**Development & Use of Progressive Techniques in Mathematics Education**

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**Abstract :**

 In this universe, the need to measure and quantify everything is executable in mathematics, from the mass of one atom and subatomic particles to the celestial objects in galaxies; from the undercurrent of water-flows below the earth-surface to the unpredictable downpours of rains; from the lake water-bodies of irregular depths to the water-currents of uneven rivers; from the hourly labour-rate to gross-national-product in economies; from the weight of microbes to the measure of mountains; from the intercellular-space between two human cells to the measure of galactic spaces in far skies; from the volts of electricity generated by one human cell to the very large electrostatic discharges through the atmosphere; from a candle-light intensity at a distance to Solar radiometry;

 The 3rd International Congress on Mathematical Education stressed to focus mainly on the need to develop a critical perception of the “Objectives of Education” finding that in every civilization the important goals of education were always: promoting creativity, helping people to fulfil their potential and rise to the highest of their capability, to promote citizenship, transmitting values, and showing rights and responsibilities in society, to cooperate in building a civilization in peace, which is free of inequity, arrogance and bigotry which gives opportunity to every individual to reach the full realization of its capabilities ; and to strategize education-system to pursue goals through the Curriculum, organized in three strands: Objectives, Contents, and Methods.

 Mathematics and its Education has proved to bring: Individual-Development, Accelerated Brain-Functions, Logical-Reasoning, Deep-Analytical-Ability, which prepare human brain to function logically and habitually even in the day-to-day real-world situations and complexities.

 Mathematics, always teaches a set order of operations, develops orderly planning and sequential processing capabilities in students, who get trained to think in a determinable yet improvable pattern. Practice and repeated maths exams increase the efficiency of their actions in life as well. Maths education also augments selective-attention and working-memory capacity in students. Attending minute details is a skill of utmost importance in Maths.

 All computation and automation processes are based on mathematical concepts, especially the ‘Binary Mathematics’ which facilitates tele-communication, transport, construction, banking, internet, entertainment, graphics, medical developments, to navigate satellites, analyse data, invent robotics and artificial-intelligence. It helps in predicting weathers, epidemics, population dynamics, glacial meltdown, and ocean decline. Maths is present in each fraction of advanced technology, from carbon dating to crime detection. The world is dynamic and innovative, hence the need for Mathematics Education is imperative.

**Key-Words: Accelerated Brain-Functions, Logical-Reasoning, Deep-Analytical-Ability, Mathematical Logic, Modeling, Mathematical Abstractions, Deep-Stream mathematics students, Main-Stream mathematics students, Interactive-Whiteboards; Virtual-Manipulatives, Computer Algebra Systems, Augmented-Reality, Virtual-Reality, Gamified-Learning, Data Visualization Tools, Collaborative Problem Solving.**

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**1.0 Importance of Mathematics & Mathematics Education to the World –**

 In this universe, the need to measure and quantify everything is executable in mathematics, from the mass of one atom and subatomic particles to the celestial objects in galaxies; from the undercurrent of water-flows below the earth-surface to the unpredictable downpours of rains; from the lake water-bodies of irregular depths to the water-currents of uneven rivers; from the hourly labour-rate to gross-national-product in economies; from the weight of microbes to the measure of mountains; from the intercellular-space between two human cells to the measure of galactic spaces in far skies; from the volts of electricity generated by one human cell to the very large electrostatic discharges through the atmosphere; from a candle-light intensity at a distance to Solar radiometry;

 The 3rd International Congress on Mathematical Education (ICME-3,1976) held in Germany observed in its session on ‘**Objectives & Goals of Mathematics Education: Why Study Mathematics’** focused mainly on the need to develop a critical perception of the objectives of education through history in different civilizations. This was based on finding that in every civilization of all the times, from initiation practices through complex education systems, the major goals of education were always:

**–** to promote creativity, helping people to fulfil their potential and rise to the highest of their capability, but being careful not to promote docile citizens. Students not to become citizens who obey and accept rules and codes which violate human dignity ;

**–** to promote citizenship, transmitting values, and showing rights and responsibilities in society, but being careful not to promote irresponsible creativity, e.g., students not to become bright scientists creating new weaponry and instruments of oppression and inequity ;

**–** to cooperate in building-up a civilization in peace, which is free of inequity, arrogance and bigotry which gives opportunity to every individual to reach the full realization of its capabilities ; and

**–** to strategize education-system to pursue goals through the Curriculum (organized in three strands: Objectives (why), Contents (what), and Methods (how)) accepting the three components in solidarity, similar to a Cartesian-Model of Curriculum. This model implies accepting the social-aims of education systems, then identifying contents that may help to reach the goals and developing methods to transmit those contents where the contents are dictated by the inner structure of mathematics, giving origin to methods, and subordinate the vague social-aims to the achievement of the contents.

 Mathematics and its Education has proved to bring: **Individual-Development** (due to advanced cognitive skills); **Accelerated Brain-Functions** (due to efficient maths teaching, learning and consistent practice) ; **Logical-Reasoning** (due to necessity of critical and analytical thinking) ; **Deep-Analytical-Ability** (due to mathematical exercises to solve questions after deep analysis, collection, differentiation and combination of data) which processes prepare human brain to function logically and habitually even in the day-to-day real-world situations and complexities.

