Approximation of signals by Matrix-Cesàro product summability means in the generalized Hölder class

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Abstract: In this paper, two results has been established to approximate functions belonging to generalized Hölder class by a more generalized $UC^{\alpha,\eta}$ means of Fourier series (F. S.) and conjugate Fourier series (C. F. S.). Very few researchers worked out in the area of generalized Hölder class. The established theorem extend and generalize the existing result by Nigam and Hadish [6]. Also, we have derived several new corollaries and useful remarks.

Keywords: Signal approximation, Generalized Hölder class, Matrix (U) mean, $C^{\alpha,\eta}$ means, $UC^{\alpha,\eta}$ means, Euler summability.

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1 Introduction

Many results on the estimation of error by single and product means in Lipchitz and Hölder classes using trigonometric polynomial have been obtained by the researchers like [2]-[3] and [7]-[19].
The purpose of this work is to find best approximation by using trigonometric polynomial. So, here we generalize the results of Nigam and Hadish [6]. We approximate the two functions $g \in H_z^{(w)}$ $(z \geq 1)$ and $\tilde{g} \in H_z^{(w)}$ $(z \geq 1)$ by $UC^{\alpha,\eta}$ method by F. S. and C. F. S. respectively. Thus, the result of Nigam and Hadish [6] become the particular cases of our Theorem 2.1.
Let $U = (a_{q,p})$ be an infinite triangular matrix satisfying the condition of regularity [15], i.e.,

$$\sum_{p=0}^{q} a_{q,p} = 1 \quad as \;\; q \to \infty,$$

$$a_{q,p} = 0 \quad for \;\; p > q, \tag{1.1}$$

$$\sum_{p=0}^{q} |a_{q,p}| \leq M, \quad \text{a finite constant.}$$

The sequence-to-sequence transformation

$$t_q^U := \sum_{p=0}^{q} a_{q,p} s_p = \sum_{p=0}^{q} a_{q,q-p} s_{q-p} \tag{1.2}$$

defines the sequence t_q^U of triangular matrix means of the sequence $\{s_q\}$ generated by the sequence of coefficients $(a_{q,p})$.

If $t_q^U \to s$ as $q \to \infty$, then the infinite series $\sum_{q=0}^{\infty} h_q$ or the sequence $\{s_q\}$ is summable to s by a triangular matrix [1].

Let

$$C_p^{\alpha,\eta} = t_p^{C^{\alpha,\eta}} = \frac{1}{B_p^{\alpha+\eta}} \sum_{h=0}^{p} B_{p-h}^{\alpha-1} B_h^{\eta} s_h.$$

If $C_p^{\alpha,\eta} \to s$ as $q \to \infty$, then the infinite series $\sum_{q=0}^{\infty} h_q$ is summable to s by $C_p^{\alpha,\eta}$ means [1].

The $UC^{\alpha,\eta}$ means (U-means of $C^{\alpha,\eta}$ means) is given by

$$\begin{aligned} t_q^{U.C^{\alpha,\eta}} &:= \sum_{p=0}^{q} a_{q,p} C_p^{\alpha,\eta} \\ &= \sum_{p=0}^{q} a_{q,p} \frac{1}{B_p^{\alpha+\eta}} \sum_{h=0}^{p} B_{p-h}^{\alpha-1} B_h^{\eta} s_h. \end{aligned}$$

If $t_q^{U.C^{\alpha,\eta}} \to s$ as $q \to \infty$, then $\sum_{q=0}^{\infty} h_q$ is summable to s by $U.C^{\alpha,\eta}$ means.

The regularity of U and $C^{\alpha,\eta}$ methods leads to the regularity of $U.C^{\alpha,\eta}$ method.

Remark 1: (Example) Consider the series

$$1 + \sum_{n=1}^{\infty} (-1)^n.2n. \tag{1.3}$$

which is not (C, α, η) summable and if we take $a_{n,k} = \frac{1}{n+1}$, then the series (1.3) is also not summable by U means. But (1.3) is summable by the $U.C^{\alpha,\eta}$ product means. That's why the product means are better than the individual means.

Remark 2: $UC^{\alpha,\eta}$ means changes to

1. $\left(H, \frac{1}{q+1}\right)C^{\alpha,\eta}$ or $H.C^{\alpha,\eta}$ means if $a_{q,p} = \frac{1}{(q-p+1)\log(q+1)}$;

2. $(N, \theta_q)C^{\alpha,\eta}$ or $N_\theta C^{\alpha,\eta}$ means if $a_{q,p} = \frac{\theta_{q-p}}{P_q}$, where $P_q = \sum_{p=0}^{q} \theta_p \neq 0$;

3. $(N, \theta, \tau)C^{\alpha,\eta}$ or $N_{\theta,\tau}C^{\alpha,\eta}$ means if $a_{q,p} = \frac{\theta_{q-p}\tau_p}{R_q}$, where $R_q = \sum_{p=0}^{q} \theta_p \tau_{q-p}$;

4. $(\overline{N}, \theta_q)C^{\alpha,\eta}$ or $\overline{N}_\theta C^{\alpha,\eta}$ if $a_{q,p} = \frac{\theta_p}{P_q}$.

Let $L^z[0, 2\pi] = \{g : [0, 2\pi] \to \mathbb{R} : \int_0^{2\pi} |g(x)|^z dx < \infty, z \geq 1\}$ be a space of functions. The norm $\|.\|_{(z)}$ is defined by

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} |g(x)|^z dx\right\}^{\frac{1}{z}}, \ z \geq 1.$$

As define in [1], $w : [0, 2\pi] \to \mathbb{R}$ is an arbitrary function with $w(l)>0$ for $0<l \leq 2\pi$ and $\lim_{l\to 0^+} w(l) = w(0) = 0$.

Now, we define

$$H_z^{(w)} = \left\{g \in L^z[0, 2\pi] : \sup_{l\neq 0}\frac{\|g(., +l) - g(.)\|_z}{w(l)} < \infty, z \geq 1\right\}$$

and

$$\|.\|_z^{(w)} = \|g\|_z^{(w)} = \|g\|_z + \sup_{l\neq 0}\frac{\|g(., +l) - g(.)\|_z}{w(l)}; \ z \geq 1.$$

Note 1: $w(l)$ and $v(l)$ denotes Zygmund moduli of Continuity [1].

