Plane Wave Propagation in functionally graded nonlocal couple stress elastic solid medium

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In this chapter, a study of plane waves in functionally graded non local micropolar couple stress elastic medium has been undertaken. We observe that there exist four waves, namely, a longitudinal displacement wave, a longitudinal microrotational wave and a set of two coupled waves with different phase velocities. The penetration depth, specific loss, attenuation coefficient and phase velocity are evaluated numerically and shown graphically.

1 Basic equation and constitutive relation

The constitutive relations for non local micropolar couple stress elastic medium are given by

$$
\begin{aligned}
(1-\epsilon^2\nabla^2)t_{mn} &= \bar{\lambda}\delta_{mn}\epsilon_{rr} + (\bar{\mu}+\bar{K})\epsilon_{mn} + \bar{\mu}\epsilon_{nm} \\
(1-\epsilon^2\nabla^2)m_{mn} &= \bar{\alpha}\delta_{mn}\gamma_{rr} + \bar{\beta}\gamma_{mn} + \bar{\gamma}\gamma_{nm}
\end{aligned}
$$

where $\gamma_{mn}(=\phi_{m,n})$ is the curvature tensor and $\epsilon_{mn}(=u_{n,m}-\epsilon_{mnl}\phi_l)$ is the relative distortion tensor where $\mathbf{u}$ denotes displacement vector and ϕ denotes microrotation vector. $\bar{\lambda}$ and $\bar{\mu}$ are Lame's constant; $\bar{K}, \bar{\alpha}, \bar{\beta}$ and $\bar{\gamma}$, are constitutive coefficients; ∇^2 is the laplacian operator. $\epsilon(=e_0 a)$ denotes non local parameter where e_0 is material constant and a is characteristic length.

Following S. K. Tomar and A. Khurana (), the basic equations for non local micropolar couple stress elastic medium given by

$$
(\bar{\lambda}+\bar{\mu})\nabla\nabla\mathbf{u} + (\bar{\mu}+\bar{K})\nabla^2\mathbf{u} + \bar{K}\times\phi = \bar{\rho}(1-\epsilon^2\nabla^2)\ddot{\mathbf{u}} \tag{1}
$$

$$
(\bar{\alpha}+\bar{\beta})\nabla\nabla\phi + \bar{\gamma}\nabla^2\phi + \bar{K}\nabla\times\mathbf{u} - 2\bar{K}\phi = \bar{\rho}\bar{j}(1-\epsilon^2\nabla^2)\ddot{\phi} \tag{2}
$$

where $\bar{\rho}$ is the density of the medium and $\bar{j}$ denotes the coefficients of equilibrated inertia.

2 Dynamics of exponentially graded non local couple stress medium

The exponentially gradedness present in the media are supposed to be in the form of exponential function with respect to x_2-direction as

$$\bar{\lambda} = \lambda^{(1)} e^{l_1 x_2}, \quad \bar{\mu} = \mu^{(1)} e^{l_1 x_2}, \quad \bar{K} = K^{(1)} e^{l_1 x_2}, \quad \bar{\rho} = \rho^{(1)} e^{l_1 x_2},$$
$$\bar{\gamma} = \gamma^{(1)} e^{l_1 x_2}, \quad \bar{\alpha} = \alpha^{(1)} e^{l_1 x_2}, \quad \bar{\beta} = \beta^{(1)} e^{l_1 x_2}, \quad \bar{j} = j^{(1)} e^{l_1 x_2}, \tag{3}$$

where $\lambda^{(1)}, \mu^{(1)}, K^{(1)}, \alpha^{(1)}, \beta^{(1)}, \gamma^{(1)}, \rho^{(1)}$ and $j^{(1)}$ are the value of the corresponding elastic constants $\bar{\lambda}, \bar{\mu}, \bar{K}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\rho}$ and $\bar{j}$ respectively associated with the media at $x_2 = 0$. l_1 represents the exponential gradient parameter of the medium.
For a 2-D problem, we have displacement components as

$$\mathbf{u} = (u_1, 0, u_3)$$

Using Helmholtz decomposition theorem on vectors, the displacement component $\mathbf{u}$ related to the potential $\sigma(x_1, x_3, t)$ and $\Sigma(x_1, x_3, t)$ are as

$$\mathbf{u} = \nabla\sigma + \nabla \times \mathbf{Q}, \qquad \nabla \cdot \mathbf{Q} = 0$$
$$\phi = \nabla\vartheta + \nabla \times \mathbf{V}, \qquad \nabla \cdot \mathbf{V} = 0 \tag{4}$$

