Closed Functional Factorial Using Generalized Difference Operators

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Abstract. This chapter uses the inverse of the generalized difference operator to derive equations for generic partial sums on infinite series of rational functions of the generalized polynomial factorial and trigonometric factorials. Suitable examples are provided to demonstrate the key findings.

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1. Introduction

Difference operators are fundamental tools in discrete mathematics, analogous to derivatives in continuous calculus. They measure the change in a function's value over discrete intervals and are essential in the study of sequences, discrete functions, and difference equations. Understanding difference operators provides a foundation for discrete calculus, which is crucial for various fields, including computer science, numerical analysis, and combinatorics.

Factorials, defined traditionally for integers, are a fundamental concept in combinatorics, algebra, and analysis. Extending factorials to non-integer values led to the gamma function, a continuous extension of the factorial function. However, various generalizations and extensions of factorials exist, one of which involves using generalized difference operators. These operators provide a powerful tool for discrete calculus and enable the definition of factorial functions for a wider range of functional forms.

Early instances of finite differences can be traced back to ancient mathematicians such as Archimedes, who used methods resembling finite differences to compute areas and volumes. However, the systematic study of finite differences began much later. In the 17th and 18th centuries, the development of calculus by Newton and Leibniz paved the way for the study of finite differences. Newton, in particular, made significant contributions with his work on interpolation and the binomial theorem, laying the groundwork for the formalization of difference operators.

The forward difference operator Δ was one of the earliest forms to be formalized, this operator became essential in numerical analysis for approximating derivatives and solving difference equations. and also the backward difference operator ∇ was introduced to complement the forward difference operator. The backward difference operator is particularly useful in situations where the initial value of a sequence is known. The central difference operator δ provides a symmetric measure of change. It is used to approximate derivatives more accurately than the forward or backward difference operators, especially when dealing with evenly spaced data points. Higher-order differences extend to n-th order, which are crucial in polynomial interpolation and numerical differentiation.

Difference operators are fundamental in numerical differentiation and integration, allowing for the approximation of derivatives and integrals when working with discrete data. Methods such as finite difference methods for solving differential equations heavily rely on these operators. Newtons forward and backward difference formulas, based on difference operators, are used for polynomial interpolation. These methods provide polynomial approximations to functions based on discrete data points, which are critical in numerical analysis and computational mathematics.

Difference operators are used in combinatorial identities, summations, and the study of discrete structures. They play a key role in the analysis of sequences, recurrence relations, and discrete dynamic systems. The study of difference operators continues to evolve, with extensions to non-uniform grids, higher dimensions, and various applications in computer science, engineering, and the natural sciences. Modern research often involves the integration of difference operators with other mathematical frameworks, such as graph theory and discrete optimization.

2. Notations

Throughout this chapter, we make use of the following assumptions:

(1) l, m, r and n are positive integers.
(2) $[\frac{k}{l}]$ denotes the integer part of $\frac{k}{l}$.
(3) $\alpha^{(j)} = (\alpha)(\alpha-1)\cdots(\alpha-j+1)$.
(4) $[0,\infty)$ is the non-negative reals.
(5) $j = k - [\frac{k}{\ell}]\ell$.

3. Preliminaries

In this section, we provide some basic definitions and results that will be relevant in the future discussion.

Definition 3.1. Let $u(k)$ be the real valued function defined on $(0,\infty)$, $n \in \mathbb{N}(1)$ and $\ell \in (0,\infty)$. Then the closed functional factorial and its reciprocal are respectively defined as

$$(u(k))_\ell^{[n]} = u(k)u(k-\ell)u(k-2\ell)...u(k-(n-1)\ell) \tag{1}$$

and when $u(k-r\ell) \neq 0$ for $r = 0, 1, 2, ..., n-1$,

$$\frac{1}{(u(k))_\ell^{[n]}} = \frac{1}{u(k)u(k-\ell)u(k-2\ell)...u(k-(n-1)\ell)}. \tag{2}$$

In particular when $u(k) = k$ (1) becomes the generalized polynomial factorial

$$k_\ell^{(n)} = k(k-\ell)(k-2\ell)...(k-(n-1)\ell) \tag{3}$$

and (2) becomes the generalized reciprocal polynomial factorial

$$\frac{1}{k_\ell^{(n)}} = \frac{1}{k(k-\ell)(k-2\ell)...(k-(n-1)\ell)}. \tag{4}$$