If we consider $\frac{w(l)}{v(l)}$ as +ive and non-decreasing,

$$\|g\|_z^{(v)} \leq max\left(1, \frac{w(2\pi)}{v(2\pi)}\right)\|g\|_z^{(w)} < \infty.$$

Thus,

$$H_z^{(w)} \subset H_z^{(v)} \subset L^z; \ z \geq 1.$$

Remark 3:

1. If $w(l) = l^\alpha$ in $H^{(w)}$, then $H^{(w)} \implies H_\alpha$ class.

2. If $w(l) = l^\alpha$ in $H_z^{(w)}$, then $H_z^{(w)} \implies H_{\alpha,z}$ class.

3. If $z \to \infty$ in $H_z^{(w)}$, then $H_z^{(w)} \implies H^{(w)}$ class and $H_{\alpha,z} \implies H_\alpha$ class.

Remark 4: The q^{th} partial sum of F. S. and C. F. S. is denoted as

$$s_q(g; x) - g(x) = \frac{1}{2\pi}\int_0^{\pi} \phi_x(l)\frac{\sin\left(q + \frac{1}{2}\right)l}{\sin\frac{l}{2}}dl$$

and

$$s_q(\tilde{g};x)-\tilde{g}(x)=\frac{1}{2\pi}\int_0^{\pi}\psi_x(l)\frac{\cos\left(q+\frac{1}{2}\right)l}{\sin\frac{l}{2}}dl$$

respectively, where

$$\tilde{g}(x)=-\frac{1}{2\pi}\int_0^{\pi}\psi_x(l)\cot\frac{l}{2}dl.$$

The error function g is given by

$$E_q(g)=min\left\|g-t_q\right\|_z,$$

where t_q is a trigonometric polynomial of degree q [1].

We write

$$\phi_x(l)=\phi(x,l)=g(x+l)+g(x-l)-2g(x),$$

$$\psi_x(l)=\psi(x,l)=g(x+l)-g(x-l),$$

$$\Delta p_m=p_m-p_{m+1},\ m\geq 0,$$

$$H_q(l)=\frac{1}{2\pi}\sum_{p=0}^{q}a_{q,p}\frac{1}{B_m^{\alpha+\eta}}\sum_{v=0}^{p}B_{p-v}^{\alpha-1}B_v^{\eta}\frac{\sin\left(v+\frac{1}{2}\right)l}{\sin\frac{l}{2}},$$

and

$$\tilde{H}_q(l)=\frac{1}{2\pi}\sum_{p=0}^{q}a_{q,p}\frac{1}{B_p^{\alpha+\eta}}\sum_{v=0}^{p}B_{p-v}^{\alpha-1}B_v^{\eta}\frac{\cos\left(v+\frac{1}{2}\right)l}{\sin\frac{l}{2}}.$$

2 Main theorems

2.1 Theorem

If $g\in H_z^{(w)}$ class; $z\geq 1$ and $\frac{w(l)}{v(l)}$ are positive and non-decreasing, then the error estimation of g by $UC^{\alpha,\eta}$ means of F.S. is

$$\left\|t_q^{U.C^{\alpha,\eta}}-g\right\|_z^{(v)}=\mathcal{O}\left(\frac{1}{q+1}\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2v(l)}dl\right),$$

where $U=(a_{q,p})$ and w, v are defined as in Note 1 provided

$$\sum_{p=0}^{q-1}|\Delta a_{q,p}|=\mathcal{O}\left(\frac{1}{q+1}\right)\quad and\ \ (q+1)a_{q,q}=\mathcal{O}(1). \tag{2.1}$$

2.2 Theorem

If $\tilde{g} \in H_z^{(w)}$ class; $z \geq 1$ and $\frac{w(l)}{v(l)}$ are positive and non-decreasing, then the error estimation of $\tilde{g}$ by $UC^{\alpha,\eta}$ means of C.F.S. is

$$\left\| \tilde{t}_q^{U.C^{\alpha,\eta}} - \tilde{g} \right\|_z^{(v)} = \mathcal{O}\left(\frac{\log(q+1)+1}{q+1} \int_{\frac{1}{q+1}}^{\pi} \frac{w(l)}{l^2 v(l)} dl \right),$$

where $U = (a_{q,p})$ and w, v are defined as in Note 1.

3 Lemmas

3.1 Lemma

Under the condition (1.1), $H_q(l) = \mathcal{O}(q+1)$ for $0 < l < \frac{1}{q+1}$.

Proof: Using $\sin\left(\frac{l}{2}\right) \geq \frac{l}{\pi}$, $\sin(ql) \leq ql$.

$$\begin{aligned}
H_q(l) &= \frac{1}{2\pi} \sum_{p=0}^{q} a_{q,p} \frac{1}{B_p^{\alpha+\eta}} \sum_{v=0}^{p} B_{p-v}^{\alpha-1} B_v^{\eta} \frac{\sin\left(v+\frac{1}{2}\right) l}{\sin \frac{l}{2}} \\
|H_q(l)| &\leq \frac{1}{2\pi} \times \frac{\pi}{l} \left| \sum_{p=0}^{q} a_{q,p} \frac{1}{B_m^{\alpha+\eta}} \sum_{v=0}^{p} B_{p-v}^{\alpha-1} B_v^{\eta} \sin\left(v+\frac{1}{2}\right) l \right| \\
&= \frac{1}{2l} \left| \sum_{p=0}^{q} a_{q,p} \frac{1}{B_p^{\alpha+\eta}} \sum_{v=0}^{p} B_{p-v}^{\alpha-1} B_v^{\eta} \sin(2v+1) \frac{l}{2} \right| \\
&\leq \frac{1}{2l} \left| \sum_{p=0}^{q} a_{q,p} \frac{1}{B_p^{\alpha+\eta}} \sum_{v=0}^{p} B_{p-v}^{\alpha-1} B_v^{\eta} (2v+1) \frac{l}{2} \right| \\
&= \frac{1}{4} \left| \sum_{p=0}^{q} a_{q,p} \frac{1}{B_p^{\alpha+\eta}} \sum_{v=0}^{p} B_{p-v}^{\alpha-1} B_v^{\eta} (2v+1) \right| \\
&\leq \frac{1}{4} \left| \sum_{p=0}^{q} a_{q,p} (2p+1) \frac{1}{B_p^{\alpha+\eta}} \sum_{v=0}^{p} B_{p-v}^{\alpha-1} B_v^{\eta} \right| \\
&= \frac{1}{4} \left| \sum_{p=0}^{q} a_{q,p} (2p+1) \right| \quad \left\{ \because \sum_{v=0}^{p} B_{p-v}^{\alpha-1} B_v^{\eta} = B_p^{\alpha+\eta} \right\} \\
&= \frac{1}{4} (2p+1) \sum_{p=0}^{q} |a_{q,p}| \\
&= \mathcal{O}(2q+1).
\end{aligned}$$