 Mathematics and its Education also help in **Problem-Solving** (by make-learning in ascertaining connections among real-world objects, their cause-and-effect relations, recognize patterns, leading to deriving logical conclusions, helping to analyze across different alternatives and scenarios to determine the optimum and make the right decision with the preparedness of logic, techniques and tools of critical thinking). While facing life-issues, mathematics always make learners look for solutions with a critical yet creative and innovative eye, to weigh the evidences from multiple perspectives and with practicality instead of imagination and inclination, thus, making ardent learners to become proficient problem-solvers.

 Mathematics, always teaches a set order of operations, develops orderly planning and sequential processing capabilities in students, who get trained to think in a determinable yet improvable pattern. Practice and repeated maths exams increase the efficiency of their actions in life as well.

 Mathematics education also augments selective-attention and working-memory capacity in students. Attending minute details is a skill of utmost importance in Maths. One cannot expect accuracy if a single decimal is displaced.

 Similarly, remembering each digit, formula and sequence to follow involves continuous use of working-memory. Thus, repetitive maths practice enriches students’ everyday life with enhanced attention and memory. Maths education also increases the Visual-Spatial Intelligence of students as with the complexity of maths increases, the use of visualization and spatial processes also rise. Geometrical applications calls for stressed imagination and visualisation abilities as Geometry boosts students’ capacity to transform and understand the mathematical concepts visually.

 Far outside the maths classes, **Life-skills** Maths is used in the most mundane day-to-day tasks, making students habitual in applying mathematical principles and concepts normally, without wilful effort in daily life-activities with enhanced reasoning and analytical abilities that results in enhancing the efficiency and effectiveness of the daily life-skill based routine transactions. **Life-skills** Maths prove to be more productive, remunerative, and providing financial-security when used for regulating all day-to-day financial transactions and deciding on investments in properties and financial products irrespective of career & profession one serves in.

Mathematics, especially the ‘Binary Mathematics’, is the heart of technology. Computation and automation processes are based on mathematical concepts. It is the foundation of the modern world. It facilitates tele-communication, transport, construction, banking, internet, entertainment, graphics, medical developments, to navigate satellites, analyse data, invent robotics and artificial-intelligence. It helps in predicting weathers, epidemics, population dynamics, glacial meltdown, and ocean decline. Maths is present in each fraction of advanced technology, from carbon dating to crime detection. The world is dynamic and innovative, hence the need for Mathematics Education is imperative. Hence, the ICME question, “Why Study Mathematics?” flashes through justifications in abundance.

 In ‘[Asia Society International Studies Schools](https://asiasociety.org/issn)’ (https://asiasociety.org/education/understanding-world-through-math), all high school graduates are expected to demonstrate a mastery of mathematics. Students work on skills and projects throughout their secondary education. At graduation, students have a portfolio of work that includes evidence of:

**Global Connections**

* Use of mathematics to model situations or events in the world;
* Explanations of how the complexity and interrelatedness of situations or events in the world are reflected in the model;
* Data generated by the model to make and defend a decision; and
* A decision or conclusion supported by the mathematics within the context of a global community.

**Problem Solving**

* The application of appropriate strategies to solve problems;
* The use of appropriate mathematical tools, procedures, and representations to solve the problem;
* The review and proof of a correct and reasonable mathematical solution given the context.

**Communication**

* The development, explanation, and justification of mathematical arguments, including concepts and procedures used;
* Coherently and clear communication using correct mathematical language and visual representations;
* The expression of mathematical ideas using the symbols and conventions of mathematics.

**2.0 Evolution of Mathematics –**

 The history of progressive and significant advancements made in mathematics during different historical segments: prehistoric, Sumerian and Babylonian, Egyptian, Greek, Hellenistic, Roman, Mayan, Chinese, Islamic, Medieval European, 16th Century, 17th Century, 18th Century, 19th Century, and 20th Century are highlighted below chronologically –

 The earliest evidence of human thinking was of ‘counting’ in the form of Ishango bone, dated back to 35,000 to 20,000 years ago.

 The Sumerian and Babylonian mathematics was based on a sexegesimal (Base-60) numeric system. In Sumer multiplication and division tables, tables of squares, square-roots, cube-roots, geometrical exercises and division problems were used from around 2600 BCE. Later Babylonian tablets (1800–1600 BCE) covered topics like, fractions, algebra, methods for solving linear, quadratic and even some cubic equations, and the calculation of regular reciprocal pairs.

 The Egyptians introduced the earliest fully-developed ‘Base-10’ numeration system (2700 BCE). The Rhind Papyrus (1650 BCE) was an explicit demonstration of how multiplication and division were carried out at that time. The Berlin Papyrus (1300 BCE) solved 2nd order algebraic (quadratic) equations. The pyramids are examples of sophistication of Egyptian mathematics as the structures of pyramid were built on the golden ratio of 1: 1.618.

 The ancient Greek numeral system was in regular use by 7th Century BCE. Greek mathematician, ‘Thales’ is known for his  Thales’ Theorem. Pythagoras’ Theorem is one of the best known of all mathematical theorems. Hippocrates’s (5th Century BCE) book “The Elements” was the first compilation on Geometry.