Definition 3.1. Let $u(k),\ k \in [0,\infty)$ be a real valued function. The generalized difference operator Δ_ℓ on $u(k)$ is defined as;

$$\Delta_\ell u(k) = u(k+\ell) - u(k),\ \ k \in [0,\infty),\ \ \ell \in (0,\infty), \tag{5}$$

and the inverse of Δ_ℓ on $u(k)$ is defined as,

$$\text{if } \Delta_\ell v(k) = u(k),\ \text{then } v(k) \ = \ \Delta_\ell^{-1} u(k) + c_{\ell(k)}. \tag{6}$$

In general,

$$\Delta_\ell^{-\nu} = \Delta_\ell^{-1}\left(\Delta_\ell^{-(\nu-1)}\right). \tag{7}$$

Lemma 3.2. If $\lim\limits_{k\to\infty} \Delta_\ell^{-1} u(k) = 0$ and $j = k - \left[\frac{k}{\ell}\right]\ell$, then

$$\Delta_\ell^{-1} u(k)|_k^\infty = (-1)\sum_{r=0}^{\infty} u(k+r\ell). \tag{8}$$

Theorem 3.3. If $\lim\limits_{k\to\infty} \Delta_\ell^{-r} u(k) = 0$ for $r = 1,2,\cdots,m$ and $k \in [m\ell,\infty)$, then

$$\Delta_\ell^{-m} u(k)|_k^\infty = (-1)^m \sum_{r=m}^{\infty} \frac{(r-1)^{(m-1)}}{(m-1)!} u(k - m\ell + r\ell). \tag{9}$$

Proof. The proof follows by taking Δ_ℓ^{-1} on (8) for $(m-1)$ times. □

Lemma 3.4. Let $u(k)$ and $v(k)$ be two real valued functions. Then

$$\Delta_\ell^{-1}[u(k)v(k)] = u(k)\Delta_\ell^{-1} v(k) - \Delta_\ell^{-1}[\Delta_\ell^{-1} v(k+\ell)\Delta_\ell u(k)]. \tag{10}$$

Proof. From definition of Δ_ℓ, we find

$$\Delta_\ell[u(k)z(k)] = z(k+\ell)\Delta_\ell u(k) + u(k)\Delta_\ell z(k). \tag{11}$$

Applying (6) in (11), we obtain

$$\Delta_\ell^{-1}[u(k)\Delta_\ell z(k)] = u(k)z(k) - \Delta_\ell^{-1}[z(k+\ell)\Delta_\ell u(k)]. \tag{12}$$

The proof follows by taking $v(k) = \Delta_\ell z(k)$ and (12). □

Theorem 3.2. *Let* $\ell > 0$ *and* $k \in [\ell,\infty)$. *Then*

$$\sum_{r=0}^{[\frac{k}{\ell}]} \binom{r+n-1}{n-1} u(k-r\ell) + \sum_{r=1}^{n-1} \frac{([\frac{k}{\ell}]+r)^{(r)}}{r!} \Delta_\ell^{-(n-r)} u(\hat{\ell}(k) + (n-1-r)\ell)$$
$$= \Delta_\ell^{-n} u(k+n\ell) - \Delta_\ell^{-n} u(\hat{\ell}(k) + (n-1)\ell). \tag{13}$$

Proof. we have

$$u(k)+u(k-\ell)+\cdots+u(\hat{\ell}(k))=\Delta_\ell^{-1}u(k+\ell)-\Delta_\ell^{-1}u(\hat{\ell}(k)). \tag{14}$$

Since $\hat{\ell}(k)=\hat{\ell}(k-r\ell)$ for $r=1,2,...[\frac{k}{\ell}]$, replacing k by $k-\ell$, $k-2\ell$,...$\hat{\ell}(k)$, in (14) and then adding all the resultant expressions, we arrive

$$\sum_{r=0}^{[\frac{k}{\ell}]}\binom{r+1}{1}u(k-r\ell)=\Delta_\ell^{-2}u(k+2\ell)-\left\{\Delta_\ell^{-2}u(\hat{\ell}(k)+\ell)+\left(\left[\frac{k}{\ell}\right]+1\right)\Delta_\ell^{-1}u(\hat{\ell}(k))\right\}. \tag{15}$$

Applying the process mentioned above to (15), we find that

$$\sum_{r=0}^{[\frac{k}{\ell}]}\binom{r+2}{2}u(k-r\ell)=\Delta_\ell^{-3}u(k+3\ell)-\Big\{\Delta_\ell^{-3}u(\hat{\ell}(k)+2\ell)$$
$$-\frac{([\frac{k}{\ell}]+1)_1^{(1)}}{1!}\Delta_\ell^{-2}u(\hat{\ell}(k)+\ell)-\frac{([\frac{k}{\ell}]+2)_1^{(2)}}{2!}\Delta_\ell^{-1}u(\hat{\ell}(k))\Big\}. \tag{16}$$

Following this procedure, we have the theorem's proof. ☐

4. Summation on Closed Functional Factorial

Here, we derive Δ_ℓ^{-1} on certain closed functional factorial to find the sum on infinite series of generalized closed functional factorial and polynomial factorial functions.