3.2 Lemma

Under the condition (1.1) and (2.1), $H_q(l) = \mathcal{O}\left(\frac{1}{(q+1)l^2}\right)$ for $\frac{1}{q+1} \leq l \leq \pi$.

Proof: Using $\sin\left(\frac{l}{2}\right) \geq \frac{l}{\pi}$ and $\sin^2(ql) \leq 1$, we have

$$
\begin{aligned}
H_q(l) &= \frac{1}{2\pi}\sum_{p=0}^{q} a_{q,p}\frac{1}{B_p^{\alpha+\eta}}\sum_{v=0}^{p} B_{p-v}^{\alpha-1}B_v^{\eta}\frac{\sin\left(v+\frac{1}{2}\right)l}{\sin\frac{l}{2}} \\
|H_r(l)| &\leq \frac{1}{2\pi}\times\frac{\pi}{l}\left|\sum_{p=0}^{q} a_{q,p}\frac{1}{B_p^{\alpha+\eta}}\sum_{v=0}^{p} B_{p-v}^{\alpha-1}B_v^{\eta}\sin\left(v+\frac{1}{2}\right)l\right| \\
&= \frac{1}{2l}\left|\sum_{p=0}^{q} a_{q,p}\frac{1}{B_p^{\alpha+\eta}}\sum_{v=0}^{p} B_{p-v}^{\alpha-1}B_v^{\eta}.\ Im\left\{\sum_{v=0}^{p} e^{\iota\left(v+\frac{1}{2}\right)l}\right\}\right| \\
&= \frac{1}{2l}\left|\sum_{p=0}^{q} a_{q,p}\ Im\left\{\sum_{v=0}^{p} e^{\iota\left(v+\frac{1}{2}\right)l}\right\}\right| \quad \left\{\because \sum_{v=0}^{p} B_{p-v}^{\alpha-1}B_v^{\eta} = B_p^{\alpha+\eta}\right\} \\
&= \frac{1}{2l}\left|\sum_{p=0}^{q} a_{q,p}\ Im\left\{e^{\iota\frac{l}{2}}\sum_{v=0}^{p} e^{\iota vl}\right\}\right| \\
&= \frac{1}{2l}\left|\sum_{p=0}^{q} a_{q,p}\ Im\left\{e^{\iota\frac{l}{2}}\left(\frac{1-e^{\iota(p+1)l}}{1-e^{\iota l}}\right)\right\}\right| \\
&= \frac{1}{2l}\left|\sum_{p=0}^{q} a_{q,p}\ Im\left\{\frac{e^{\iota(p+1)l}-1}{2\iota\sin\left(\frac{l}{2}\right)}\right\}\right| \\
&= \frac{1}{2l}\times\frac{\pi}{l}\left|\sum_{p=0}^{q} a_{q,p}\sin^2(p+1)\frac{l}{2}\right| \\
&\leq \frac{\pi}{2l^2}\left[\left|\sum_{p=0}^{q-1}\Delta a_{q,p}\sum_{v=0}^{p}\sin^2(v+1)\frac{l}{2}\right| + a_{q,q}\left|\sum_{p=0}^{q}\sin^2(p+1)\frac{l}{2}\right|\right] \\
&\leq \frac{\pi}{2l^2}\left[\sum_{p=0}^{q-1}|\Delta a_{q,p}| + a_{q,q}\right] \\
&= \mathcal{O}\left(\frac{1}{(q+1)l^2}\right).
\end{aligned}
$$

3.3 Lemma

Under the condition (1.1), $\tilde{H}_q(l) = \mathcal{O}\left(\frac{1}{l}\right)$ for $0 < l < \frac{1}{q+1}$.

Proof: Using $\sin\left(\frac{l}{2}\right) \geq \frac{l}{\pi}$ and $|\cos(ql)| \leq 1$, we get

$$\begin{aligned}
\tilde{H}_q(l) &= \frac{1}{2\pi}\sum_{p=0}^{q} a_{q,p}\frac{1}{B_p^{\alpha+\eta}}\sum_{v=0}^{p} B_{p-v}^{\alpha-1}B_v^{\eta}\frac{\cos\left(v+\frac{1}{2}\right)l}{\sin\frac{l}{2}} \\
|\tilde{H}_q(l)| &\leq \frac{1}{2\pi}\times\frac{\pi}{l}\left|\sum_{p=0}^{q} a_{q,p}\frac{1}{B_p^{\alpha+\eta}}\sum_{v=0}^{p} B_{p-v}^{\alpha-1}B_v^{\eta}\cos\left(v+\frac{1}{2}\right)l\right| \\
&\leq \frac{1}{2l}\sum_{p=0}^{q} a_{q,p}\frac{1}{B_p^{\alpha+\eta}}\sum_{v=0}^{p} B_{p-v}^{\alpha-1}B_v^{\eta}\left|\cos\left(v+\frac{1}{2}\right)l\right| \\
&\leq \frac{1}{2l}\sum_{p=0}^{q} a_{q,p}\frac{1}{B_p^{\alpha+\eta}}\sum_{v=0}^{p} B_{p-v}^{\alpha-1}B_v^{\eta} \\
&= \frac{1}{2l}\sum_{p=0}^{q} a_{q,p} \quad \left\{\because \sum_{v=0}^{p} B_{p-v}^{\alpha-1}B_v^{\eta} = B_p^{\alpha+\eta}\right\} \\
&\mathcal{O}\left(\frac{1}{l}\right).
\end{aligned}$$

3.4 Lemma

Under the condition (1.1) and (2.1), $\tilde{H}_q(l) = \mathcal{O}\left(\frac{1}{(q+1)l^2}\right)$ for $\frac{1}{q+1} \leq l \leq \pi$.