Using equations (3)-(4) in equations (1)-(2), we obtain

$$(\lambda^{(1)} + 2\mu^{(1)} + K^{(1)})\nabla^2\sigma - \rho^{(1)}(1 - \epsilon^2\nabla^2)\ddot{\sigma} = 0 \tag{5}$$

$$(\mu^{(1)} + K^{(1)})\nabla^2\mathbf{Q} + K^{(1)}\nabla \times \mathbf{V} - \rho^{(1)}(1 - \epsilon^2\nabla^2)\ddot{\mathbf{V}} = 0 \tag{6}$$

$$(\alpha^{(1)} + \beta^{(1)} + \gamma^{(1)})\nabla^2\vartheta - 2K^{(1)}\vartheta - \rho^{(1)} j^{(1)}(1 - \epsilon^2\nabla^2)\ddot{\vartheta} = 0 \tag{7}$$

$$\gamma^{(1)}\nabla^2\mathbf{V} + K^{(1)}\nabla \times \mathbf{Q} - 2K^{(1)}\mathbf{V} - \rho^{(1)} j^{(1)}(1 - \epsilon^2\nabla^2)\ddot{\mathbf{V}} = 0 \tag{8}$$

Equations (6) and (8) show that $\mathbf{Q}$ and $\mathbf{V}$ are coupled and equations (5) and (7) show that σ and ϑ are independent. We consider a plane wave propagation in a homogeneous isotropic non local couple stress micropolar elastic medium. For this we assume the solution of the form

$$\{\sigma, \vartheta, \mathbf{Q}, \mathbf{V}\}(a, b, \mathbf{A}, \mathbf{B}) \exp\{il(\mathbf{n} \cdot \mathbf{r} - ct)\} \tag{9}$$

where $\omega = lc$ is the frequency, l is the wave number and c is the phase velocity, a, b, $\mathbf{A}$ and $\mathbf{B}$ are undetermined amplitudes that are dependent on time and coordinate $\mathbf{r} = x_m(m = 1, 3)$, $\mathbf{n}$ is the unit vector.

Using equation (9) in equation (5), we obtain

$$C_1^2 = \left[\frac{(\lambda^{(1)} + \mu^{(1)} + K^{(1)})}{\rho^{(1)}} - \epsilon^2\omega^2\right] \tag{10}$$

Using equation (9) in equation (7), we obtain

$$C_2^2 = \left[\frac{(\alpha^{(1)} + \beta^{(1)} + \gamma^{(1)})}{\rho^{(1)} j^{(1)}}\right]\left[1 - \frac{2K^{(1)}}{\rho^{(1)} j^{(1)} \omega^2}\right] \tag{11}$$

Using equation (9) in equations (6) and (8), we obtain

$$\left[\left(\mu^{(1)} + K^{(1)}\right) l^2 - \rho^{(1)}\omega^2 - \rho^{(1)}\omega^2 - \rho^{(1)}\omega^2\epsilon^2 l^2\right] \mathbf{A} - i\, l\, K^{(1)}\, \mathbf{B} = 0 \tag{12}$$

$$i\, l\, K^{(1)}\, \mathbf{A} - \left[\gamma^{(1)} l^2 + 2K^{(1)} - \rho^{(1)} j^{(1)}\omega^2 - \rho^{(1)} j^{(1)}\omega^2\epsilon^2 l^2\right] \mathbf{B} = 0 \tag{13}$$

which yield the following polynomial equation

$$A_1 C^4 + A_2 C^2 + A_3 = 0 \tag{14}$$

where

$$A_1 = 1 - \Omega; \quad A_2 = \omega^2\epsilon^2 - b_4^2 - \frac{1}{2} b_3^2 \Omega + (1 - \Omega)(\omega^2\epsilon^2 - b_2^2 b_3^2); \quad A_3 = (\omega^2\epsilon^2 - b_2^2 b_3^2)(\omega^2\epsilon^2 - b_4^2);$$

$$\Omega = \frac{2\omega_0^2}{\omega^2}; \quad \omega_0^2 = \frac{K^{(1)}}{\rho^{(1)} j^{(1)}}; \quad b_2^2 = \frac{\mu^{(1)}}{\rho^{(1)}}; \quad b_4^2 = \frac{\gamma^{(1)}}{\rho^{(1)} j^{(1)}}; \quad b_3^2 = \frac{K^{(1)}}{\rho^{(1)}}$$