Theorem 4.1. Let $n\in\mathbb{N}(1)$ and $\ell\in(0,\infty)$. Then

$$\sum_{r=1}^{[\frac{k}{\ell}]}\Big[u(k-r\ell)_\ell^{[n]}\Delta_{(n+1)\ell}u(k-(r+n)\ell)\Big]=[u(k)]_\ell^{[n+1]} \tag{17}$$

Proof. From the Definition (3.1) we find

$$\Delta_\ell\,(u(k))_\ell^{[n]}=(u(k))_\ell^{[n-1]}\,\Delta_{n\ell}u(k-(n-1)\ell) \tag{18}$$

The proof follows by taking Δ_ℓ^{-1} on both sides and replacing n by $n+1$ in (18). ☐

Corollary 4.2. For $\ell\in(0,\infty), k\in[0,\infty)$ and $j=k-\left[\frac{k}{\ell}\right]$, we have

$$\sum_{r=1}^{\left[\frac{k}{\ell}\right]}\Big((k-r\ell)_{(n+1)\ell}^{[m]}\Big)_\ell^{[n]}m(n+1)\ell(k-(n+r)\ell)_{(n+1)\ell}^{(m+1)}=\Big(k_{(n+1)\ell}^{[m]}\Big)_\ell^{[n+1]}\Big|_j^k \tag{19}$$

Proof. The proof follows by taking $u(k) = k^{[m]}_{(n+1)\ell}$ in (4.1) and applying (6). $\square$

The following example illustrate Corollary (4.2).

Example 4.1. *Taking $k = 10, \ell = 5, m = 2, n = 2$ in (19) we get*

$$\sum_{r=1}^{\left[\frac{10}{5}\right]} \left((10-5r)^{[2]}_{2(3)}\right)^{[2]}_{2} 2(3)2(10-(2+r)2)^{(3)}_{(3)2} = \left(10^{[2]}_{(3)2}\right)^{[3]}_{2} - \left(8^{[2]}_{(3)2}\right)^{[3]}_{2}$$

Theorem 4.3. For $n \in \mathbb{N}(1), \ell \in (0, \infty)$, we have

$$\Delta_\ell^{-1}\left[\frac{\Delta_{(n-1)\ell}u(k-(n-1)\ell)}{(u(k))^{[n]}_\ell}\right] = -\frac{1}{(u(k-\ell))^{[n-1]}_\ell} \tag{20}$$

and hence

$$\sum_{r=0}^{\infty}\left[\frac{\Delta_{(n-1)\ell}u(k+r\ell-(n-1)\ell)}{(u(k+r\ell))^{[n]}_\ell}\right] = \frac{1}{(u(k-\ell))^{[n-1]}_\ell} \tag{21}$$

Proof. From the definition (3.1), we find that

$$\Delta_\ell \frac{1}{(u(k))^{[n]}_\ell} = -\frac{\Delta_{n\ell}u(k-(n-1)\ell)}{(u(k+\ell))^{[n+1]}_\ell} \tag{22}$$

The proof follows by taking Δ_ℓ^{-1} on both sides and replacing k by $k-\ell$ then replacing n by $n-1$ in (22).
The relation (21) is obtained by applying (8) and taking limit k to ∞ in (20) $\square$

Corollary 4.4. For $\ell \in (0, \infty)$, $k \in [0, \infty)$, we have

$$\sum_{r=0}^{\infty}\frac{((k+r\ell)^3)^{[4]}_\ell - ((k+r\ell-3\ell))^{[4]}_\ell}{((k+r\ell)^3)^{[4]}_\ell} = \frac{1}{((k-\ell)^3)^{[3]}_\ell} \tag{23}$$

Proof. The proof follows by taking $u(k) = k^3$ in (20) and applying (8) $\square$

The following example is an illustration of Corollary (4.4)

Example 4.2. *Taking $k = 5, \ell = 2$ in (23), we get*

$$\sum_{r=1}^{\infty}\frac{((5+2r)^3)^{[4]}_2 - ((5+2r-6))^{[4]}_2}{((5+2r)^3)^{[4]}_2} = \frac{1}{(9)^{[3]}_2} \tag{24}$$

The following theorem is a specified form of discrete summation by parts.

