial z}{\partial b} = 0$ and (5) .

Example 34: Find the general and singular solution of

$$z = px + qy + p^2 + q^2$$

Sol: The equation is of Clairaut's form therefore its complete solution is given by

$z = ax + by + a^2 + b^2$ ……………. (1)

Differtiating (1) partially with respect to a and b we get

$\frac{\partial z}{\partial a} = x + 2a$ and $\frac{\partial z}{\partial b} = y + 2b$

Now $\frac{\partial z}{\partial a} = 0 \Rightarrow a = -\frac{x}{2}$

and $\frac{\partial z}{\partial b} = 0 \Rightarrow b = -\frac{y}{2}$

Using these values of a and b in (1) we get

$z = -\frac{x^2}{2} - \frac{y^2}{2} + \frac{x^2}{4} + \frac{x^2}{4}$ which is the required singular solution .

Exercises

1. $p + q = pq$
2. $z = p^2 - q^2$
3. $p^2 q(x^2 + y^2) = p^2 + q$
4. $p^2 q^2 + x^2 y^2 = x^2 q^2 (x^2 + y^2)$
5. $z = p^2 - q^2$
6. $p(1 + q) = qz$

7. $p^2 + q^2 = x + y$

8. $zpq = p^2(xq + p^2) + q^2(yp + q^2)$

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