The roots of the equation (14) are given by

$$C_3^2 = \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1}; \quad C_4^2 = \frac{-A_2 - \sqrt{A_2^2 - 4A_1A_3}}{2A_1}$$

Equations (14) is cubic in C^2 with complex coefficients, whose roots will provide us the speed of the propagation waves on solving equation (14), we obtain equation (15) with four complex roots i.e. $C_i; (i = 3, 4)$. Corresponding to each of these roots, we found a set of two couples waves namely P_3 wave propagating with speed C_3 and P_4 wave propagating with speed C_4. The phase speeds (S_i), Specific loss (R_i) and Attenuation coefficients (Q_i) for waves travelling with speeds C_i can be obtained from the formulae given by Kumar et al. ()

$$S_i = \left[\frac{\omega}{|\Re(C_i)|}\right]; \quad R_i = \left(\frac{W_i'}{W_i}\right) = 4\pi\left|\frac{\Re(C_i)}{\Im(C_i)}\right|; \quad Q_i = \Im(C_i); \quad i = (1, 3, 4) \tag{15}$$

where $\Re(C_i)$ denotes the real part of C_i and $\Im(C_i)$ denotes the imaginary part of C_i.

In order to investigate the nature of these waves, inserting equation (9) in equation (4), we observed that the particle motion associated with potential $\mathbf{Q}$ is normal to the direction of wave propagation $\mathbf{n}$ and the wave associated with $\mathbf{Q}$ is transverse in nature. Also, the wave associated with $\mathbf{V}$ is transverse in nature (by using equations (6) and (8)). Since, both the waves are coupled and hence, known as coupled transverse waves. Thus, there exists two sets of coupled transverse waves, namely, transverse displacement wave and transverse microrotational wave.

3 Numerical Analysis

Following Eringen (1984), the numerical values of parameters are taken as

$\bar{K} = 1.0 \times 10^{10} Nm^{-2}; \quad \bar{\mu} = 4.0 \times 10^{10} Nm^{-2}; \quad \bar{\lambda} = 9.4 \times 10^{4} Nm^{-2}; \quad \bar{\rho} = 1.73 \times 10^{3} Kgm^{-3};$

$\bar{\gamma} = 0.779 \times 10^{-9} N; \quad j = 0.2 \times 10^{-19} m^2; \quad \bar{\alpha} = 2.33 \times 10^{-5} N; \quad \bar{\beta} = 2.48 \times 10^{-5} N; \; e_0 = 0.39$