Theorem 4.5. For $k \in [0,\infty), \ell \in (0,\infty)$ and $n \in \mathbb{N}(1)$, we have

$$\Delta_\ell^{-1}\left[\left(u(k)+\frac{\Delta_\ell u(k)}{2}\right)\Delta_\ell u(k)\right]=\frac{(u(k))^2}{2} \tag{25}$$

Proof. Taking $z(k)=u(k)$ in (12), we get

$$\Delta_\ell^{-1}[u(k)\Delta_\ell u(k)] = u(k)u(k) - \Delta_\ell^{-1}[u(k+\ell)\Delta_\ell u(k)]$$

$$\Delta_\ell^{-1}[u(k)\Delta_\ell u(k)] = (u(k))^2 - \Delta_\ell^{-1}[(\Delta_\ell u(k)+u(k))\Delta_\ell u(k)]$$

The proof follows by combining the Δ_ℓ^{-1} terms. □

Corollary 4.6. For $\ell \in (0,\infty), k \in [0,\infty), j = k - \left[\frac{k}{\ell}\right]$, we have

$$\sum_{r=1}^{\left[\frac{k}{\ell}\right]}\left\{\left((k-r\ell)_\ell^{[n]}+\frac{n\ell(k-r\ell)_\ell^{[n-1]}}{2}\right)n\ell(k-r\ell)_\ell^{[n-1]}\right\}=\frac{(k_\ell^{[n]})^2}{2}-\frac{(j_\ell^{[n]})^2}{2} \tag{26}$$

Proof. The proof follows by taking $u(k) = k_\ell^{[n]}$ in Theorem (4.5) and applying Definition (3.1). □

Example 4.3. *Taking $k=5, \ell=2, n=2$ in (26), we get*

$$\sum_{r=1}^{2}\left\{\left((5-2r)_2^{[2]}+\frac{4(5-2r)_2^{[1]}}{2}\right)4(5-2r)_2^{[1]}\right\}=\frac{(5_2^{[2]})^2}{2}-\frac{(1_2^{[2]})^2}{2}$$

$$112 = 112$$

5. Higher Order Summation

In this section we find the higher order summation on infinite series of generalized closed factorial function.

Theorem 5.1. Let $m,n \in \mathbb{N}(1)$, $(m+1)\le n$, $k_\ell^n \neq 0$. Then

$$\sum_{r=m}^{\infty}\frac{(r-1)^{(m-1)}}{(m-1)!}\frac{1}{(k-m\ell+r\ell)_\ell^{(n)}}=\frac{1}{(n-1)^{(m)}\ell^m(k-m\ell)_\ell^{(n-m)}}. \tag{27}$$

Proof. From the Definitions 5 and 6, we obtain

$$\Delta_\ell^{-1}\frac{1}{k_\ell^{(n)}}\Big|_k^\infty=\frac{1}{(n-1)\ell(k-\ell)_\ell^{(n-1)}},\ n\ge 2. \tag{28}$$

The proof follows by taking Δ_ℓ^{-1} on (28) for $(m-1)$ times and (9). □

The following example illustrate's Theorem 5.1 for m=3.

Example 5.1. *For $n \geq 4$, $k \in [3\ell, \infty)$, $\ell \in (0, \infty)$ and $m = 3$, equation (27) becomes*

$$\sum_{r=3}^{\infty} \frac{(r-1)^{(2)}}{(2)!(k-(r-2)\ell)_{\ell}^{(n)}} = \frac{1}{(n-1)^{(3)}\ell^3(k-3\ell)_{\ell}^{(n-3)}}. \tag{29}$$

In particular, $k = 2$, $\ell = 0.5, n = 4$ in (29), we obtain

$$\frac{1}{2 \times 1.5 \times 1 \times 0.5} + \frac{3}{2.5 \times 2 \times 1.5 \times 1} + \cdots = \frac{1}{6 \times 0.5^4}.$$

Theorem 5.2. For $k \neq 0$ and $(k + r\ell)_{\ell}^{(2)} \neq 0$, we have

$$\sum_{r=1}^{\infty} \frac{2k + (2r-1)\ell}{\left((k+r\ell)_{\ell}^{(2)}\right)^2} = \frac{1}{\ell k^2}. \tag{30}$$

Proof. From Theorem (3.1) and (6), we find

$$\Delta_{\ell}^{-1} \frac{2k + \ell}{\left((k+\ell)_{\ell}^{(2)}\right)^2} = -\frac{1}{\ell k^2}. \tag{31}$$

The proof follows by applying (8) on (31) for the limit k to ∞. □

The following is an illustration of Theorem 5.2.