Proof: Using $\sin\left(\frac{l}{2}\right) \geq \frac{l}{\pi}$ and $|\sin ql| \leq 1$, we have

$$\begin{aligned}
|\tilde{H}_q(l)| &\leq \frac{1}{2\pi}\times\frac{\pi}{l}\left|\sum_{p=0}^{q} a_{q,p}\frac{1}{B_p^{\alpha+\eta}}\sum_{v=0}^{p}\left(B_{p-v}^{\alpha-1}B_v^{\eta}\cos\left(v+\frac{1}{2}\right)l\right)\right| \\
&= \frac{1}{2l}\left|\sum_{p=0}^{q} a_{q,p}\frac{1}{B_p^{\alpha+\eta}}\sum_{v=0}^{p} B_{p-v}^{\alpha-1}B_v^{\eta}.\sum_{v=0}^{p}\cos\left(v+\frac{1}{2}\right)l\right| \\
&= \frac{1}{2l}\left|\sum_{p=0}^{q} a_{q,p}\sum_{v=0}^{p}\cos\left(v+\frac{1}{2}\right)l\right| \quad \left\{\because \sum_{v=0}^{p} B_{p-v}^{\alpha-1}B_v^{\eta} = B_p^{\alpha+\eta}\right\} \\
&\leq \frac{1}{2l}\left|\sum_{p=0}^{q} a_{q,p}\left\{\frac{2\sin\frac{l}{2}\cos\frac{l}{2}+2\sin\frac{l}{2}\cos\frac{3l}{2}+\ldots 2\sin\frac{l}{2}\cos\frac{(2p+1)l}{2}}{2\sin\frac{l}{2}}\right\}\right| \\
&\leq \frac{\pi}{4l^2}\left|\sum_{p=0}^{q} a_{q,p}\sin(p+1)l\right| \\
&\leq \frac{\pi}{4l^2}\left|\sum_{p=0}^{q-1}(a_{q,p}-a_{q,p+1})\sum_{v=0}^{p}\sin(v+1)l + a_{q,q}\sum_{p=0}^{q}\sin(p+1)l\right| \\
&\leq \frac{\pi}{4l^2}\left[\left(\sum_{p=0}^{q-1}|\Delta a_{q,p}| + a_{q,q}\right)\sum_{v=0}^{p}1\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\pi}{4l^2}\left[\left(\sum_{p=0}^{q-1}|\Delta a_{q,p}|+a_{q,q}\right)(p+1)\right]\\
&= \frac{\pi(q+1)}{4l^2}\left\{\mathcal{O}\left(\frac{1}{q+1}\right)+\mathcal{O}\left(\frac{1}{q+1}\right)\right\}\\
&= \mathcal{O}\left(\frac{1}{l^2}\right).
\end{aligned}
$$

3.5 Lemma

([20], p. 93) Let $g \in H_z^{(w)}$, then for $0<l \leq \pi$:

1. $\|\phi(\cdot,l)\|_z = \mathcal{O}(w(l))$;
2. $\|\phi(\cdot+y,l)-\phi(\cdot,l)\|_z = \begin{cases}\mathcal{O}(w(l)),\\ \mathcal{O}(w(|y|));\end{cases}$
3. If w(l) and v(l) are defined as in Note 1, then $\|\phi(\cdot+y,l)-\phi(\cdot,l)\|_z = \mathcal{O}\left(v(|y|)\left(\frac{w(l)}{v(l)}\right)\right)$.

3.6 Lemma

Let $\tilde{g} \in H_z^{(w)}$, then for $0<l \leq \pi$:

1. $\|\psi(\cdot,l)\|_z = \mathcal{O}(w(l))$;
2. $\|\psi(\cdot+y,l)-\psi(\cdot,l)\|_z = \begin{cases}\mathcal{O}(w(l)),\\ \mathcal{O}(w(|y|));\end{cases}$
3. If w(l) and v(l) are defined as in Note 1, then $\|\psi(\cdot+y,l)-\psi(\cdot,l)\|_z = \mathcal{O}\left(v(|y|)\left(\frac{w(l)}{v(l)}\right)\right)$.

4 Proof of the main Theorems

Proof of the Theorem 2.1: Using Titchmarsh [5], we have

$$s_q(g;x)-g(x) = \frac{1}{2\pi}\int_0^l \phi_x(l)\frac{\sin(q+\frac{1}{2})l}{\sin\frac{l}{2}}dl.$$

Now, denoting $U.C^{\alpha,\eta}$ transform of $s_q(g;x)$ by $t_q^{U.C^{\alpha,\eta}}$,

$$
\begin{aligned}
t_q^{U.C^{\alpha,\eta}}(x)-g(x) &= \sum_{p=0}^{q} a_{q,p}\left(\frac{1}{B_p^{\alpha+\eta}}\sum_{v=0}^{p}B_{p-v}^{\alpha-1}B_v^{\eta}(s_v(g;x)-g(x))\right)\\
&= \int_0^{\pi}\phi_x(l)\frac{1}{2\pi}\sum_{p=0}^{q}a_{q,p}\frac{1}{B_p^{\alpha+\eta}}\sum_{v=0}^{p}B_{p-v}^{\alpha-1}B_v^{\eta}\frac{\sin\left(v+\frac{1}{2}\right)l}{\sin\frac{l}{2}}dl
\end{aligned}
$$

$$= \int_0^{\pi} \phi_x(l) H_q(l) dl. \tag{4.1}$$

Let

$$R_q(x) = t_q^{U.C^{\alpha,\eta}}(x) - g(x) = \int_0^{\pi} \phi_x(l) H_q(l) dl. \tag{4.2}$$

