Modeling of Gauss Elimination Method for Non-Linear Programming Problem

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Introduction

Decision making play a vital role in any administration, politics, industry, business, education and many more sectors. Most favorable decision is to be taken by the higher management personnel / stakeholders in the process of decision making. Mathematical Programming is the mathematical technique used for taking the most favorable (optimal) decisions in case of many problems involved in business, industry, administration, politics, education etc. Mathematical Programming Problem (MPP) or an Optimization Problem is defined as a problem which seeks to minimize or maximize a numerical function of one or more variables when the variables can be independent or related in some way through the specification of certain constraints. The function which is to be optimized is said to be an objective function or cost function. In particular, when the objective function and constraints both are linear, the problem is called a Linear Programming Problem (LPP). All programming problems that are not linear are called non-linear i.e. a programming problem in which either the objective function or the constraints (some or all) or both are non-linear, is called a Non-Linear Programming Problem (NLPP).

Mathematical formulation of an NLPP

Mathematical Programming is an important branch of Optimization Theory. In mathematical programming, One has to optimized (maximized / minimized) a real valued objective function f of n real variables $x_1, x_2, \ldots\ldots, x_n$ associated with a finite number of constraints, which are generally written as inequalities or equations.

Generally we can define a mathematical program such as:

Min. $f(x)$

Subject to $g_i(x) \geq 0$, i = 1,2,……,m

$h_j(x) = 0$, j = 1,2,……,p

where x denotes the column vector whose components are $x_1, x_2, \dots \dots, x_n$. In various disciplines such as Economics, Business Administration, Mathematics, Engineering and Physical Sciences or in any other area where decisions must be taken in some complex (or conflicting situation) that can be represented by a mathematical model; one has to deal with NLPP.

The purpose of the chapter is to describe a brief taxonomy of an important class of programming problems which encounter in some branches of applied mathematics. In this chapter, we are focusing on Separable Programming Problems and Quadratic Programming Problems.

Separable Programming Problem (SPP)

An indirect method which is used to solve a non-linear programming problem is Separable Programming. In this process, we deal one or more linear problems that are extracted from an original problem. If an objective function and constraints are separable in any non-linear programming problem then separable programming may be very powerful technique to solve such type of problems. If there exist some functions which are not separable then one can make separable by using approximations. The single variable non-linear function reduces into piece-wise linear functions with the help of such approximations.

Quadratic Programming Problem (QPP)

Quadratic Programming Problem (QPP) is one of an important class of Non-Linear Programming and used to solve the problems of optimizing a quadratic objective function of the form (CX+α)(C'X+β).

Gauss Elimination Method for linear algebraic equations

Gauss elimination method proposes a systematic strategy for reducing the system of equations to the upper triangular form using the forward elimination approach and then for obtaining values of unknowns using the back substitution process. Let us consider the system :-

$a_1x + b_1y + c_1z = d_1$ ……………………(1)

$a_2x + b_2y + c_2z = d_2$ ……………………(2)

$a_3x + b_3y + c_3z = d_3$ ……………………(3)

Step I :- Elimination of x from equations (2) and (3)

Assuming $a_1 \neq 0$, we eliminate x from equation (2) by subtracting $\frac{a_2}{a_1}$ times the equation (1) from the equation (2). Similarly, we eliminate x from equation (3) by subtracting $\frac{a_3}{a_1}$ times the equation (1). Thus, we get the new system as given below :-

$a_1x + b_1y + c_1z = d_1$ ……………………(4)

${b_2}'y + {c_2}'z = {d_2}'$ ……………………(5)

${b_3}'y + {c_3}'z = {d_3}'$ ……………………(6)

Equation (4) is known as pivotal equation and $a_1 \neq 0$ is said to be first pivot.

Step II :- Elimination of y from equation (6)

To eliminate y from the equation (6), we multiply the equation (5) by $\frac{{b_3}'}{{b_2}'}$ and subtract it from the equation (6). Thus, we get the new system as :-

$a_1x + b_1y + c_1z = d_1$ ……………………(7)

${b_2}'y + {c_2}'z = {d_2}'$ ……………………(8)

${c_3}''z = {d_3}''$ ……………………(9)

Step III :- Now, we can find the values of unknowns $x, y\ and\ z$ from an upper triangular system defined by equations (7), (8) and (9) obtained above by using back substitution.

Gauss elimination method for inequalities

The solution of an inequality provides possibility of many solutions in bounded form. One has to select minimum or maximum value out of the solutions obtained above as per the requirement of the problem under consideration. This is the basic theme of this method for inequalities.

Maximum or minimum value of an objective function ($\sum c_i x_i$,where all $c_i s$ are positive) depends upon the maximum or minimum values of the decision variables.