 The Hellenistic influential mathematicians (4th–3rd Century BCE) included Euclid (known for Euclidean Geometry),  Archimedes (discoverer of the Principle of Buoyancy), Eratosthenes (who measured Earth's circumference mathematically), Heron (inventor of Heron’s Fountain), Menelaus (famous for his Menelaus's theorem on spherical triangle), Diophantus (known as the Father of Algebra), Apollonius (derived ellipse, parabola, and hyperbola from a cone), and Hipparchus (who measured Earth-Moon distance using the properties of triangles).

 In Roman empire, no mathematicians of historical note developed, except Vitruvius, and Marcus Terentius Varro.

 The Mayan and other Mesoamerican cultures (250 – 900 CE) used a sophisticated number system on Base-20. They did advanced astronomical calculations, and measured Solar-Year (365.242 days), compared to the modern value of 365.242198 days.), and Lunar-Month (29.5308 days) compared to the modern value of 29.53059 days.

 The Chinese mathematicians developed (2nd millennium BCE) an efficient Number-System using units, tens, hundreds, thousands, and a decimal place-value system. They developed the magic-squares, circles and triangles of Yang Hui and produced a triangular representation of binomial coefficients. By 13th Century, the Golden Age of Chinese mathematics, there were over 30 prestigious mathematics schools in China.

 The Islamic science and mathematics flourished in the medieval period (9th–15th Centuries). The House of Wisdom, set up in Baghdad translated the major Greek and Indian mathematical and astronomy works into Arabic. Muslim artists discovered all the different forms of symmetry that could be depicted on a 2-dimensional surface.

 In Europe (13th–15th Centuries), Leonardo of Pisa, Nicole Oresme, Regiomontatus, and Nicolaus Cusanus spread use of Hindu-Arabic numeral-system, opened ways for great advances in European mathematics; used a system of rectangular coordinates; established Trigonometry as an independent branch of mathematics; and opened research potentials based on infinite and infinitesimal ideas.

 Albrecht Dürer (16th C.) invented “super magic square”. In 16th Century, the equals, multiplication, division, root, decimal, inequality symbols, decimal fractions were introduced and standardized.

 The early 17th Century brought the invention of ‘Logarithm’ by John Napier which greatly helped Kepler and Newton to perform complex calculations needed for their innovations.

 René Descartes’ developed analytic geometry and Cartesian coordinates allowing orbits of planets to be plotted on a graph, laying the foundations for the development of calculus, and multi-dimensional geometry. French mathematicians, Pierre de Fermat, Blaise Pascal, and Girard Desargues formulated several theorems in number theory and infinitesimal calculus, and founded the field of projective geometry.

 In 18th Century, the Bernoulli’s of Basel advanced Leibniz’s infinitesimal calculus to calculus of variations, and the probability and number theory of Pascal and Fermat. Leonhard Euler contributed in all aspects of mathematics, from geometry to calculus to trigonometry to algebra to number theory. Christian Goldbach proposed the Goldbach Conjecture, and also proved other theorems in number theory such as the Goldbach-Euler Theorem. Abraham de Moivre is known for ‘Moivre’s Formula’, which links complex numbers and trigonometry.

French mathematician, Joseph Louis Lagrange, contributed to differential equations and number theory, and originated the theory of groups which became very important in 19th and 20th Century mathematics. Pierre-Simon Laplace is known for his monumental work Celestial Mechanics. Adrien-Marie Legendre made important contributions to statistics, number theory, abstract algebra and mathematical analysis. Gaspard Monge was the inventor of descriptive geometry.

 The 19th Century Mathematicians, who made important advances in analytical mathematics were: Joseph Fourier (known for his Fourier-Series); Jean-Robert Argand (represented complex numbers on geometric diagrams); and Évariste Galois proved that there is no general algebraic method for solving polynomial equations of any degree greater than four. Carl Friedrich Gauss (Germany) is regarded as one of the three greatest mathematicians of all times, others being Archimedes and Newton. Bernhard Riemann (Germany) is known for his famous Riemann Hypothesis.

 Charles Babbage (England) designed a machine that performed computations based on stored program of instructions. His large “difference engine” of 1823 was able to calculate logarithms and trigonometric functions. George Peacock (England) invented symbolic algebra, toward future developments in abstract algebra. George Boole devised ‘Boolean-Algebra’; and Arthur Cayley was the pioneer of modern group theory, matrix algebra, theory of higher singularities, and higher dimensional geometry.

 The early 20th Century brought two great mathematicians, G.H. Hardy and Srinivasa Ramanujan; saw rise of mathematical logic and philosophical logicism. The agenda for 20th Century mathematics started with “Hilbert-Problems” (23 greatest unsolved mathematical problems).

 John Neumann (USA) made major contributions in quantum theory, games theory, and a new design-model to hold both instructions and data for electronic computers. Soviet mathematician, Andrey Kolmogorov, is known for contributions to probability theory, topology, intuitionistic logic, turbulence, classical mechanics, algorithmic information theory. André Weil’s theorems allowed connections to be made between number theory, algebra, geometry and topology, are considered among the greatest achievements of modern mathematics. Alexander Grothendieck, known for his “Theory of Schemes”, enabled mathematical structures to be seen in a new way.

