**Modeling of Gauss Elimination Method for Non-Linear Programming Problem**

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**Modeling of Gauss Elimination Method for Non-Linear Programming Problem**

Decision making play a vital role in any administration, politics, industry, business, education and many more sectors. Most favorable decision is to be adopted by the decision maker in the process of decision making. Mathematical Programming is the mathematical technique used for taking the most favorable (optimal) decisions in case of many problems involved in business, industry, administration, politics, education etc. Mathematical Programming Problem (MPP) or an Optimization Problem is defined as a problem which seeks to minimize or maximize a numerical function of one or more variables when the variables can be independent or related in some way through the specification of certain constraints. The function which is to be maximized or minimized is called objective function or cost function. In particular, when the objective function and constraints both are linear, the problem is called a Linear Programming Problem (LPP). All programming problems that are not linear are called non-linear i.e. a programming problem in which either the objective function or the constraints (some or all) or both are non-linear, is called a Non-Linear Programming Problem (NLPP).

Mathematical Programming is an important branch of Optimization Theory. In mathematical programming, One has to optimized (maximized / minimized) a real valued objective function f of n real variables $x\_{1},x\_{2},……,x\_{n}$ associated with a finite number of constraints, which are generally written as inequalities or equations.

Generally we can define a mathematical program such as:

Min. $f(x)$

Subject to $g\_{i}\left(x\right)\geq 0$, i = 1,2,……,m

$h\_{j}\left(x\right)= 0$, j = 1,2,……,p

Where $x$ denotes the column vector whose components are $x\_{1},x\_{2},……,x\_{n}$. In various disciplines such as Economics, Business Administration, Mathematics, Engineering and Physical Sciences or in any other area where decisions must be taken in some complex ( or conflicting situation) that can be represented by a mathematical model; one has to deal with Non-Linear Programming Problem (NLPP).

The purpose of the chapter is to describe a brief taxonomy of an important class of programming problems which encounter in some branches of applied mathematics. In this chapter, we are focusing on Non-Linear Separable Programming Problems and Quadratic Programming Problems.

**Separable Programming Problem**

One of the indirect methods used to solve a non-linear programming problem is separable programming. A non-linear programming problem can be solved by indirect methods in which we deal one or more linear problems that are extracted from the original problem. If an objective function and constraints are separable in any non-linear programming problem then separable programming may be very powerful technique to solve such type of problems. Sometimes, functions that are not separable can be made separable by using simplified approximation. Such approximation reduces the single variable non-linear function into piece-wise linear functions.

**Quadratic Programming Problem**

Quadratic Programming Problem (QPP) is one of an important class of Non-Linear Programming and used to solve the problems of optimizing a quadratic objective function of the form (CX+α)(C’X+β).

**Gauss elimination technique for simultaneous linear algebraic equations**

In this method, the given system of simultaneous linear algebraic equations can be reduced to an upper triangular system by eliminating the variables successively and the knowns are found by back substitution. For the sake of clarity and simplicity, we consider only the system of three equations. Let the system be

$$a\_{11}x\_{1}+ a\_{12}x\_{2}+ a\_{13}x\_{3}= b\_{1}$$

$$a\_{21}x\_{1}+ a\_{22}x\_{2}+ a\_{23}x\_{3}= b\_{2}$$

$$a\_{31}x\_{1}+ a\_{32}x\_{2}+ a\_{33}x\_{3}= b\_{3}$$

To eliminate $x\_{1}$ from the second equation, multiply the first equation by - $\frac{a\_{21}}{a\_{11}}$ and add to the second equation. Similarly to eliminate $x\_{1}$ from the third equation, multiply the first equation by - $\frac{a\_{31}}{a\_{11}}$ and add to the third equation. These elements - $\frac{a\_{21}}{a\_{11}}$ and - $\frac{a\_{31}}{a\_{11}}$ are called the multipliers for the first stage of elimination, assuming $a\_{11}\ne 0$.

The first equation is called the pivotal equation and $a\_{11}\ne 0$ is called the first pivot.