We are applying Gauss elimination method on the system of linear inequalities of the same sign in nature i.e., either ≤ or ≥. The process of elimination is done by combining the inequalities in such a way that at each iteration one variable and an inequality are eliminated. At the end of the process, there remains only one inequality with one variable remains. The last inequality thus obtained gives the value of last variable in bounded form. One can select the value of that particular variable as per requirement of the objective function. At the end, we can obtain the values of remaining variables by the methos of back substitution. Let us consider the system of m inequalities and n variables:-

$a_{11}x_1 + \dots\dots\dots. + a_{1n}x_n \leq b_1$ ....(10)

$a_{21}x_1 + \dots\dots\dots. + a_{2n}x_n \leq b_2$ ....(11)

....

....

$a_{m1}x_1 + \dots\dots\dots. + a_{mn}x_n \leq b_m$.........................(12)

Elimination of x_1

Assuming $a_{11} \neq 0$, we multiply the first row by corresponding factors and then subtract them as laid down in case of linear simultaneous equations stated earlier. Thus, we get the new system as given below :-

$$a_{22}'\,x_2 + \dots\dots\dots.. + a_{2n}'\,x_n \leq b_2'$$

$$a_{32}'\,x_2 + \dots\dots\dots.. + a_{3n}'\,x_n \leq b_3'$$

....

....

$$a_{m2}'\,x_2 + \dots\dots\dots.. + a_{mn}'\,x_n \leq b_m'$$

where $\qquad a_{m2}' = a_{mn} - \frac{a_{m1}a_{1n}}{a_{11}}$

There remain (n-1) variables as well as inequalities at the end of first iteration. There remains only one variable after repeating this process to (n-1) iterations. One can easily obtain the value of the last variable. There may be some redundant constraints present in the system.

Algorithm of Gauss elimination method

The steps to apply Gauss elimination method are as follows:

1. One has to reformulate the given problem as per requirement of the method. In this method, an objective function of the problem under consideration is also considered as constraint. The nature of all inequalities including constraint inequalities and an objective function inequality must be same either ≤ or ≥.
2. The process of elimination is done by combining the inequalities in such a way that at each iteration one variable and an inequality are eliminated. If an absurd inequality appears like $0 \leq d$ where d is not a positive number at any step, then one can easily conclude that the given problem has an infeasible solution otherwise a feasible solution.
3. One has to form the following pairwise disjoint sets as given below.

$$l_j^{+} = \{i : A_{i1} > 0\}$$
$$l_j^{-} = \{i : A_{i1} < 0\}$$
$$l_j^{0} = \{i : A_{i1} = 0\} \text{ ; } i = 1,2, \ldots\ldots, m \text{ and } j = 1,2,\ldots,n$$

4. If the set l_j^{+} or l_j^{-} comes out as an empty set for a variable at any step, then it indicates that the given problem has an unbounded solution.

Problem Formulation

Separable Non-linear Programming Problem

Let us consider the problem

Max. (or Min.) $f_j\ (x_1, x_2, \ldots \ldots \ldots .., x_n)$

$$g_{ij}(x_1, x_2, \ldots \ldots \ldots .., x_n) \leq b_i$$

$$x_j \geq 0$$

Above problem can be written as given below if an objective function and constraints are separable:

$$f_j\ (x_1, x_2, \ldots \ldots \ldots .., x_n) = \sum_{j=1}^{n} f_j(x_j)$$

$$g_{ij}(x_1, x_2, \ldots \ldots \ldots .., x_n) = \sum_{j=1}^{n} g_{ij}(x_j)$$

It can also be rewritten as

$$\text{Max. (or Min.) } \sum_{j=1}^{n} f_j(x_j)$$

$$\sum_{j=1}^{n} g_{ij}(x_j) \leq b_i$$

$$x_j \geq 0$$

where some or all g_{ij} , x_{ij} , $f(x_j)$ are not linear. Now, one can easily approximate g_{ij} through a set of arbitrary break points. Let K_j be the breaking points in number and a_{jk} be its k^{th} breking value for the j^{th} variable. Also, let the weight be w_{jk} associated with the k^{th} breaking point of j^{th} variable.

The reduced separable non-linear programming problem can be rewritten as

Max. (or Min.) $\sum_{j=1}^{n} \sum_{k=1}^{K_j} f_j(a_{jk}) w_{jk}$

$\sum_{j=1}^{n} \sum_{k=1}^{K_j} g_{ij}(a_{jk}) w_{jk} \leq b_i$

$0 \leq w_{j1} \leq y_{j1}$

$0 \leq w_{jk} \leq y_{j,k-1} + y_{jk}$

$0 \leq w_{jK_j} \leq y_{j,K_j-1}$

$\sum_{k=1}^{K_j} w_{jk} = 1$, $\sum_{k=1}^{K_j-1} y_{jk} = 1$

$y_{jk=}$ 0 or 1

The process of break points transforms the separable non-linear programming problem into a LPP. The decision variables for this approximating problem are considered as w_{jk} and y_{jk}.