Example 5.2. *By taking $\ell = 0.4$ in (30), we get*

$$\sum_{r=1}^{\infty} \frac{2k + (2r-1)(0.4)}{\left((k+r(0.4))_{0.4}^{(2)}\right)^2} = \frac{1}{(0.4)k^2}$$

In particular, when $k = 1$,

$$\frac{2 + 1(0.4)}{(1.4)^2(1)^2} + \frac{2 + 3(0.4)}{(1.8)^2(1.4)^2} + \frac{2 + 5(0.4)}{(2.2)^2(1.8)^2} + \cdots = \frac{1}{0.4}.$$

6. Summation on Trigonometric Factorials

In this section, we derive some results on closed functional factorial in trigonometric functions.

Lemma 6.1. *Let p be any real numbers. Then,*

$$\Delta_{\ell}^{-1} \sin pk = \frac{\sin p(k-\ell) - \sin pk}{1 - \cos p\ell} + c_{\ell(k)} \tag{32}$$

and

$$\Delta_{\ell}^{-1} \cos pk = \frac{\cos p(k-\ell) - \cos pk}{1 - \cos p\ell} + c_{\ell(k)} \tag{33}$$

are solutions of equation (6) when $m = 1$ and for $u(k) =$ sinpk and cospk respectively .

Proof. Replacing u(k) by $\sin pk$ and $\cos pk$ in (5), we find that

$$\Delta_\ell \sin pk = (\cos p\ell - 1)\sin pk + \sin p\ell \cos pk, \tag{34}$$

and

$$\Delta_\ell \cos pk = (\cos p\ell - 1)\cos pk - \sin p\ell \sin pk. \tag{35}$$

Since Δ_ℓ is linear, i.e., $c\Delta_\ell u(k) = \Delta_\ell cu(k)$ and $(\cos p\ell - 1)$ and $\sin p\ell$ are constants, multiplying (34) by $(\cos p\ell - 1)$, (35) by $\sin p\ell$ and then subtracting the second resultant from the first one, we find that

$$\Delta_\ell[(\cos p\ell - 1)\sin pk - \sin p\ell \cos pk] = (2 - 2\cos p\ell)\sin pk \tag{36}$$

Now (32) follows from (6) and dividing (36) by $(2 - 2\cos p\ell)$.
Similarly multiplying (34) by $\sin p\ell$, (35) by $(\cos p\ell - 1)$ and then adding them, we arrive

$$\Delta_\ell[\sin p\ell \sin pk - (\cos p\ell - 1)\cos pk] = (2 - 2\cos p\ell)\cos pk \tag{37}$$

Now (33) follows from (6) and dividing (37) by $(2 - 2\cos p\ell)$. □

Theorem 6.2. *Let $n \in N(1)$, $k \in [0, \infty)$ and p be any real number. Then,*

$$\Delta_\ell^{-(2n-1)}\big(\sin pk\big)_\ell^{[2]} = \frac{\cos p\ell}{2(2n-1)!}\frac{k_\ell^{[2n-1]}}{\ell^{2n-1}} + \frac{(-1)^n}{2^{2n}}\frac{\sin p(2k - 2n\ell)}{\sin^{2n-1} p\ell} \tag{38}$$

and

$$\Delta_\ell^{-(2n)}\big(\sin pk\big)_\ell^{[2]} = \frac{\cos p\ell}{2(2n)!}\frac{k_\ell^{[2n]}}{\ell^{2n}} + \frac{(-1)^{n+1}}{2^{2n+1}}\frac{\cos p(2k - (2n+1)\ell)}{\sin^{2n} p\ell} \tag{39}$$

Proof. From the definition of closed functional factorial, we have

$$\Delta_\ell^{-1}\big(\sin pk\big)_\ell^{[2]} = \Delta_\ell^{-1} \sin pk \sin p(k - \ell)$$

Applying the formula of $\sin C \sin D$ and Lemma 6.1 we get

$$\Delta_\ell^{-1}\big(\sin pk\big)_\ell^{[2]} = \frac{\cos p\ell}{2}\frac{k_\ell^{[1]}}{\ell} - \frac{1}{2}\frac{\cos p(2k - 3\ell) - \cos p(2k - \ell)}{2(1 - \cos 2p\ell)}$$