Therefore the second and third equations become

 $a\_{22}^{(2)}$ $x\_{2}$ + $a\_{23}^{(2)}$ $x\_{3}$ = $b\_{2}^{(2)}$

 $a\_{32}^{(2)}$ $x\_{2}$ + $a\_{33}^{(2)}$ $x\_{3}$ = $b\_{3}^{(2)}$

where $a\_{22}^{(2)}$ = $a\_{22}$ - $\frac{a\_{21}}{a\_{11}}$ $a\_{12}$

 $a\_{23}^{(2)}$ = $a\_{23}$ - $\frac{a\_{21}}{a\_{11}}$ $a\_{13}$

 $a\_{32}^{(2)}$ = $a\_{32}$ - $\frac{a\_{31}}{a\_{11}}$ $a\_{12}$

 $a\_{33}^{(2)}$ = $a\_{33}$ - $\frac{a\_{31}}{a\_{11}}$ $a\_{13}$

$ b\_{2}^{(2)}$= $b\_{2}$ - $\frac{a\_{21}}{a\_{11}}$ $b\_{1}$

$ b\_{3}^{(2)}$= $b\_{3}$ - $\frac{a\_{31}}{a\_{11}}$ $b\_{1}$

In the second stage of elimination, we multiply the fourth equation by - $\frac{a\_{33}^{(2)} }{a\_{22}^{(2)}}$ and adding to the fifth equation. We have $a\_{33}^{(3)}$ $x\_{3}$ = $b\_{3}^{(3)}$

 where $a\_{33}^{(3)}$ = $a\_{33}^{(2)}$ - $\frac{a\_{33}^{(2)}}{a\_{22}^{(2)}}$ $a\_{23}^{(2)}$

 and $b\_{3}^{(3)}$ = $b\_{3}^{(2)}$ - $\frac{a\_{32}^{(2)}}{a\_{22}^{(2)}}$ $b\_{2}^{(2)}$

Now collecting the first equation from each stage, we get the system :

 $a\_{11}x\_{1}+ a\_{12}x\_{2}+ a\_{13}x\_{3}= b\_{1}$

 $a\_{22}^{(2)}$ $x\_{2}$ + $a\_{23}^{(2)}$ $x\_{3}$ = $b\_{2}^{(2)}$ ……………………………….(A)

 $a\_{33}^{(3)}$ $x\_{3}$ = $b\_{3}^{(3)}$

The above system (A) is known as upper triangular system and can be solved by using back substitution method.

**Gauss elimination technique for inequalities**

In Numerical Analysis, the system of simultaneous equations is solved by Gauss elimination technique with the help of elimination of variables one by one and finally reduce to upper triangular system of equations, which can be solved by back substitution. Equation gives only one solution while inequality gives possibility of many solutions in bounded/ region form, out of which we select maximum or minimum value according to the problem to optimize. This is the main theme to apply Gauss elimination technique for inequality in place of equation. It can easily verify that max. / mini. Value of linear variables gives, max. / mini. Value of objective function of $\sum\_{}^{}C\_{i}X\_{i}$ form where all $C\_{i}$ are positive. If some $C\_{i}$ are not positive, then we take mini. / max. value of corresponding linear variables so it gives max. / mini. Value of objective function.

Here we apply Gauss elimination technique for a system of inequalities of the same sign i.e., either less or equal to ($\leq )$ or greater than or equal to ($\geq )$ in nature. Here variables are eliminated by combining inequalities in such a way that the inequalities and variables reduced one by one in every iteration i.e., one variable and one inequality reduce in one iteration so at last there remains only one inequality with one variable remains. This last inequality gives value of last variable in bounded form and finally taking the maximum or minimum value of last variable according to objective function of Mathematical Programming Problem. Finally, we get value of other variables by back substitution of value of the last variable.

For the sake of clarity and simplicity, we consider the system of inequalities of n variables and m inequalities:

$$a\_{11}x\_{1}+ ………. + a\_{1n}x\_{n} \leq b\_{1}$$

$$a\_{21}x\_{1}+ ………. + a\_{2n}x\_{n} \leq b\_{2}$$

 ……………………………………

 ……………………………………

 $a\_{m1}x\_{1}+ ………. + a\_{mn}x\_{n} \leq b\_{m}$

First of all to eliminate the first variable say $x\_{1}$, multiply the first row by $\frac{a\_{21}}{a\_{11}} , \frac{a\_{31}}{a\_{11}} ,………., \frac{a\_{m1}}{a\_{11}}$ respectively and then subtract them from second, third and so on up to the $m^{th}$ row respectively. Then we get the first iteration as:

 $a\_{22}^{(2)} x\_{2}+ ……….. + a\_{2n}^{(2)} x\_{n} \leq b\_{2}^{(2)} $

 $a\_{32}^{(2)} x\_{2}+ ……….. + a\_{3n}^{(2)} x\_{n} \leq b\_{3}^{(2)} $

 …………………………………………………

 …………………………………………………

 $a\_{m2}^{(2)} x\_{2}+ ……….. + a\_{mn}^{(2)} x\_{n} \leq b\_{m}^{(2)} $

 where $a\_{m2}^{(2)}$ = $a\_{mn}$ - $\frac{a\_{m1}a\_{1n}}{a\_{11}}$

Now after first iteration, we get (n-1) variables and (m-1) inequalities. Repeating this process or after (n-1) iteration we have only one variable remains. Finally, we can get the value of last variable. There may be some redundant constraints present in the system.

**Algorithm of Gauss elimination technique for Mathematical Programming Problem**

In this chapter, main emphasis is given on several programming problems viz. Linear Programming problem, Fractional Programming Problem, Extended Integer Solution for Fractional Programming Problem, Multi-objective Linear Programming Problem and Multi-objective Fractional Programming Problem to find an optimal solution by Gauss elimination technique. The steps to apply Gauss elimination technique on MPP are as follows:

1. To apply Gauss elimination technique on Mathematical Programming Problem (MPP), we have to formulate this MPP again by taking objective function as constraints and all constraints of having same sign of inequality.
2. Now variables eliminated by combining inequalities in such a way that the inequalities and variables reduced one by one in each iteration. If at any stage we get an absurd inequality like 0 $\leq d$ where $d$ is a negative number then we conclude that the given MPP has an infeasible solution otherwise MPP has a feasible solution.
3. In case of having feasible solution, we form the following pairwise disjoint sets for all the variables like,

 $l\_{j}^{+}$ = $\left\{i : A\_{i1} >0\}\right.$

 $l\_{j}^{-}$ = $\left\{i : A\_{i1} <0\}\right.$

 $l\_{j}^{0}$ = $\left\{i : A\_{i1} =0\}\right.$ ; i = 1,2, …….., m and j = 1,2,…..,n

1. If at any stage the set $l\_{j}^{+}$ or $l\_{j}^{-}$ becomes empty for a variable, then the given MPP possess an unbounded solution. Hence, optimal / feasible solution of MPP exists when neither absurd inequality nor empty set of $l\_{j}^{+}$ or $l\_{j}^{-}$ exist in the system.

**Problem Formulation for Various Mathematical Programming Problems**

**Separable Non-linear Programming Problem**

Let us consider the problem

Max. (or Min.) $f\_{j}$ ($x\_{1},x\_{2},………..,x\_{n}$)

 $g\_{ij}$($x\_{1},x\_{2},………..,x\_{n}$) ≤ $b\_{i}$

$$ x\_{j}\geq 0$$

Above non-linear programming problem can be written in the following form if the objective function and constraints are separable:

$ f\_{j}$ ($x\_{1},x\_{2},………..,x\_{n}$) = $\sum\_{j=1}^{n}f\_{j}$($x\_{j}$)

 $g\_{ij}$($x\_{1},x\_{2},………..,x\_{n}$) = $\sum\_{j=1}^{n}g\_{ij}$($x\_{j}$)

It can also be rewritten as

Max. (or Min.) $\sum\_{j=1}^{n}f\_{j}$($x\_{j}$)

 $\sum\_{j=1}^{n}g\_{ij}$($x\_{j}$) ≤ $b\_{i}$

$$ x\_{j}\geq 0$$

Where some or all $g\_{ij} ,x\_{ij} ,f(x\_{j})$ are non-linear. Now, we can approximate $g\_{ij}$ (the sub function) through a set of arbitrary break points. Let $K\_{j}$ be the breaking points in number and $a\_{jk} $be its $k^{th}$ breking value for the $j^{th}$ variable. Also, let the weight be $w\_{jk}$ associated with the $k^{th }$ breaking point of $j^{th}$ variable.