Quadratic Programming Problem (QPP)

Let us consider QPP as

Max. Z = $(a_1\, x + \alpha)(\, a_2\, x + \beta)$

Subject to A $x \leq b$

and $x \geq 0$

where it is assumed that $(a_1\, x + \alpha)$,$(\, a_2\, x + \beta)$ are positive in case of all feasible solutions.

We have to reformulate the given problem as per requirement of the method. In this method, an objective function of the problem under consideration is also considered as constraint. The nature of all inequalities including constraint inequalities and an objective function inequality must be same either ≤ or ≥.

Max. Z = $(a_1\, x + \alpha)(\, a_2\, x + \beta)$ = Max. Z_1 Max. Z_2

$a_1\, x + \alpha$ - $Z_1 \leq 0$

$a_2\, x + \beta$ - $Z_2 \leq 0$

A $x \leq b$

- $x \leq 0$

Numerical Example

1. Solve the SPPP given below:-

Max. Z = $x_1 + x_2{}^4$

$3x_1 + 2\, x_2{}^2 \leq 9$

$x_1, x_2 \geq 0$

The separable functions of the given problem are as follows:-

$$f_1(x_1) = x_1 \, , f_2(x_2) = x_2{}^4$$

$$g_{11}(x_1) = 3\, x_1 \, , \; g_{12}(x_2) = 2x_2{}^4$$

Considering the break points for $f_2(x_2)$ and $g_{12}(x_2)$ only as $f_1(x_1)$ and $g_{11}(x_1)$ are in linear form. The conditions for concavity-convexity in case of maximization problem are satisfied by the above separable functions.

Now, one can easily obtain from the constraint set

$x_1 \leq \frac{9}{3} = 3 \, , x_2 \leq \sqrt{\frac{9}{2}}$ = 2.13 and $x_1, x_2 \geq 0$

$0 \leq x_1 \leq 3, 0 \leq x_2 \leq 3$

The upper and lower limits for the variables x_1 and x_2 are 3 and 0 respectively. The interval [0,3] can be subdivided into 4 equal parts to obtain the breaking points. Now, we can tabulate the breaking points as :-

K	a_{2k}	$f_2(a_{2k}) = x_2{}^4$	$g_{12}(a_{2k}) = 2\, x_2{}^2$
1	0	0	0
2	1	1	2
3	2	16	8
4	3	81	18

From the above tabulation, we have

$$f_2\,(x_2) \cong w_{21}\, f_2(a_{21}) + w_{22}\, f_2(a_{22}) + w_{23}\, f_2(a_{23}) + w_{24}\, f_2(a_{24})$$

$$= w_{21}(\,0) + w_{22}\,(1) + w_{23}\,(16) + w_{24}\,(81)$$

$= w_{22} + 16\, w_{23} + 81\, w_{24}$

$g_{12}\,(x_2) \cong w_{21}\, g_{12}(a_{21}) + w_{22}\, g_{12}(a_{22}) + w_{23}\, g_{12}(a_{23}) + w_{24}\, g_{12}(a_{24})$

$= w_{21}(\,0) + w_{22}\,(2) + w_{23}\,(8) + w_{24}\,(18)$

$= 2\, w_{22} + 8\, w_{23} + 18\, w_{24}$

Thus, reduced linear programming problem is

Max. Z = $x_1 + w_{22} + 16\, w_{23} + 81\, w_{24}$

$3\, x_1 + 2\, w_{22} + 8\, w_{23} + 18\, w_{24} \leq 9$

$w_{21} + w_{22} + w_{23} + w_{24} = 1$

$w_{21}, w_{22}, w_{23}, w_{24} \geq 0$

The additional restrictions must also be noted along with

(a) More than two w_{jk} for each j = 1,2 are positive, and

(b) If two w_{jk} are positive, then they must correspond to adjacent points.