Then

$$R_q(x+y) - R_q(x) = \int_0^{\pi} \left(\phi(x+y,l) - \phi(x,l)\right) H_q(l) dl.$$

Using generalized Minkowski's inequality, Chui [4], we get

$$\begin{aligned}
\|R_q(\cdot,+y) - R_q(\cdot)\|_z &\leq \int_0^{\pi} \|\phi(\cdot+y,l) - \phi(\cdot,l)\|_z H_q(l) dl \\
&= \left(\int_0^{\frac{1}{q+1}} + \int_{\frac{1}{q+1}}^{\pi}\right) \|\phi(\cdot+y,l) - \phi(\cdot,l)\|_z H_q(l) dl \\
&= V^{(1)} + V^{(2)}. \qquad (4.3)
\end{aligned}$$

Using Lemmas 3.1 and serial number (3) of Lemma 3.5

$$\begin{aligned}
V^{(1)} &= \mathcal{O}(2q+1)\left(v(|y|)\int_0^{\frac{1}{q+1}} \frac{w(l)}{v(l)} dl\right) \\
&= \mathcal{O}\left(v(|y|)\frac{w(\frac{1}{q+1})}{v(\frac{1}{q+1})}\right). \qquad (4.4)
\end{aligned}$$

Also, using Lemmas 3.2 and serial number (3) of Lemma 3.5, we get

$$V^{(2)} = \mathcal{O}\left(\frac{1}{q+1}\int_{\frac{1}{q+1}}^{\pi} v(|y|)\frac{w(l)}{l^2 v(l)} dl\right). \tag{4.5}$$

By (4.3), (4.4), and (4.5), we have

$$\sup_{y\neq 0}\frac{\|R_q(\cdot,+y) - R_q(\cdot)\|_z}{v(|y|)} = \mathcal{O}\left(\frac{w(\frac{1}{q+1})}{v(\frac{1}{q+1})}\right) + \mathcal{O}\left(\frac{1}{q+1}\int_{\frac{1}{q+1}}^{\pi} \frac{w(l)}{l^2 v(l)} dl\right). \tag{4.6}$$

Again applying Minkowski's inequality, Lemma 3.1, Lemma 3.2, and $\|\phi(\cdot,l)\|_z = \mathcal{O}(w(l))$, we get

$$\begin{aligned}
\|R_q(\cdot)\|_z &= \left\|t_q^{U.C^{\alpha,\eta}} - g\right\|_z \\
&\leq \left(\int_0^{\frac{1}{q+1}} + \int_{\frac{1}{q+1}}^{\pi}\right) \|\phi(\cdot,l)\|_z H_q(l) dl \\
&= \mathcal{O}\left((2q+1)\int_0^{\frac{1}{q+1}} w(l) dl\right) + \mathcal{O}\left(\frac{1}{q+1}\int_{\frac{1}{q+1}}^{\pi} \frac{w(l)}{l^2} dl\right)
\end{aligned}$$

$$= \mathcal{O}\left(w\left(\frac{1}{q+1}\right)\right) + \mathcal{O}\left(\frac{1}{q+1}\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2}dl\right). \tag{4.7}$$

Now, we have

$$\|R_q(\centerdot)\|_z^v = \|R_q(\centerdot)\| + \sup_{y\neq 0}\frac{\|R_q(\centerdot,+y)-R_q(\centerdot)\|_z}{v(|y|)}. \tag{4.8}$$

Putting the values of (4.6) and (4.7) in (4.8), we get

$$\begin{aligned}\|R_q(\cdot)\|_z^v &= \mathcal{O}\left(w\left(\frac{1}{q+1}\right)\right) + \mathcal{O}\left(\frac{1}{q+1}\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2}dl\right)\\ &\quad + \mathcal{O}\left(\frac{w(\frac{1}{q+1})}{v(\frac{1}{q+1})}\right) + \mathcal{O}\left(\frac{1}{q+1}\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2v(l)}dl\right). \end{aligned}\tag{4.9}$$

By the monotonicity of $v(l)$, $w(l) = v(l)\frac{w(l)}{v(l)} \le v(\pi)\frac{w(l)}{v(l)}$ for $0<l\le\pi$, we get

$$\|R_q(\cdot)\|_z^v = \mathcal{O}\left(\frac{w(\frac{1}{q+1})}{v(\frac{1}{q+1})}\right) + \mathcal{O}\left(\frac{1}{q+1}\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2v(l)}dl\right). \tag{4.10}$$

Since $\frac{w(l)}{v(l)}$ is +ive and non-decreasing, therefore

$$\begin{aligned}\frac{1}{q+1}\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2v(l)}dl &\ge \frac{w\left(\frac{1}{q+1}\right)}{v\left(\frac{1}{q+1}\right)}\left(\frac{1}{q+1}\right)\int_{\frac{1}{q+1}}^{\pi}\frac{1}{l^2}dl\\ &\ge \frac{w\left(\frac{1}{q+1}\right)}{v\left(\frac{1}{q+1}\right)}.\end{aligned}$$

Then

$$\frac{w\left(\frac{1}{q+1}\right)}{v\left(\frac{1}{q+1}\right)} = \mathcal{O}\left(\frac{1}{q+1}\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2v(l)}dl\right). \tag{4.11}$$

From (4.10), (4.11), we get

$$\|R_q(\cdot)\|_z^v = \mathcal{O}\left(\frac{1}{q+1}\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2v(l)}dl\right),$$

$$\left\|t_q^{U.C^{\alpha,\eta}} - g\right\|_z^v = \mathcal{O}\left(\frac{1}{q+1}\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2v(l)}dl\right). \tag{4.12}$$

Proof of Theorem 2.2: The $s_q(\tilde{g};x)$ of C.F.S. is given by

$$s_q(\tilde{g};x)-\tilde{g}(x)=\frac{1}{2\pi}\int_0^{\pi}\psi_x(l)\frac{\cos(q+\frac{1}{2})l}{\sin\frac{l}{2}}dl.$$