The reduced separable non-linear programming problem can be rewritten as

Max. (or Min.) $\sum\_{j=1}^{n}\sum\_{k=1}^{K\_{j}}f\_{j}(a\_{jk})w\_{jk}$

$\sum\_{j=1}^{n}\sum\_{k=1}^{K\_{j}}g\_{ij}(a\_{jk})w\_{jk}$ $\leq $ $b\_{i}$

0 $\leq $ $w\_{j1} \leq y\_{j1}$

0$\leq w\_{jk} \leq y\_{j,k-1}+ y\_{jk}$

0$\leq w\_{jK\_{j}} \leq y\_{j,K\_{j}-1}$

$\sum\_{k=1}^{K\_{j}}w\_{jk}$ = 1, $\sum\_{k=1}^{K\_{j}-1}y\_{jk}$ = 1

 $y\_{jk= }$ 0 or 1

The technique of break points transforms the separable non-linear programming problem into a linear programming problem. The decision variables for this approximating problem are considered as $w\_{jk}$ and $y\_{jk}.$

**Quadratic Programming Problem**

Here we consider the Quadratic Programming Problem as

Max. Z = $(a\_{1} x+α)$($ a\_{2} x+β)$

Subject to A$ x\leq b$

and $x \geq 0$

where it is assumed that $\left(a\_{1} x+α\right),$($ a\_{2} x+β)$ are positive for all feasible solutions. The set comprises all the feasible solutions is bounded closed convex polyhedron.

To apply Gauss elimination technique, we formulate this QPP by taking objective function as constraints and all constraints of the same sign of inequality. So reduced form of QPP for Gauss elimination technique is as follows:

Max. Z = $(a\_{1} x+α)$($ a\_{2} x+β)$ = Max. $Z\_{1}$ Max. $Z\_{2}$

$a\_{1} x+α $ - $Z\_{1}$ $\leq 0$

$a\_{2} x+β$ - $Z\_{2}$ $\leq 0$

A$ x\leq b$

- $ x\leq 0$

# Numerical Example

1. Consider the separable non-linear programming problem

Max. Z = $x\_{1}+ x\_{2}^{4}$

3$x\_{1}+ 2 x\_{2}^{2}$ $\leq $ 9

$x\_{1},x\_{2}\geq 0$

Now, we have to collect the separable functions of the given problem

$f\_{1}\left(x\_{1}\right)= x\_{1}$ , $f\_{2}\left(x\_{2}\right)= x\_{2}^{4}$

$g\_{11}\left(x\_{1}\right)= 3 x\_{1}$ , $g\_{12}\left(x\_{2}\right)= 2x\_{2}^{4}$

Considering the break points for $f\_{2}\left(x\_{2}\right)$ and $g\_{12}\left(x\_{2}\right)$ only as $f\_{1}\left(x\_{1}\right)$ and $g\_{11}\left(x\_{1}\right)$ are in linear form. It can be observed that the above separable functions satisfy the concavity-convexity conditions for the maximization problem.

Now, it can be observed from the constraint set that

$x\_{1} \leq \frac{9}{3}=3 , x\_{2} \leq \sqrt{\frac{9}{2}}$ = 2.13 and $x\_{1},x\_{2}\geq 0$

0 $\leq x\_{1} \leq 3$, $0 \leq x\_{2} \leq 3$

Since the upper and lower limits for the variables $x\_{1}$ and $x\_{2}$ are 3 and 0 respectively, therefore, we can divide the interval [0,3] into four equal parts for breaking points.

Now, we can tabulate the breaking points as

|  |  |  |  |
| --- | --- | --- | --- |
| K | $$a\_{2k}$$ | $$f\_{2}\left(a\_{2k}\right)= x\_{2}^{4}$$ | $$g\_{12}\left(a\_{2k}\right)=2 x\_{2}^{2}$$ |
| 1 | 0 | 0 | 0 |
| 2 | 1 | 1 | 2 |
| 3 | 2 | 16 | 8 |
| 4 | 3 | 81 | 18 |

From the above tabulation, we have

$$f\_{2} \left(x\_{2}\right)≅ w\_{21} f\_{2}\left(a\_{21}\right)+ w\_{22} f\_{2}\left(a\_{22}\right)+ w\_{23} f\_{2}\left(a\_{23}\right)+ w\_{24} f\_{2}(a\_{24})$$