From the formula of $\cos C - \cos D$, we get

$$\Delta_\ell^{-1}\big(\sin pk\big)_\ell^{[2]} = \frac{\cos p\ell}{2}\frac{k_\ell^{[1]}}{\ell} - \frac{1}{2^2}\frac{\sin p(2k - 2\ell)}{\sin p\ell} \tag{40}$$

Taking Δ_ℓ^{-1} on both sides in (40) and apply the Formula we get

$$\Delta_\ell^{-2}\big(\sin pk\big)_\ell^{[2]} = \frac{\cos p\ell}{2^2}\frac{k_\ell^{[2]}}{\ell^2} - \frac{1}{2^3}\frac{\cos p(2k-3\ell)}{\sin^2 p\ell} \tag{41}$$

Continuing this process upto n times we get (38) and (39). □

Theorem 6.3. *Let $n \in N(1)$, $k \in [0,\infty)$. Then,*

$$\Delta_\ell^{-(2n-1)}\big(\sin pk\big)_\ell^{[3]} = (-1)^n \frac{\cos p\ell \cos p\left(k - \dfrac{(2n+3)\ell}{2}\right)}{2^{2n}\sin^{(2n-1)}\left(\frac{p\ell}{2}\right)}$$
$$+\frac{(-1)^{n+1}\cos p\left(3k - \dfrac{(6n+3)\ell}{2}\right)}{2^{2n+1}\sin^{(2n-1)}\left(\frac{3p\ell}{2}\right)} + \frac{(-1)^n \cos p\left(k + \dfrac{(3-2n)\ell}{2}\right)}{2^{2n+1}\sin^{(2n-1)}\left(\frac{p\ell}{2}\right)} \tag{42}$$

and

$$\Delta_\ell^{-(2n)}\big(\sin pk\big)_\ell^{[3]} = (-1)^n \frac{\cos p\ell \sin p\,(k-(n+2)\ell)}{2^{2n+1}\sin^{(2n)}\left(\frac{p\ell}{2}\right)}$$
$$+\frac{(-1)^{n+1}\sin p\,(3k-(3n+3)\ell)}{2^{2n+2}\sin^{(2n)}\left(\frac{3p\ell}{2}\right)} + \frac{(-1)^n \sin p\,(k-(n-1)\ell)}{2^{2n+2}\sin^{(2n)}\left(\frac{p\ell}{2}\right)} \tag{43}$$

Proof. Now, $\Delta_\ell^{-1}\big(\sin pk\big)_\ell^{[3]} = \Delta_\ell^{-1} \sin pk \sin p(k-\ell)\sin p(k-2\ell)$
Applying the formula of $\sin C \sin D$ and Lemma 6.1, we get

$$\Delta_\ell^{-1}\big(\sin pk\big)_\ell^{[3]} = -\frac{\cos p\ell \cos p\left(k-\dfrac{5\ell}{2}\right)}{2^2 \sin\left(\frac{p\ell}{2}\right)} + \frac{\cos p\left(3k - \dfrac{9\ell}{2}\right)}{2^3\sin\left(\frac{3p\ell}{2}\right)} - \frac{\cos p\left(k+\dfrac{\ell}{2}\right)}{2^3 \sin\left(\frac{p\ell}{2}\right)} \tag{44}$$

Taking Δ_ℓ^{-1} on both sides in (44) and apply Lemma (6.1) we get

$$\Delta_\ell^{-2}\big(\sin pk\big)_\ell^{[3]} = -\frac{\cos p\ell \sin p\,(k-3\ell)}{2^3\sin^2\left(\frac{p\ell}{2}\right)} + \frac{\sin p\,(3k-6\ell)}{2^4\sin^2\left(\frac{3p\ell}{2}\right)} - \frac{\sin p\,(k)}{2^4 \sin^2\left(\frac{p\ell}{2}\right)} \tag{45}$$

Continuing this process upto n times we get (42) and (43). □

Theorem 6.4. *Let $n \in N(1)$, $k \in [0,\infty)$. Then,*

$$\Delta_\ell^{-(2n-1)}\big(\sin pk\big)_\ell^{[4]} = \frac{\cos^2 p\ell}{2^2(2n-1)!}\frac{k_\ell^{[2n-1]}}{\ell^{(2n-1)}} + \frac{(-1)^n \cos p\ell \sin p(2k-(2n+4)\ell)}{2^{(2n+1)}\sin^{(2n-1)} p\ell}$$
$$+\frac{(-1)^n \sin p(2k-(2n-1)\ell)}{2^{(2n+2)}\sin^{(2n-1)} p\ell} + \frac{(-1)^{(n+1)}\sin p(4k-(4n+4)\ell)}{2^{(2n+2)}\sin^{(2n-1)} 2p\ell}$$
$$+\frac{\cos(4p\ell)}{2^3(2n-1)!}\frac{k_\ell^{[2n-1]}}{\ell^{(2n-1)}} + \frac{(-1)^n \sin p(2k-(2n+1)\ell)}{2^{(2n+2)}\sin^{2n-1} p\ell} \tag{46}$$