Now, denoting $U.C^{\alpha,\eta}$ transform of $s_q(\tilde{g};x)$ by $\tilde{t}_q^{U.C^{\alpha,\eta}}$, we get

$$\begin{aligned}\tilde{t}_q^{U.C^{\alpha,\eta}}(x)-\tilde{g}(x) &= \sum_{p=0}^{q}a_{q,p}\left(\frac{1}{B_p^{\alpha+\eta}}\sum_{v=0}^{p}B_{p-v}^{\alpha-1}B_v^{\eta}(s_v(\tilde{g};x)-\tilde{g}(x))\right)\\ &= \int_0^{\pi}\psi_x(l)\left[\frac{1}{2\pi}\sum_{p=0}^{q}a_{q,p}\frac{1}{B_p^{\alpha+\eta}}\sum_{v=0}^{p}B_{p-v}^{\alpha-1}B_v^{\eta}\frac{\cos\left(v+\frac{1}{2}\right)l}{\sin\frac{l}{2}}\right]dl\\ &= \int_0^{\pi}\psi_x(l)\tilde{H}_q(l)dl.\end{aligned}$$

Let

$$\tilde{R}_q(x)=\tilde{t}_q^{U.C^{\alpha,\eta}}(x)-\tilde{g}(x)=\int_0^{\pi}\psi_x(l)\tilde{H}_q(l)dl.$$

Then

$$\tilde{R}_q(x+y)-\tilde{R}_q(x)=\int_0^{\pi}\left(\psi_x(x+y,l)-\psi_x(x,l)\right)\tilde{H}_q(l)dl.$$

Using generalized Minkowski's inequality Chui [4], we get

$$\begin{aligned}\left\|\tilde{R}_q(\cdot,+y)-\tilde{R}_q(\cdot)\right\|_z &\leq \int_0^{\pi}\left\|\psi_x(\cdot+y,l)-\psi_x(\cdot,l)\right\|_z\tilde{H}_q(l)dl\\ &= \left(\int_0^{\frac{1}{q+1}}+\int_{\frac{1}{q+1}}^{\pi}\right)\left\|\psi(\cdot+y,l)-\psi(\cdot,l)\right\|_z\tilde{H}_q(l)dl\\ &= J^{(1)}+J^{(2)}. \end{aligned}\tag{4.13}$$

Using Lemmas 3.3 and serial number (3) of Lemma 3.6,

$$\begin{aligned}J^{(1)} &= \mathcal{O}\left(v(|y|)\frac{w(\frac{1}{q+1})}{v(\frac{1}{q+1})}\int_0^{\frac{1}{q+1}}\frac{1}{l}dl\right)\\ &= \mathcal{O}\left(v(|y|)\frac{w(\frac{1}{q+1})}{v(\frac{1}{q+1})}\log(q+1)\right). \end{aligned}\tag{4.14}$$

Also, using Lemmas 3.4 and serial number (3) of Lemma 3.6,

$$J^{(2)} = \mathcal{O}\left(\int_{\frac{1}{q+1}}^{\pi}v(|y|)\frac{w(l)}{l^2v(l)}dl\right). \tag{4.15}$$

By (4.13), (4.14), and (4.15), we have

$$\underset{y\neq 0}{sup}\frac{\left\|\tilde{R}_q(\cdot,+y)-\tilde{R}_q(\cdot)\right\|_z}{v(|y|)}=\mathcal{O}\left(\frac{w(\frac{1}{q+1})}{v(\frac{1}{q+1})}\log(q+1)\right)+\mathcal{O}\left(\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2v(l)}dl\right). \tag{4.16}$$

Again applying Minkowski's inequality, Lemma 3.3, Lemma 3.4, and $\|\psi(\cdot,l)\|_z = \mathcal{O}(w(l))$, we get

$$\begin{aligned}\left\|\tilde{R}_q(\cdot)\right\|_z &= \left\|\tilde{t}_q^{U.C^{\alpha,\eta}}-\tilde{g}\right\|_z\\ &\leq \left(\int_0^{\frac{1}{q+1}}+\int_{\frac{1}{q+1}}^{\pi}\right)\|\psi(\cdot,l)\|_z\,\tilde{H}_q(l)dl\\ &= \mathcal{O}\left(\int_0^{\frac{1}{q+1}}\frac{w(l)}{l}dl\right)+\mathcal{O}\left(\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2}dl\right)\\ &= \mathcal{O}\left(w\left(\frac{1}{q+1}\right)\log(q+1)\right)+\mathcal{O}\left(\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2}dl\right). \end{aligned}\tag{4.17}$$

Now,

$$\left\|\tilde{R}_q(\cdot)\right\|_z^{(v)}=\left\|\tilde{R}_q(\cdot)\right\|+\underset{y\neq 0}{sup}\frac{\left\|\tilde{R}_q(\cdot,+y)-\tilde{R}_q(\cdot)\right\|_z}{v(|y|)}. \tag{4.18}$$

Putting the values of (4.16) and (4.17) in (4.18), we get

$$\begin{aligned}\left\|\tilde{R}_q(\cdot)\right\|_z^{v} &= \mathcal{O}\left(w\left(\frac{1}{q+1}\right)\log(q+1)\right)+\mathcal{O}\left(\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2}dl\right)\\ &+ \mathcal{O}\left(\frac{w(\frac{1}{q+1})}{v(\frac{1}{q+1})}\log(q+1)\right)+\mathcal{O}\left(\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2v(l)}dl\right).\end{aligned}$$

$$\left\|\tilde{R}_q(\cdot)\right\|_z^{v}=\mathcal{O}\left(\frac{w(\frac{1}{q+1})}{v(\frac{1}{q+1})}\log(q+1)\right)+\mathcal{O}\left(\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2v(l)}dl\right). \tag{4.19}$$