Standard form for applying the proposed method, we have

Max. Z

Z - $x_1 - w_{22} - 16\, w_{23} - 81\, w_{24} \leq 0$

$3x_1 + 2\, w_{22} + 8\, w_{23} + 18\, w_{24} \leq 9$

$w_{21} + w_{22} + w_{23} + w_{24} \leq 1$

$-w_{21} \leq 0$

$-w_{22} \leq 0$

$-w_{23} \leq 0$

$-w_{24} \leq 0$

$-x_1 \leq 0$

After eliminating w_{21}, we get

Max. Z

Z - $x_1 - w_{22} - 16\, w_{23} - 81\, w_{24} \leq 0$

$3x_1 + 2\, w_{22} + 8\, w_{23} + 18\, w_{24} \leq 9$

$w_{22} + w_{23} + w_{24} \leq 1$

$-w_{22} \leq 0$

$-w_{23} \leq 0$

$-w_{24} \leq 0$

$-x_1 \leq 0$

After eliminating w_{23}, we get

Max. Z

Z - $x_1 - 16\, w_{23} - 81\, w_{24} \leq 0$

$3x_1 + 8\, w_{23} + 18\, w_{24} \leq 9$

$w_{23} + w_{24} \leq 1$

$-w_{23} \leq 0$

$-w_{24} \leq 0$

$-x_1 \leq 0$

After eliminating x_1 we get

Max. Z

$Z - 16\, w_{23} - 81\, w_{24} \leq 0$

$8\, w_{23} + 18\, w_{24} \leq 9$

$w_{23} + w_{24} \leq 1$

$-w_{23} \leq 0$

$-w_{24} \leq 0$

After eliminating w_{23}, we get

Max. Z

$Z - 65\, w_{24} \leq 16$

$10\, w_{24} \leq 1$

$-w_{24} \leq 1$

$-w_{24} \leq 0$

Now, it can be rewritten as

Max. Z

$Z - 65\, w_{24} \ \leq 16$

$w_{24} \ \leq \ \frac{1}{10}$

$-w_{24} \ \leq 0$

After eliminating w_{24}, we get

Max. Z

$Z \leq \ \frac{45}{2} = 22.5$

$0 \leq \ \frac{1}{10}$

It indicates that $Z = \frac{45}{2}$. By using the back substitution method and by putting $Z = \frac{45}{2}$ into the above inequalities we get $w_{24} = \frac{1}{10}$, $w_{23} = \frac{9}{10}$, $w_{22} = x_1 = w_{21} = 0$. Hence, the solution of above linear programming problem is $Z = \frac{45}{2}$, $w_{24} = \frac{1}{10}$, $w_{23} = \frac{9}{10}$, $w_{22} = x_1 = w_{21} = 0$.

Now, an optimal solution of original SPP is as follows:-

$x_2 = 2\, w_{23} + 3\, w_{24} = 2\left(\frac{9}{10}\right) + 3\left(\frac{1}{10}\right) = \frac{21}{10} = 2.1$ and $x_1 = 0$

Hence, an optimal solution of original SPP is

$x_1 = 0, x_2 = 2.1$

Max. $Z = x_1 + x_2{}^4 = (0) + (2.1)^4 = 19.45$

2. Solve the QPP given below:-

Max. $Z = (2x_1+3x_2+2)\,(x_2-5)$

Subject to $x_1+x_2+ \leq 1$

$4x_1+x_2+ \geq 2$

and $x_1, x_2 \geq 0$

Two linear functions are involved in the objective function of QPP given under consideration. We can assume them as Z_1 and Z_2.

Standard form for applying the proposed method, we have

Max z = $Z_1\ \ Z_2$

$2x_1 + 3x_2 - Z_1 \leq -2$

$0x_1 + x_2 - Z_2 \leq 5$

$x_1 + x_2 \leq 1$

$-4x_1 - x_2 \leq -2$

$-x_1 \leq 0$

$-x_2 \leq 0$

After eliminating x_1, we get

Max z = $Z_1\ \ Z_2$

$x_2 - Z_2 \leq 5$

$- x_2 + Z_1 \leq 4$

$5\ x_2 - 2Z_1 \leq -6$

$3\ x_2 - Z_1 \leq -2$

$- x_2 \leq 0$

After eliminating x_2, we get

$-2Z_1 + 5\ Z_2 \leq -31$

$-Z_1 + 3\ Z_2 \leq 29$

$Z_2 \geq -5$

$Z_1 \geq 3$

One can easily obtain the values of Z_1 and Z_2 as $Z_1 = 3$ and $Z_2 = -5$. By using the process of back substitution, one can get the various bounded values for the variable x_2. Out of these values, $x_2 = 0$ is an only value that satisfies all the inequalities simultaneously. After that, proceeding on the similar fashion, one can get the various bounded values for the variable x_1. Out of these values, $x_1 = \frac{1}{2}$ is an only value that satisfies all the inequalities simultaneously.

Hence, the solution of above QPP is as follows : x_1 = ½ , x_2 = 0, Z_1 = 3, Z_2 = -5 and z = -15.

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