 Paul Erdös (Hungary), with hundreds of mathematical collaborators, contributed to solve problems in combinatorics, graph theory, number theory, classical analysis, approximation theory, set theory, and probability theory. Paul Cohen  (1960s) rocked the mathematical world by proving that Cantor‘s continuum hypothesis (one of Hilbert’s original 23 problems) could be both ‘True’ AND ‘Not-True’, and that there were effectively two completely separate but valid mathematical worlds, one in which the continuum hypothesis was true and one where it was not.

 The British mathematician, Andrew Wiles, finally proved Fermat’s Last Theorem for all numbers, 350 years after Fermat’s initial posing. John Horton Conway (England) established the rules for the “Game of Life” in 1970, an early example of a “cellular automaton” in which patterns of cells evolve and grow in a grid, which became extremely popular among computer scientists.

**3.0 Mathematics in 21st Century :**

 Upto 20th Century, the two most important inventions were: ‘Language’ and ‘Mathematics’. In 18th & 19th century, mathematical advancements were mainly driven by the mechanical needs of the world, so electricity, steam-power, machinery, etc. were invented; use of mathematics-discipline like operations-research and statistics were for military, production, and transportation; and for advancing it for improvised applications in computer science and information technology.

 The digital revolution in the past decades providing a great chance to solve complex problems with the help of Mathematics, Computer-Science, and Algorithms, Big-Data, Machine-Learning and Artificial-Intelligence are now driving forward mathematics and mathematics education in relevant direction that will include more of mathematical modelling, more of mathematical applications in physical sciences, machine learning algorithm and modelling, and more complex non-linear optimization algorithms.

**3.1 Crisis in Mathematical Research in 21st Century –**

 As the mathematical communities of world face Pandora’s box of problems, in their Research-Paper (“**Mathematics and Mathematics Education in 21st Century**”) Alexandre Boroviki, Zoltan Kocsis, and Vladimir Kondratiev concluded that Mathematics has entered the period of a revolution-change in its history, due to the need for a switch to use of computers as assistants and checkers in production of proofs.

 Mathematics as a scientific discipline has entered the period of crisis as the research level mathematics has become too complex for human comprehension which has serious implications also for mathematics education. They estimated that much over 2 lakhs/year new theorems would be published in mathematical journals around the world which, as a rule, explicitly refers to other theorems and definitions and is integrated into the huge systems of mathematical knowledge which are unified, tightly connected, and cohesive; despite the diversity of mathematics; despite mathematical researches would continue to grow further with mathematical analysis, concepts, and results involved in practical applications which are much deeper, more abstract and more difficult than ever before.

 The above situation warrants a need that the subject of “Mathematics” is hardwired and digitally coded as assistants and checkers without which progress in mathematical researches will remain constrained and unauthenticated. They projected the future role of proof assistants as to making mathematics more computer-friendly and making mathematics more human-friendly where all mathematical results should be computer-verified, easy to access world-wide, and annotated in various ways.

**3.2 Crisis of Mathematical Education in 21st Century –**

The main highlights of crisis of mathematical education in 21st Century will take new form due to :

**–** Merging of big parts of mathematics as well as Computer-Science & Information-Technology;

**–** Development and Application of the new form of Auto-Math;

**–** Development of Mathematical Thinking in real-world situations ;

**–** Development of Thinking like Computer Scientists and Programmers ;

**–** Separate Curriculum for the deep-stream mathematics students ;

**–** Separate Curriculum for the main-stream mathematics students;

**–** Increase in level and scope of mathematical abstractions ;

**–** Industrial Applications of Applied Mathematics **;**

**–** Education of new teachers and re-education of existing teachers;

**–** Increase in the use & application of mathematical logic and modeling; and

**–** Development of Teaching-Learning Tools & Aids in above lines.

**4.0 Definition, Nature, & Characteristics of Mathematics :**

**4.1 Definition of Mathematics** **–**

 The dictionary meaning of “Mathematics” (derived from two Greek words: ‘Manthanein’ (means ‘learning’ and ‘Techne’ (means ‘technique’) stands as the science of number, space, measurement, quantity and magnitude, and spatial relations. It is a systematised, organised and exact branch of science dealing with quantitative facts, shape, arrangement, relationships and problems involving space and form. It is a logical study of shape, arrangement, and quantity.

 Mathematics is defined in diverse ways by different mathematicians. Bell, opposing the traditional definitions, emphasised mathematics to be an entirely abstract science which can reduce all mathematics to postulation forms (mathematical postulates are statements that are accepted as true without being proven). Similarly, “mathematics” was diversely defined as the science of indirect measurement (Comte); the indispensable instrument of all physical sciences (Kant); mathematics is the queen of sciences and arithmetic is the queen of all mathematics (Gauss); the gateway and key to all sciences (Bacon); mathematics is the profound study of art and the expression of beauty (J.B. Shaw).

**4.2 Nature of Mathematics –**

 J.B. Shaw stressed on four significant methods of mathematics giving more insight into the nature of mathematics, which are : **Scientific** (leading to generalisations of widening scope); **Intuitive** (leading to an insight into subtler depths); **Deductive** (leading to a permanent statement and rigorous form)’ and **Inventive** (leading to the ideal element, and creation of new realms). Mathematics, therefore, is not only ‘number work’ or ‘computation’, but is more about forming generalisations, seeing relationships, and developing logical thinking and reasoning.