Using the fact that $\frac{w(l)}{v(l)}$ is +ive and non-decreasing,

$$\begin{aligned}\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2v(l)}dl &\geq \frac{w\left(\frac{1}{q+1}\right)}{v\left(\frac{1}{q+1}\right)}\int_{\frac{1}{q+1}}^{\pi}\frac{1}{l^2}dl\\ &\geq \frac{w\left(\frac{1}{q+1}\right)}{v\left(\frac{1}{q+1}\right)}.\end{aligned}$$

Then

$$\frac{w\left(\frac{1}{q+1}\right)}{v\left(\frac{1}{q+1}\right)} = \mathcal{O}\left(\int_{\frac{1}{q+1}}^{\pi} \frac{w(l)}{l^2 v(l)} dl\right). \tag{4.20}$$

From (4.19), (4.20), we get

$$\left\|\tilde{R}_q(\cdot)\right\|_z^v = \mathcal{O}\left(\log(q+1)\int_{\frac{1}{q+1}}^{\pi} \frac{w(l)}{l^2 v(l)} dl\right) + \mathcal{O}\left(\int_{\frac{1}{q+1}}^{\pi} \frac{w(l)}{l^2 v(l)} dl\right),$$

$$\therefore \left\|\tilde{t}_q^{U.C^{\alpha,\eta}} - \tilde{g}\right\|_z^v = \mathcal{O}\left((\log(q+1)+1)\int_{\frac{1}{q+1}}^{\pi} \frac{w(l)}{l^2 v(l)} dl\right). \tag{4.21}$$

5 Corollaries

Several known and previous results can be derived from the main results as:

5.1 Corollary

Let $0 \le \rho < \zeta \le 1$ and $\tilde{g} \in H_{(\zeta),z}$; $z \ge 1$. Then

$$\left\|\tilde{t}_q^{U.C^{\alpha,\eta}} - \tilde{g}\right\|_{(\rho),z} = \begin{cases} \mathcal{O}[(\log(q+1)e)(q+1)^{\rho-\zeta-1}] & if\, 0 \le \rho < \zeta \le 1, \\ \mathcal{O}\left[(\log(q+1)e)(\log(q+1)\pi)\right] & if\, \rho = 0,\, \zeta = 1. \end{cases} \tag{5.1}$$

Proof: Putting $w(l) = l^\zeta$, $v(l) = l^\rho$, $0 \le \rho < \zeta \le 1$ in (4.21)

$$\begin{aligned} \left\|\tilde{t}_q^{U.C^{\alpha,\eta}} - \tilde{g}\right\|_{(\rho),z} &= \mathcal{O}\left[\log(q+1)e\int_{\frac{1}{q+1}}^{\pi} l^{\zeta-\rho-2} dl\right] \\ &= \begin{cases} \mathcal{O}\left((\log(q+1)e)\int_{\frac{1}{q+1}}^{\pi} l^{\zeta-\rho-2} dl\right) & if\, 0 \le \rho < \zeta \le 1, \\ \mathcal{O}\left((\log(q+1)e)\int_{\frac{1}{q+1}}^{\pi} l^{-1} dl\right) & if\, \rho = 0,\, \zeta = 1. \end{cases} \end{aligned}$$

By solving it, we easily get the result in condition (5.1).

5.2 Corollary

Let $0 \le \rho < \zeta \le 1$, $a, b \in \mathbb{R}$ and suppose $w(l) = \frac{l^\zeta}{\left(\log\frac{1}{l}\right)^a}$, $v(l) = \frac{l^\rho}{\left(\log\frac{1}{l}\right)^b}$, $0 < l \le \pi$, $\tilde{g} \in H_z^{(w)}$, $z \ge 1$. Then

$$\left\|\tilde{t}_q^{U.C^{\alpha,\eta}} - \tilde{g}\right\|_z^{(v)} = \begin{cases} \mathcal{O}\left[\frac{\log(q+1)e(q+1)}{\{\log(q+1)\}^{b-a}}\right] & if\, \zeta = \rho \text{ and } a - b \ge -1, \\ \mathcal{O}\left[\frac{\log(q+1)e(q+1)}{\{\log(q+1)\}}\right] & if\, \zeta = \rho \text{ and } a - b = -1. \end{cases}$$

Proof: We have

$$\begin{aligned}
\left\|\tilde{t}_q^{U.C^{\alpha,\eta}}-\tilde{g}\right\|_z^{(v)} &= \mathcal{O}\left[\log(q+1)e\int_{\frac{1}{q+1}}^{\pi}\frac{l^{\zeta}}{l^2\left(\log\frac{1}{l}\right)^a\frac{l^{\rho}}{\left(\log\frac{1}{l}\right)^b}}dl\right]\\
&= \mathcal{O}\left[\log(q+1)e\int_{\frac{1}{r+1}}^{\pi}l^{\zeta-\rho-2}\left(\log\frac{1}{l}\right)^{b-a}dl\right]\\
&= \begin{cases}\mathcal{O}\left[\frac{\log(q+1)e(q+1)}{\{\log(q+1)\}^{b-a}}\right] & if\,\zeta=\rho\ and\ a-b\geq -1,\\ \mathcal{O}\left[\frac{\log(q+1)e(q+1)}{\{\log(q+1)\}}\right] & if\,\zeta=\rho\ and\ a-b=-1.\end{cases}
\end{aligned}$$

5.3 Corrolary

If $a_{q,p}=\frac{1}{(q-p+1)\log(q+1)}$, then $U.C^{\alpha,\eta}$ means reduces to $\left(H,\frac{1}{q+1}\right)C^{\alpha,\eta}$ means and the function $g\in H_z^{(w)}$ is approximated by $\left(H,\frac{1}{q+1}\right)C^{\alpha,\eta}$ means of F. S. as

$$\left\|t_q^{H.C^{\alpha,\eta}}-g\right\|_z^{(v)}=\mathcal{O}\left(\frac{1}{q+1}\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2v(l)}dl\right).$$