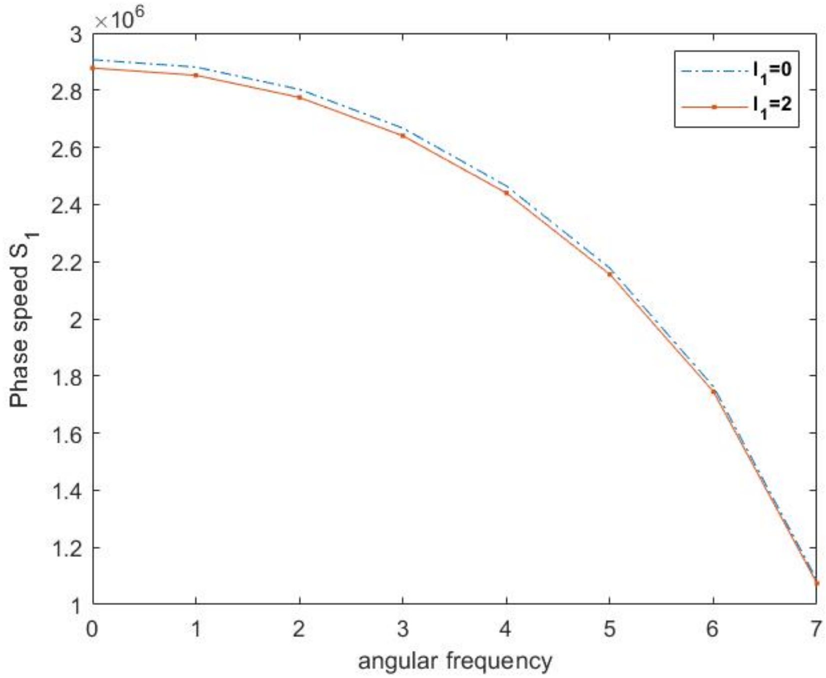

Figure 1: Distinction of magnitude of phase velocity S_1 with angular frequency

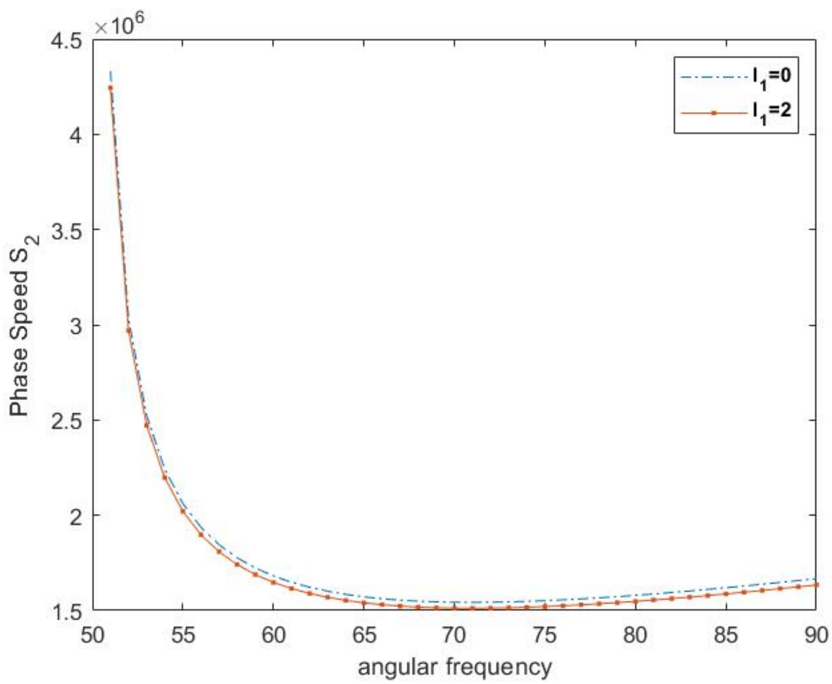

Figure 2: Distinction of magnitude of phase velocity S_2 with angular frequency

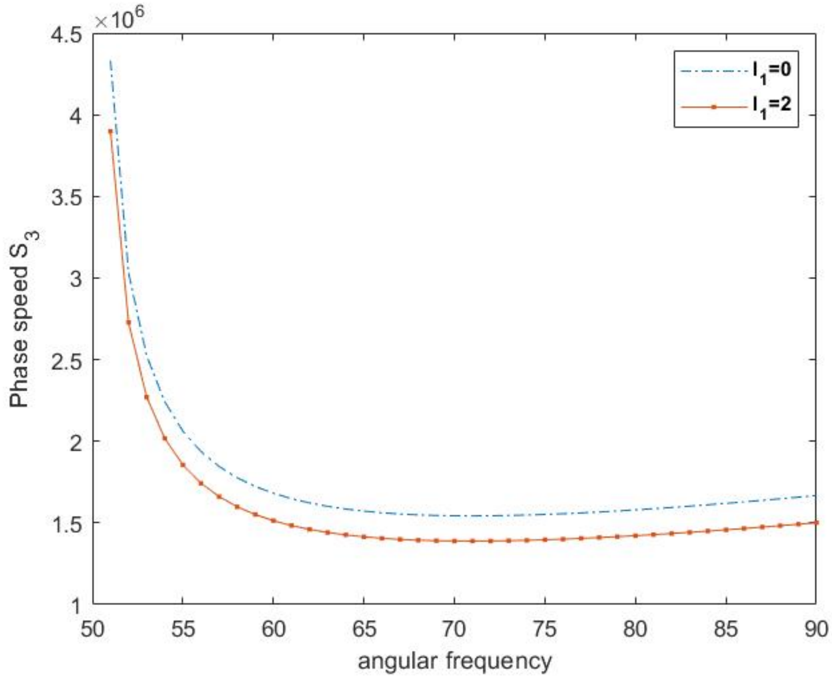