**4.3 Characteristics of Mathematics –**

 Thus, from above definitions, it can be concluded that i) mathematics is an abstract science ; ii) mathematics is a science that deals with number and space ; iii) mathematics deals with quantitative facts and relationships ; iv) mathematics deals with problem involving space and form ; v) mathematics establishes various relationships between phenomena in space ; vi) mathematics helps man to give exact interpretation to his ideas and conclusions ; vii) mathematics explains that science is by-product of our empirical knowledge ; viii) mathematics involves man's high cognitive powers ; ix) mathematics has its own tools like intuition, logical reasoning, analysis, construction, generalities and individuality ; x) mathematics is a science of logical-reasoning ; and xi) mathematics is a tool especially suited for dealing with scientific concepts as well as industrial applications.

 The central and focal points of the ‘characteristics’ of ‘Mathematics’ can be further described under the key-terms : Objectivity, Logical Structure, Abstractness, Symbolism, Applicability, and Precision and Accuracy as explained below –

**4.3.1 Objectivity –** Objectivity is a central philosophical concept which has been differently defined by various sources. A proposition is generally considered to be objectively true when its truth conditions are met and are “mind-independent” that is, not met by the judgment of a conscious entity or subject. Objectivity in mathematics is traditionally thought of as one of the more desirable necessities for its own credibility.

**4.3.2 Logical Structure** – Mathematics uses an unfailing logical structure, where each step in a problem has to be arranged in a logical sequence, each step or statement to be mathematical must be supported by a proper reason. If not done, then, even an ordinary equation may lead to fallacy. For example, when a box which is half open and half closed, its equation form cannot be, ½ open = ½ closed; or Box-Open = Box-Closed, which will be a mistake arising due to use of a wrong logic as statements not supported by proper reason and logic is not acceptable in mathematics.

**4.3.3 Abstractness –**  Abstractness is the basis of mathematics. All the principles of mathematics are based on abstractness. Even at the elementary stage, mathematics is at the abstract level. For example, number is an abstract concept. When we teach children about numbers we make use of fingers, dolls etc,. We say two and show two balls. Two stands for an abstract concept. Showing balls is a concrete activity. All the numbers and operations such as 1,2,3,4,5,…etc. and union [U] and intersection[∩] are examples of abstractions; percent, length, volumes are abstractions; sum, difference, product, average are also examples of abstractness.

**4.3.4 Symbolism** – Mathematics is a language of symbols. Symbols are tools for conveying ideas. It is a technical way of expressing range of ideas in forms which are convenient for applying the processes of reasoning. It has its own precision, conciseness an accuracy which are absent from any other language. In mathematics, symbols are used to bring out the mathematical relationships clear and specific. Thus, it is essential to insert the appropriate mathematical symbols (punctuation, brackets, parenthesis, etc.). So, all mathematical statements should have the following characteristics: a) making use of conventional signs and symbols; b) every symbol used should have a definite purpose; c) mathematical language should be clear, concise, unambiguous and simple.

**4.3.5 Applicability** – Study of mathematics requires the learners to apply the skill acquired to new situations appropriately and verify the validity of the mathematical rules and relationships after applying them to novel situations. The teachers should always help the students in applying and verifying the mathematical ideas. Mathematical concepts and principles become more functional and meaningful only when they are related to actual practical applications.

**4.3.6 Precision and Accuracy –**  Mathematics is known as an exact science because of its precision and accuracy. It is perhaps the only subject which can claim certainty of results. In mathematics the results are either right or wrong, accepted or rejected. There is no midway possible between right and wrong. Precision and accuracy are different criteria for measures of approximation which can be effectively discussed even in contrast. The most effective measures of both precision and accuracy are in terms of the errors (positive or negative) involved.

**5.0 Development of Mathematics in India:**

**5.1 Historic Development of Mathematics in India –**

 In India, the “mantras” from the early Vedic period (before 1000 BCE) invoked powers of ten from a hundred all the way up to a trillion, and provided evidence of using arithmetic operations, viz., addition, subtraction, multiplication, fractions, squares, cubes and roots. As early as the 8th Century BCE, long before Pythagoras, a text known as the “Sulva-Sutras” listed several simple Pythagorean triples, also contained geometric solutions of linear and quadratic equations.

 The Jain mathematicians recognized (3rd/2nd Century BCE) five different types of infinities. Ancient Buddhist demonstrated a prescient awareness of indeterminate and infinite numbers, with numbers deemed to be of three types: countable, uncountable and infinite. Brahmagupta (c. 598 – c. 668 CE)  established the basic mathematical rules for dealing with zero, and negative numbers.

 The Golden Age of Indian mathematics (5th to 12th Centuries) produced many mathematical discoveries. They used trigonometry to calculate the relative distances between the Earth and the Moon and the Earth and the Sun. The important contributions of Bhaskara-I (600–680 BC) included: numbers and symbolism, the categorization of mathematics, the names and solution of the first degree equations, quadratic equations, cubic equations and equations having multiple unknown values, symbolic algebra, the algorithm method to solve linear indeterminate equations, discovery of Circle for Zero, positional arithmetic, the approximation of sine; the three treatises that were written by him on the works of mathematician, Aryabhata were: the Mahabhaskariya (“Great Book of Bhaskara”), the Laghubhaskariya (“Small Book of Bhaskara”), and the Aryabhatiyabhashya.