5.4 Corrolary

If $a_{q,p}=\frac{\theta_{q-p}}{P_q}$, then $U.C^{\alpha,\eta}$ means reduces to $N_\theta.C^{\alpha,\eta}$ and the function $g\in H_v^{(w)}$ is approximated by $N_\theta.C^{\alpha,\eta}$ means of F. S. as

$$\left\|t_q^{N_\theta.C^{\alpha,\eta}}-g\right\|_z^{(v)}=\mathcal{O}\left(\frac{1}{q+1}\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2v(l)}dl\right).$$

5.5 Corrolary

If $a_{q,p}=\frac{\theta_{q-p}\tau_p}{R_q}$, then $U.C^{\alpha,\eta}$ means reduces to $N_{\theta,\tau}.C^{\alpha,\eta}$ and the function $g\in H_v^{(w)}$ is approximated by $N_{\theta,\tau}.C^{\alpha,\eta}$ means of F. S. as

$$\left\|t_q^{N_{\theta,\tau}.C^{\alpha,\eta}}-g\right\|_z^{(v)}=\mathcal{O}\left(\frac{1}{q+1}\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2v(l)}dl\right).$$

5.6 Corollary

By using the conditions as in Corollary 5.3, the function $\tilde{g}\in H_z^{(w)}$ is approximated by $\left(H,\frac{1}{q+1}\right)C^{\alpha,\eta}$ means of C. F. S. as

$$\left\|\tilde{t}_q^{H.C^{\alpha,\eta}}-\tilde{g}\right\|_z^{(v)}=\mathcal{O}\left((\log(q+1)+1)\int_{\frac{1}{q+1}}^{\pi}\frac{w(l)}{l^2v(l)}dl\right).$$

5.7 Corrolary

If $a_{q,p} = \frac{\theta_{q-p}}{P_q}$, then $U.C^{\alpha,\eta}$ means reduces to $N_\theta.C^{\alpha,\eta}$ and the function $\tilde{g} \in H_v^{(w)}$ is approximated by $N_\theta.C^{\alpha,\eta}$ means of C. F. S. as

$$\left\| \tilde{t}_q^{N_\theta.C^{\alpha,\eta}} - \tilde{g} \right\|_z^{(v)} = \mathcal{O}\left((\log(q+1)+1) \int_{\frac{1}{q+1}}^{\pi} \frac{w(l)}{l^2 v(l)} dl \right).$$

5.8 Corrolary

If $a_{q,p} = \frac{\theta_{q-p}\tau_p}{R_q}$, then $U.C^{\alpha,\eta}$ means reduces to $N_{\theta,\tau}.C^{\alpha,\eta}$ and the function $\tilde{g} \in H_v^{(w)}$ is approximated by $N_{\theta,\tau}.C^{\alpha,\eta}$ means of C. F. S. as

$$\left\| \tilde{t}_q^{N_{\theta,\tau}.C^{\alpha,\eta}} - \tilde{g} \right\|_z^{(v)} = \mathcal{O}\left((\log(q+1)+1) \int_{\frac{1}{q+1}}^{\pi} \frac{w(l)}{l^2 v(l)} dl \right).$$

Remark 5:

1. If $z \to \infty$ then $H_z^{(w)}$ reduces to $H^{(w)}$ class. Also taking $w(l) = l^\zeta$ and $v(l) = l^\rho$ in $H^{(w)}$ class, it reduces to H_ζ class; then by taking $\rho = 0$ in H_ζ class, it reduces to $Lip\zeta$ class.

2. By taking $w(l) = l^\zeta$ and $v(l) = l^\rho$ in Result 2.1, $H_z^{(w)}$ class reduces to $H_{\zeta,z}$; then, by taking $\rho = 0$ in $H_{\zeta,z}$ class, it reduces to $Lip(\zeta, z)$ class.

6 Particular cases

1. By putting $\eta = 0$ and $\alpha = 1$ in the Theorem 2.1 and Theorem 2.2, our results reduces to the result of Nigam and Hadish [6].

2. By using Remark 5(1), $\eta = 0$, and $\alpha = 1$, in the main Theorem 2.1, our results reduces to the result of Dhakal [2].

3. By using Remark 5(2) and taking $\eta = 0$, $\alpha = 1$, $a_{r,m} = \frac{p_{r-m}q_m}{R_r}$, where $R_r = \sum_{m=0}^{r} p_m q_{r-m}$ in the main Theorem 2.1, our results reduces to the result of Kushwaha and Dhakal [8].

4. By using Remark 5(1) and taking $\eta = 0$, $\alpha = 1$, $a_{r,m} = \frac{p_{r-m}q_m}{R_r}$, where $R_r = \sum_{m=0}^{r} p_m q_{r-m}$ in the main Theorem 2.1, our results reduces to the result of Dhakal [3].

7 Conclusion

The approximation theory is a field of great practical significance. Analysis of periodic functions are important because of its applications in the engineering fields like digital communication, digital signal processing, mechanical engineering etc. These functions are derived using a polynomial approximation function and a Fourier truncated series. The closest estimate can be carried out using either a Fourier approximation or a polynomial approximation. The polynomial approximation of the function is done using Taylor series expansion, and the quality of the approximation is dependent on the number of terms utilized. Naturally, a function must be infinite times differentiable in some interval in order to have a Taylor series, which is a fairly strict requirement. Nevertheless, sines and cosines functions are used in Fourier approximation and act as far more adaptable components than powers of any variable. Sines and cosines are useful in approximating non-analytical functions as well as wildly discontinuous ones. Due to their numerous uses, Fourier approximation has taken on significant new dimensions in signal analysis.

The result focuses on approximation of functions g and $\tilde{g}$ belongings to generalized $H_z^{(w)}$, $z \geq 1$ Hölder class using Matrix-$C^{\alpha,\eta}$ ($U.C^{\alpha,\eta}$) methods of F. S. and C.F.S. respectively. As we know that product summability methods are better than the individual methods. So, here we introduce the product method ($U.C^{\alpha,\eta}$), which is better than the individual Matrix-U method and $C^{\alpha,\eta}$ method. In summary, the aim of all these measures is to minimize approximation errors and improve accuracy. The more the authors reduce the error, the stronger the results will be. The concept of product summarizability is very useful. Moreover, some previous known results become the particular cases of our Result 2.1.

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