 = $w\_{21}\left( 0\right)+ w\_{22} \left(1\right)+ w\_{23} \left(16\right)+ w\_{24} (81)$

 = $w\_{22}+ 16 w\_{23}+81 w\_{24}$

 $g\_{12} \left(x\_{2}\right)$ $≅ w\_{21} g\_{12}\left(a\_{21}\right)+ w\_{22} g\_{12}\left(a\_{22}\right)+ w\_{23} g\_{12}\left(a\_{23}\right)+ w\_{24} g\_{12}(a\_{24})$

 = $w\_{21}\left( 0\right)+ w\_{22} \left(2\right)+ w\_{23} \left(8\right)+ w\_{24} (18)$

= 2 $w\_{22}+ 8 w\_{23}+ 18 w\_{24}$

Thus, the reduced linear programming problem is

Max. Z = $x\_{1}+ w\_{22}+ 16 w\_{23}+ 81 w\_{24}$

3 $x\_{1}+ 2 w\_{22}+ 8 w\_{23}+ 18 w\_{24} \leq 9$

$w\_{21}+ w\_{22}+ w\_{23}+ w\_{24}=1$

$w\_{21}, w\_{22}, w\_{23}, w\_{24}\geq 0$

The additional restrictions must also be noted along with

1. More than two $w\_{jk}$ for each j = 1,2 are positive, and
2. If two $w\_{jk}$ are positive, then they must correspond to adjacent points.

Making standard form by treating objective function as constraints and all inequalities of same sign for Gauss elimination technique, we have

Max. Z

Z - $x\_{1}- w\_{22}- 16 w\_{23}- 81 w\_{24} \leq 0$

$$3x\_{1}+2 w\_{22}+8 w\_{23}+18 w\_{24} \leq 9$$

$$w\_{21}+ w\_{22}+ w\_{23}+ w\_{24}\leq 1$$

$$-w\_{21} \leq 0$$

$$-w\_{22} \leq 0$$

$$-w\_{23} \leq 0$$

$$-w\_{24} \leq 0$$

$$-x\_{1}\leq 0$$

In the first stage of Gauss elimination technique, eliminating $w\_{21}$ we have,

Max. Z

Z - $x\_{1}- w\_{22}- 16 w\_{23}- 81 w\_{24} \leq 0$

$$3x\_{1}+2 w\_{22}+8 w\_{23}+18 w\_{24} \leq 9$$

$$ w\_{22}+ w\_{23}+ w\_{24}\leq 1$$

$$-w\_{22} \leq 0$$

$$-w\_{23} \leq 0$$

$$-w\_{24} \leq 0$$

$$-x\_{1}\leq 0$$

After eliminating $w\_{23}$ by using Gauss elimination technique in the second stage, we have

Max. Z

Z - $x\_{1}- 16 w\_{23}- 81 w\_{24} \leq 0$

$$3x\_{1}+8 w\_{23}+18 w\_{24} \leq 9$$

$$ w\_{23}+ w\_{24}\leq 1$$

$$-w\_{23} \leq 0$$

$$-w\_{24} \leq 0$$

$$-x\_{1}\leq 0$$

After eliminating $x\_{1}$ by using Gauss elimination technique in the third stage, we have

Max. Z

Z $- 16 w\_{23}- 81 w\_{24} \leq 0$

$$8 w\_{23}+18 w\_{24} \leq 9$$

$$ w\_{23}+ w\_{24}\leq 1$$

$$-w\_{23} \leq 0$$

$$-w\_{24} \leq 0$$

After eliminating $w\_{23}$ by using Gauss elimination technique in the fourth stage, we have

Max. Z

Z $- 65 w\_{24} \leq 16$

$$10 w\_{24} \leq 1$$

$$-w\_{24} \leq 1$$

$$-w\_{24} \leq 0$$

Now, it can be rewritten as

Max. Z

Z $- 65 w\_{24} \leq 16$

$w\_{24} \leq \frac{1}{10}$

$$-w\_{24} \leq 0$$

After eliminating $w\_{24}$ by using Gauss elimination technique in the fifth stage, we have