 Aryabhatta (476–550 CE) produced definitions of sine, cosine, versine and inverse sine, and specified complete sine and versine tables, demonstrated solutions to simultaneous quadratic equations, and produced an approximation for the value of π. Bhaskara-II (12th Century) explained the previously misunderstood operation of division by zero, indicating that 1 ÷ 0 = ∞ (symbol of infinity). The Kerala School of Astronomy and Mathematics (14th Century) developed the infinite series approximations for a range of trigonometric functions, including π, sine, etc.

**5.2 National Education Policy 1986 (NEP 1986) –**

 The ‘National Policy on Education’ (NPE-1986) also emphasises that mathematics should be visualised as the vehicle to train learners to think, reason, analyse and articulate logically, and apart from treating it as a specific subject, it should be treated as concomitant to any subject involving analysis and reasoning.

**5.3 National Curriculum Framework 2005 (NCF 2005) –**

 According to National Curriculum Framework 2005, the higher goal of teaching ‘mathematics’ is to develop the children's ability to mathematization. Mathematics should be taught through application rather than emphasizing formal algorithms. Real-world examples should be incorporated into the classroom to increase student engagement. According to the NCF 2005, “Developing children's abilities for mathematization is the main goal of mathematics education.” NEP’ 2005 sets two kinds of aims in mathematics for school education e.g., Broader Aims and Narrower Aims. Broader aims include: problem-solving, optimization, use of patterns, etc.; and Narrower aims include: developing numeracy-related skills, to develop useful capabilities, etc..

**5.4 National Education Policy 2020 (NEP 2020) –**

 The National Education Policy’ 2020(NEP-2020) recommends that the “Mathematics Curricula” should aim for holistic development of learners, equipping them with 21st century skills, reduction in curricular content to enhance essential learning and critical thinking and greater focus on experiential learning. NEP-2020 recommends that there should be a shift from summative assessment to regular and formative assessment, which is more competency-based, promotes learning and development, and tests higher-order skills, such as analysis, critical thinking, and conceptual clarity by reforming the assessment and evaluation. NEP’2020 recognises ‘Foundational Literacy and Numeracy’ as an urgent and essential prerequisite to learning and calls for building the National Mission on Foundational Literacy and Numeracy.

**6.0 Building-Blocks of Mathematics**

 Mathematics is completely built on basic mathematical concepts and principles. A complete and strong understanding of foundational elements which function as Building-Blocks for more complex topics in Mathematics is of foremost critical essentiality which alone can provide a well-built foundation for intelligent reasoning, logical thinking, and skill required for complex problem-solving. Finding ‘Mathematics’ as challenging and intimidating, the key to overcoming these challenges lies in adopting the right approach and mindset towards maths learning. This paper explores practical strategy to develop and enhance mathematics skills effectively.

 Jeremy Avigad (2016) observed that the ‘Objects-of-Knowledge’ in Mathematics can be grouped under two categories : (i) Concrete/Syntactic: Definitions, Theorems, Proofs, Theories, Questions, Conjectures, etc. and (ii) Abstract/Quasi-Algorithmic: Methods, Concepts, Heuristics, Intuitions, etc.

 The Building-Blocks of the mathematical science contributing to the ‘Body-of-Knowledge & Practice’ of Mathematics includes all its foundational components. A review of literature revealed that the Building-Blocks of Mathematics, as listed and explained below, includes Mathematical Terminologies, Conceptual Constructs, Operators, Formulas, Notations, Symbols, Rules, Properties, Mathematical-Tables, Mathematical Constants, Mathematical Logic, Mathematical Sets, Mathematical Functions, Mathematical Relations, and Mathematical Paradoxes.

**7.0 Mathematical Pedagogy**

 The Mathematical ‘Education’ can be compared with a ‘Mother-Board’, the main Printed-C ircuit-Board (PCB) in a computer, a computer's central communications backbone and connectivity-station, through which all components and external peripherals stand connected to each other to operationalise its purpose. Similarly, the mathematical ‘Pedagogy’ is the umbrella component for ‘mathematics-education’ which receives, interprets, processes, disseminates, reacts, exchanges, and concludes the mathematical messages and instructions among and through logics, processes, learners, learning-psychology, teachers, teaching-methodology, information-systems, information-technology, communication-networks, learning contents, and teaching-learning aids to provide authenticated propagation of knowledge in order to achieve particular purpose(s) of education.

 The University Grants Commission, Govt. of India recognising the importance of Innovation in Pedagogy issued a detailed Guidelines (<https://www.ugc.gov.in/pdfnews/1031121_Guidelines-Innovative-Pedagogical-Approaches-Evaluation-Reforms.pdf> ) reiterating that the curricula and pedagogies have to be reoriented and revamped including raising the standard of curricula and using appropriate pedagogies to deliver effectively to the learners for developing competent individuals with 21st-century skills following India’s National Education Policy (NEP) 2020.