Figure 3: Distinction of magnitude of phase velocity S_3 with angular frequency

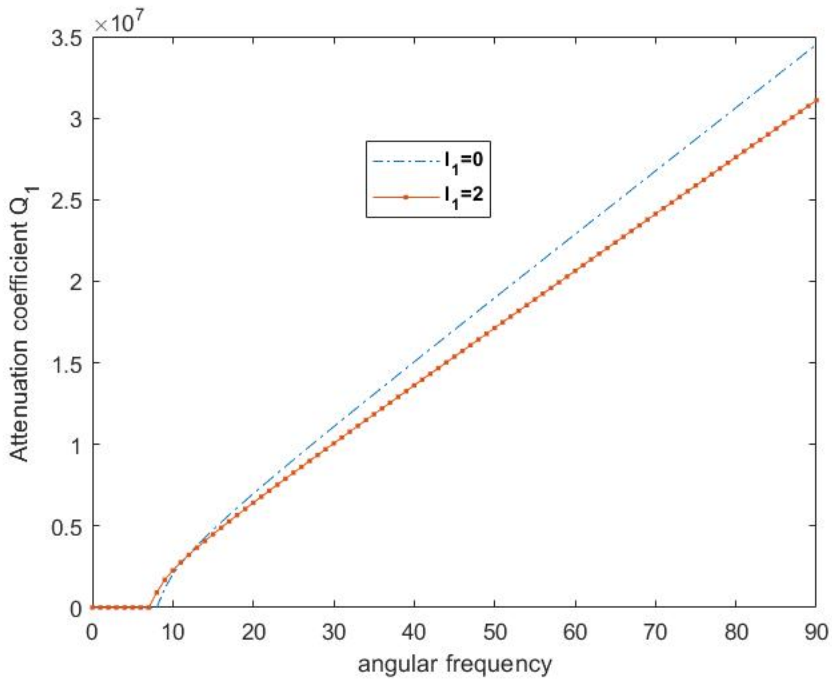

Figure 4: Distinction of magnitude of attenuation coefficients Q_1 with angular frequency

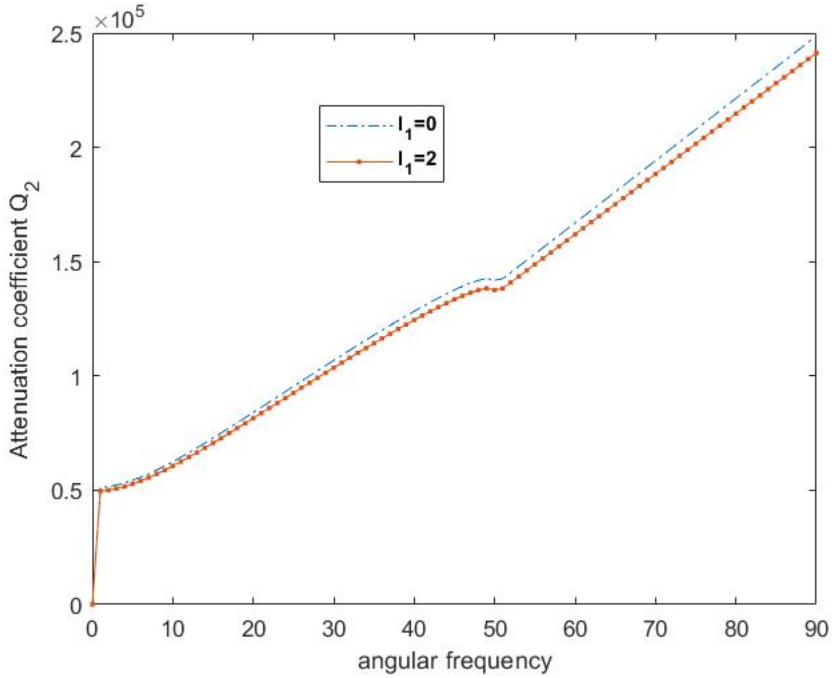

Figure 5: Distinction of magnitude of attenuation coefficients Q_2 with angular frequency