Max. Z

Z $\leq \frac{45}{2}=22.5$

0 $\leq \frac{1}{10}$

It indicates that Z = $\frac{45}{2}$ . By using the back substitution method and by putting Z = $\frac{45}{2}$ into the above inequalities we get $w\_{24}= \frac{1}{10}$ , $w\_{23}= \frac{9}{10}$ , $w\_{22}= x\_{1}= w\_{21}=0.$ Hence the solution of above linear programming problem by Gauss elimination technique is Z = $\frac{45}{2}$ , $w\_{24}= \frac{1}{10}$ , $w\_{23}= \frac{9}{10}$ , $w\_{22}= x\_{1}= w\_{21}=0$

Now, we have to find an optimal solution of original Separable Non-Linear Programming Problem . We have,

$x\_{2}=2 w\_{23}+ 3 w\_{24}=2 \left(\frac{9}{10} \right)+ 3 \left(\frac{1}{10}\right)= \frac{21}{10}=2.1$ and $x\_{1}=0$

Hence, an optimal solution of original separable non-linear programming problem is

$x\_{1}=0$, $x\_{2}=2.1$

Max. Z = $x\_{1}+ x\_{2}^{4}$ = (0) + $(2.1)^{4}$ = 19.45

1. Consider the Quadratic Programming Problem

Max. Z = (2$x\_{1}$+3$x\_{2}$+2) ($x\_{2}$-5)

 Subject to $x\_{1}$+$x\_{2}$+ ≤ 1

 4$x\_{1}$+$x\_{2}$+ ≥ 2

and $x\_{1}$, $x\_{2}$ $\geq 0$

Here, we assume both the linear functions involved in objective function as a constraint, say $Z\_{1}$ and$ Z\_{2}$.

Making standard form by treating objective function as constraints and all inequalities of same sign for Gauss elimination technique, we have

Max z = $Z\_{1}$ $ Z\_{2}$

2$x\_{1}$ + 3$x\_{2}$ -$Z\_{1}$ ≤ -2

 0$x\_{1}$ +$ x\_{2}$- $ Z\_{2}$ ≤ 5

 $x\_{1}$+$ x\_{2}$ ≤ 1

 -4$x\_{1}$ -$ x\_{2}$ ≤ -2

 -$x\_{1}$ ≤ 0

 -$x\_{2}$ ≤ 0

After first stage of elimination, we have

Max z = $Z\_{1}$ $ Z\_{2}$

$ x\_{2}$ – $ Z\_{2}$ ≤ 5

 -$ x\_{2}$ + $Z\_{1}$ ≤ 4

 5 $ x\_{2}$ – 2$Z\_{1}$ ≤ -6

 3$ x\_{2}$ – $Z\_{1}$ ≤ -2

 -$ x\_{2}$≤ 0

After second stage of elimination, we have

-2$Z\_{1}$ + 5$ Z\_{2}$ ≤ -31

 -$Z\_{1}$ + 3$ Z\_{2}$ ≤ 29

$ Z\_{2}$≥ -5

 $Z\_{1}$ ≥ 3

We have some values of $Z\_{1}$ and $ Z\_{2}$ but our object is to maximize $Z\_{1}$ and $ Z\_{2}$. Values of $Z\_{1}$ and $ Z\_{2}$from the above inequalities, we have $Z\_{1}$ = 3 and $ Z\_{2}$ = -5 Now putting $Z\_{1}$ = 3 and $ Z\_{2}$ = -5 in the inequalities , we get different bounded values for variable $ x\_{2}$. Out of this, $ x\_{2}$ = 0 is the only value that satisfies all the inequalities simultaneously. Therefore $ x\_{2}$ = 0. Now putting $Z\_{1}$ = 3, $ Z\_{2}$ = -5 and $ x\_{2}$ =0 in the inequalities , we get different bounded values for variable $x\_{1}$. Out of this$ x\_{1} $= ½ is the only value that satisfies all the inequalities simultaneously. Therefore $x\_{1}$ = ½ . Hence, the solution of above Quadratic programming Problem is as follows : $x\_{1}$ = ½ ,$ x\_{2}$ = 0, $Z\_{1}$ = 3, $ Z\_{2}$ = -5 and z = -15.

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