and

$$\Delta_\ell^{-(2n)}\big(\sin pk\big)_\ell^{[4]} = \frac{\cos^2 p\ell}{2^2(2n)!}\frac{k_\ell^{[2n]}}{\ell^{(2n)}} + \frac{(-1)^{n+1}\cos p\ell\cos p(2k-(2n+5)\ell)}{2^{(2n+2)}\sin^{(2n)} p\ell}$$
$$+ \frac{(-1)^{n+1}\cos p(2k-(2n)\ell)}{2^{(2n+3)}\sin^{(2n)} p\ell} + \frac{(-1)^{n}\cos p(4k-(4n+6)\ell)}{2^{(2n+3)}\sin^{(2n)} 2p\ell}$$
$$+ \frac{\cos(4p\ell)}{2^3(2n)!}\frac{k_\ell^{[2n]}}{\ell^{(2n)}} + \frac{(-1)^{n+1}\cos p(2k-(2n+2)\ell)}{2^{(2n+3)}\sin^{2n} p\ell} \tag{47}$$

Proof. Since $\Delta_\ell^{-1}\big(\sin pk\big)_\ell^{[4]} = \Delta_\ell^{-1}\sin pk\sin p(k-\ell)\sin p(k-2\ell)\sin p(k-3\ell)$
By applying Lemma (6.1) to the linera expression of above product, we get,

$$\Delta_\ell^{-1}\big(\sin pk\big)_\ell^{[4]} = \frac{\cos^2 p\ell}{2^2}\frac{k_\ell^{[1]}}{\ell} - \frac{\cos p\ell\sin p(2k-6\ell)}{2^3\sin p\ell} - \frac{\sin p(2k-\ell)}{2^4\sin p\ell}$$
$$+ \frac{\sin p(4k-8\ell)}{2^4\sin 2p\ell} + \frac{\cos(4p\ell)}{2^3}\frac{k_\ell^{[1]}}{\ell^1} - \frac{\sin p(2k-3\ell)}{2^4\sin p\ell} \tag{48}$$

Now taking Δ_ℓ^{-1} in above equation, we get

$$\Delta_\ell^{-2}\big(\sin pk\big)_\ell^{[4]} = \frac{\cos^2 p\ell}{2^2(2)!}\frac{k_\ell^{[2]}}{\ell^{(2)}} + \frac{\cos p\ell\cos p(2k-7\ell)}{2^4\sin^{(2)} p\ell} + \frac{\cos p(2k-2\ell)}{2^5\sin^{(2)} p\ell}$$
$$- \frac{\cos p(4k-10\ell)}{2^5\sin^{(2)} 2p\ell} + \frac{\cos(4p\ell)}{2^3(2)!}\frac{k_\ell^{[2]}}{\ell^{(2)}} + \frac{\cos p(2k-4\ell)}{2^5\sin^2 p\ell} \tag{49}$$

Taking Δ_ℓ^{-1} upto n times we get (46) and (47) □

2.6 Numerical Example

In this section we present some numerical examples to the finite closed factorial trigonometric functions.

Example 6.5. *In Theorem (3.2) taking* $n = 3, u(k) = \big(\sin pk\big)_\ell^{[2]}$ *gives*

$$\sum_{r=0}^{[\frac{k}{\ell}]}\binom{r+2}{2}\sin p(k-r\ell)_\ell^{[2]} + \sum_{r=1}^{2}\frac{\left(\left[\frac{k}{\ell}\right]+r\right)^{(r)}}{r!}\Delta_\ell^{-(3-r)}\sin p(j+(2-r)\ell)_\ell^{(2)}$$
$$= \Delta_\ell^{-3}\sin p(k+3\ell)_\ell^{[2]} - \Delta_\ell^{-3}\sin p(j+2\ell))_\ell^{[2]}$$

In particular, taking $k = 3, \ell = 2, p = 2$ *and applying Theorem (6.2) we get*

$$-2.283395695 = -2.283395695$$

Example 6.6. *In Theorem (3.2) taking* $n = 4, u(k) = \big(\sin pk\big)_\ell^{[3]}$ *gives*

$$\sum_{r=0}^{[\frac{k}{\ell}]} \binom{r+3}{2} \sin p(k - r\ell)_\ell^{[3]} + \sum_{r=1}^{3} \frac{\left(\left[\frac{k}{\ell}\right] + r\right)^{(r)}}{r!} \Delta_\ell^{-(4-r)} \sin p(j + (2 - r)\ell)_\ell^{(3)}$$
$$= \Delta_\ell^{-4} \sin p(k + 3\ell)_\ell^{[3]} - \Delta_\ell^{-4} \sin p(j + 2\ell))_\ell^{[3]}$$

In particular taking $k = 4, \ell = 3, p = 1$ *and applying Theorem (6.3) we get*

$$-0.4409404378 = -0.4409404378$$

Like this one can get more example in all the cases.