 The NEP’2020 (Para 4.4, Para 9.3 (d), Para 11.6, Para 12.1, Para 12.2 and Para 12.6) envisions innovative pedagogical approaches and their role in higher education; emphasises holistic development of the learners through experiential learning, cutting edge pedagogy, art integrated learning, flipped classroom etc. The NEP’2020 is learner centric in its approach, and teachers play a pivotal role in its implementation. The policy gives teachers more autonomy in choosing aspects of pedagogy so that they may plan teaching-learning in the manner they find most effective for the students in their classrooms.

**8.0 Use of Progressive Techniques & Digital Technology in Mathematical Pedagogy**

 To make ‘Mathematical Pedagogy’ both efficient and effective the use of Progressive Techniques and Digital Technology is imperative. Use of Progressive Techniques in Mathematical Pedagogy focus on exploring innovative and effective methods of teaching mathematics. It would involve researching and developing new approaches, strategies, and tools to enhance the teaching and learning of mathematics at various levels, such as schools, colleges, and universities. This field combines aspects of mathematics education, instructional design, curriculum development, and educational technology. Here are some potential areas of research and study within this field:

**Constructivist Approaches:** Investigating and implementing constructivist pedagogical strategies that emphasize active learning, problem-solving, and student engagement in mathematical learning.

**Technology-Enhanced Learning:** Exploring the integration of technology, such as educational software, online platforms, virtual manipulatives, and interactive simulations, to facilitate mathematical understanding and engagement.

**Game-Based Learning:** Examining the use of educational games and gamification techniques to make mathematics learning more enjoyable, motivating, and effective.

**Inquiry-Based Learning**: Investigating inquiry-based instructional methods that encourage students to explore mathematical concepts, develop reasoning skills, and construct their own knowledge.

**Differentiated Instruction:** Exploring techniques for adapting mathematical instruction to cater to diverse learning needs, abilities, and preferences of students.

**Assessment and Feedback:** Studying innovative assessment methods and providing timely and constructive feedback to enhance students' mathematical understanding and performance.

**Teacher Professional Development:** Investigating effective professional development programs and strategies for mathematics teachers to enhance their pedagogical content knowledge and teaching skills.

**Curriculum Design and Innovation:** Developing and evaluating curriculum materials and frameworks that promote conceptual understanding, critical thinking, and problem-solving skills in mathematics.

**Cultural and Contextual Factors:** Examining the influence of cultural, social, and contextual factors on mathematical learning and designing pedagogical approaches that are responsive to diverse learners and cultural contexts.

**Educational Policy and Reform:** Analyzing educational policies and reforms related to mathematics education and exploring their implications on pedagogical practices and student outcomes.

**8.1 Examples of technology-enhanced learning tools that used in Mathematics Education –**

 There are various technology-enhanced learning tools that can be used in mathematics education to enhance students' understanding, engagement, and problem-solving skills. Here are some examples:

**Interactive Whiteboards:** Interactive whiteboards allow teachers to display and manipulate mathematical content, annotate examples, and engage students in collaborative problem-solving activities. They often come with software that provides interactive features and tools for creating dynamic and visually appealing math lessons.

**Educational Software and Apps:** There are numerous software programs and mobile applications designed specifically for teaching and learning mathematics. These tools provide interactive lessons, tutorials, practice exercises, and simulations to help students explore mathematical concepts and reinforce their understanding.

**Online Learning Platforms:** Online platforms, such as Learning-Management-Systems (LMS) or dedicated Mathematics Learning Platforms, offer a range of resources for mathematics education. They can include video lessons, interactive quizzes, virtual manipulatives, discussion forums, and progress tracking features.

**Virtual Manipulatives:** Virtual manipulatives are digital representations of physical objects used in mathematics education. They allow students to manipulate objects, visualize concepts, and explore mathematical relationships. Examples include virtual fraction bars, base-ten blocks, geometric shapes, and algebra tiles.

**Computer Algebra Systems (CAS):** CAS software, such as Mathematica, Maple, or MATLAB, enables students to perform complex mathematical computations, symbolic calculations, graphing, and data analysis. CAS tools are particularly useful for advanced mathematics topics and modeling.

**Online Math Problem-Solving Platforms:** There are web-based platforms that provide a collection of math problems for students to solve individually or collaboratively. These platforms often offer adaptive learning features, provide instant feedback, and track students' progress.

**Augmented Reality (AR) and Virtual Reality (VR):** AR and VR technologies can create immersive and interactive mathematical experiences. They can be used to visualize 3D objects, explore geometry in virtual environments, or simulate real-world math applications. AR uses a real-world setting while VR is completely virtual. AR users can control their presence in the real world; VR users are controlled by the system. VR requires a headset device, but AR can be accessed with a smartphone. AR enhances both the virtual and real world while VR only enhances a fictional reality.

**Math Games and Gamified Learning:** Digital math games and gamified learning platforms engage students through challenges, rewards, and competitive elements while reinforcing mathematical concepts and skills.

**Online Math Communities and Tutoring Platforms:** Online communities and tutoring platforms connect students with math tutors, peers, and experts for collaborative learning, problem-solving support, and personalized instruction.

**Data Visualization Tools:** Tools for data visualization, such as graphing calculators, spreadsheet software, and data analysis platforms, enable students to explore and analyze data sets, make connections between mathematical concepts and real-world data, and develop statistical reasoning skills.