7. Conclusion

The closed functional form of the factorial using generalized difference operators illustrates the factorial's combinatorial and analytic properties. By leveraging the difference operators, forward differences, and the gamma function, we gain a deeper understanding of general partial sums on infinite series of rational functions of the generalized polynomial factorial and trigonometric factorials. Numerical results are also presented.

References

[1] Ahmed E, Elgazzar As, *On fractional order differential equations model for nonlocal epidemics*, Physica A, 379 (2007), 607-614.

[2] Atici FM, Eloe PW, *A transform method in discrete fractional calculus*, International Journal of Difference Equations, 2(2), 2007,165-176.

[3] Baliarsingh P,*Some new difference sequence spaces of fractional order and their dual spaces*, Applied Mathematics and Computation, 219 (2013) 97379742.

[4] Butzer PL, Westphal U, *An introduction to fractional calculus*, World Scientific Press, Singapore, 2000.

[5] Bastos NRO, Ferreira RAC and Torres DFM, *Necessary optimality conditions for fractional difference problems of the calculus of variations*, Discrete Contin. Dyn. Syst. , 29(2):417437, 2011.

[6] Cheng JF, Chu YM, *Fractional Difference Equations with Real Variable*, Abstract and Applied Analysis, 2012.

[7] Et M, olak R, *On some generalized difference sequence spaces*, Soochow J. Math. 21 (4) (1995) 377386.

[8] Davis PJ, *Leonhard Euler Integral Historical Profile of the Gamma function*, American Mathematical Society, 66(10), 849-869.

[9] Gao Z, *A computing method on stability intervals of time-delay for fractional-order retarded systems with commensurate time-delays*, Automatica 50 (2014), 1611-1616.

[10] Hamamci SE, *An algoritheorem for stabilization of fractional-order time delay systems using fractional-order PID controllers*, IEEE Trans. Automat. Control 52 (2007), 1964-1969.

[11] Hilfer R, *Applications of Fractional calculus in physics*, Word Scientific Press, Singapore,2000.

[12] Kilbas AAA, Srivastava HM, Trujillo JJ, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

[13] Kzmaz H, *On certain sequence spaces*, Canad. Math. Bull. 24 (2) (1981) 169176.

[14] Laskin N, *Fractional quantum mechanics and Levy path integrals*, Phys. Lett. A 268 (2000), 298-305.

[15] Li Y, Chen YQ, Podlubny I, *Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffer stability*, Comput. Math. Appl. 59(2010), 1810-1821.

[16] Li Y, Chen YQ, Podlubny I, *Mittag-Leffer stability of fractional order nonlinear dynamic systems*, Automatica 45 (2009), 1965-1969.

[17] Malti R, Aoun M, Levron F, et al, *Analytical computation of the H 2 -norm of fractional commensurate transfer functions, Automatica*, 47 (2011), 2425-2432.

[18] Malti R, Moreau X, Khemane F, et al, *Stability and resonance conditions of elementary fractional transfer functions, Automatica*, 47 (2011), 2462-2467.

[19] Mesbahi A, Haeri M, *Robust non-fragile fractional order PID controller for linear time invariant fractional delay systems*, J. Process Contr, 24 (2014), 1489-1494.

[20] Miller KS, Ross M, *Fractional Difference Calculus in Univalent functions, fractional calculus and their applications(Koriyama, 1988)*, Ellis Horwood Ser. Math. Appl, 1989, 139-152.

[21] Ozalp N, Demirci E, *A fractional order SEIR model with vertical transmission*, Math. Comput. Model 54 (2011), 1-6.

[22] Podlubny I, *Fractional differential equations*, Academic Press, New York, 1999.

[23] Samko SG, Kilbas AA, Marichev OI, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[24] Ye H, Gao J, Ding Y, *A generalized Gronwall inequality and its application to a fractional differential equation*, J. Math. Anal. Appl, 328 (2007), 1075-1081.