 These are just a few examples of technology-enhanced learning tools used in mathematics education. The choice of tools depends on the specific learning objectives, grade level, and available resources. It's important for educators to select tools that align with their instructional goals and provide meaningful and engaging learning experiences for students.

**8.2 Details of Interactive-Whiteboards Uses in enhancing Mathematics Teaching and Learning –**

 Interactive-Whiteboards, also known as Smart-Boards, can significantly enhance mathematics education by providing dynamic and interactive learning experiences. Here are some ways interactive whiteboards can be used to enhance mathematics teaching and learning :--

**Visual Representation:** Interactive whiteboards allow teachers to display mathematical concepts, equations, graphs, and diagrams on a large, visible surface. This visual representation helps students better understand and visualize abstract mathematical ideas, making them more accessible.

**Dynamic Manipulation:** Teachers can use the interactive features of the whiteboard to manipulate mathematical objects, such as dragging and resizing shapes, moving points on graphs, or highlighting specific elements. This dynamic manipulation enables teachers to demonstrate mathematical transformations, illustrate concepts, and engage students in hands-on exploration.

**Annotation and Explanation:** Teachers can annotate examples and explanations directly on the whiteboard, providing step-by-step solutions, highlighting key points, or illustrating problem-solving strategies. This real-time annotation helps students follow along, grasp mathematical processes, and reinforce their understanding.

**Collaborative Problem Solving:** Interactive whiteboards encourage collaboration and active participation. Students can come to the board and work together on mathematical problems, present their solutions, and explain their reasoning to their peers. This collaborative problem-solving fosters mathematical discourse and peer learning.

**Interactive Tools and Widgets:** Interactive whiteboards often come with a variety of tools and widgets specifically designed for mathematics, such as rulers, protractors, compasses, and graphing tools. These interactive tools enable students to explore geometric concepts, measure angles, construct shapes, plot graphs, and perform calculations directly on the board.

**Integration of Multimedia**: Interactive whiteboards allow teachers to integrate multimedia resources, such as videos, animations, and interactive simulations, into their lessons. This multimedia integration can provide dynamic visualizations, real-world applications, and interactive experiences that enhance students' engagement and understanding of mathematical concepts.

**Instant Feedback and Assessment:** Interactive whiteboards can facilitate formative assessment by providing instant feedback to students. Teachers can use the whiteboard to display quizzes, polls, or multiple-choice questions, and students can respond using interactive response systems. This immediate feedback helps identify misconceptions, assess learning progress, and adapt instruction accordingly.

**Access to Online Resources:** Interactive whiteboards often have internet connectivity, enabling teachers to access online math resources, educational websites, and interactive math games directly from the board. This access to a vast array of online resources expands the range of mathematical materials available to teachers and students.

**Differentiation and Personalization:** Interactive whiteboards allow teachers to adapt their instruction to meet the diverse needs of students. Teachers can use the interactive features to differentiate instruction by adjusting the pace, complexity, or level of support provided. For example, they can hide or reveal elements of a problem, customize the difficulty level, or provide additional resources for students who need extra support or challenge.

**Scaffolding and Guided Practice:** Interactive whiteboards provide opportunities for teachers to scaffold students' learning by gradually releasing responsibility. Teachers can model problem-solving strategies, guide students through examples, and provide step-by-step solutions. This guided practice helps students develop procedural fluency, mathematical reasoning, and problem-solving skills.

**Error Analysis and Reflection:** Through interactive whiteboards, teachers can highlight common errors, misconceptions, or alternative approaches to problem-solving. This error analysis helps students identify and correct mistakes, analyze their thinking, and deepen their understanding of mathematical concepts.

**Historical and Cultural Context:** Interactive whiteboards can display historical mathematical documents, cultural artifacts, or real-world applications of mathematics. This integration of historical and cultural context helps students understand the relevance and significance of mathematics in various contexts, fostering a deeper appreciation for the subject.

**Remote and Online Learning:** Interactive whiteboards can be particularly useful in remote or online learning environments. Teachers can use screen-sharing and video conference tools to deliver live lessons, share their whiteboard screens, and engage students in real-time discussions. This interaction and visual representation help bridge the physical distance and maintain student engagement.

**Virtual Field Trips and Guest Speakers:** Interactive whiteboards can facilitate virtual field trips or guest speaker sessions, where students can explore mathematical concepts in real-world contexts or interact with experts remotely. This use of interactive whiteboards broadens students' exposure to different perspectives, applications, and career opportunities related to mathematics.

**Data Analysis and Visualization:** Interactive whiteboards can be used to display and manipulate data sets, create graphs and charts, and explore statistical concepts. This data analysis and visualization capability supports students in developing skills related to data interpretation, making predictions, and drawing conclusions.

 It's important to note that while interactive whiteboards offer numerous benefits, effective pedagogy and instructional practices play a crucial role in leveraging the technology to its fullest potential. Teachers need to design and facilitate engaging and meaningful learning experiences that integrate the interactive whiteboard features appropriately. This can involve careful planning, teacher professional development, and ongoing reflection on instructional strategies.

 In an overall assessment, the Interactive-Whiteboards are powerful tools, among other tools, in mathematics education, enabling teachers to create interactive, visual, and collaborative learning environments that promote mathematical understanding, engagement, and critical thinking skills.

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