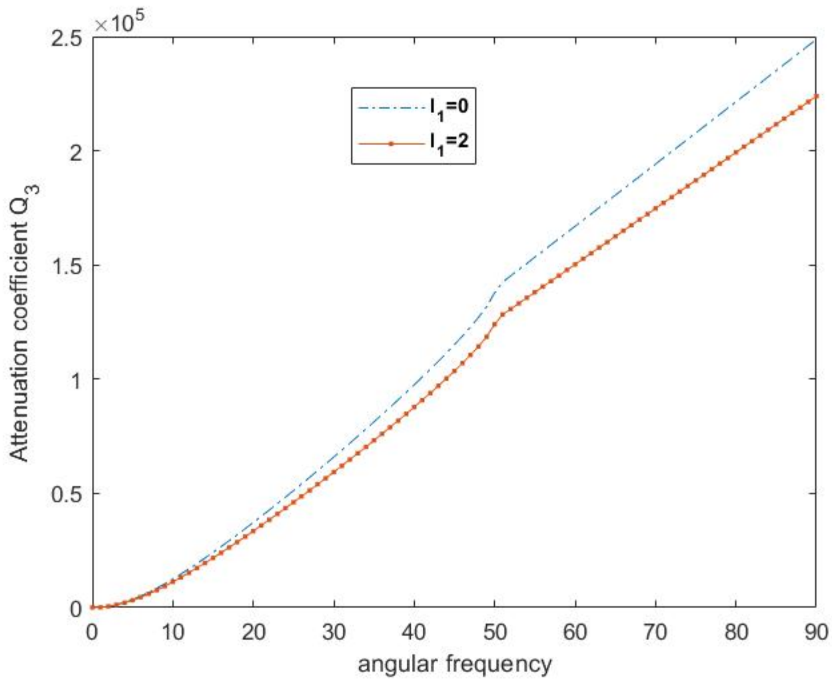

Figure 6: Distinction of magnitude of attenuation coefficients Q_3 with angular frequency

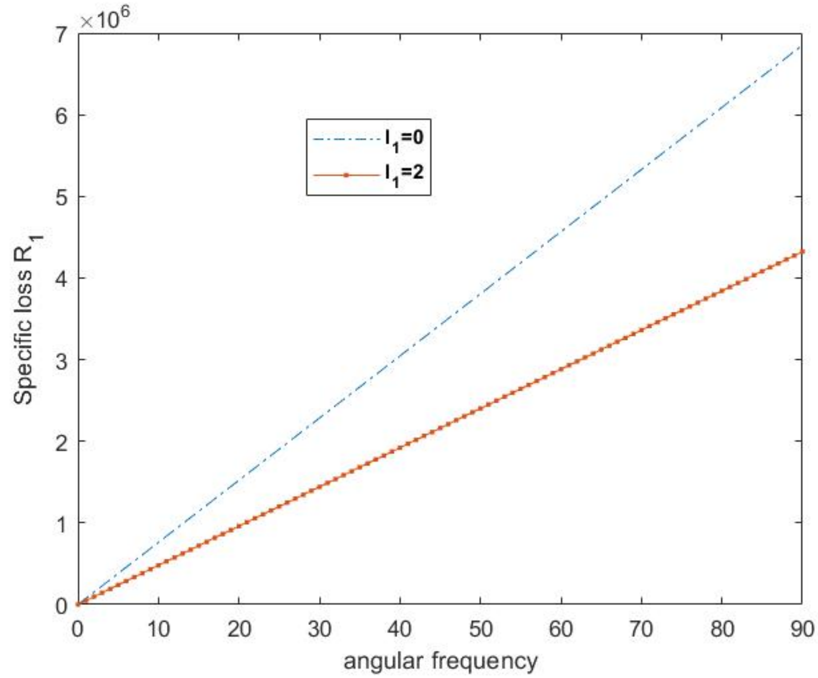

Figure 7: Distinction of magnitude of specific loss R_1 with angular frequency

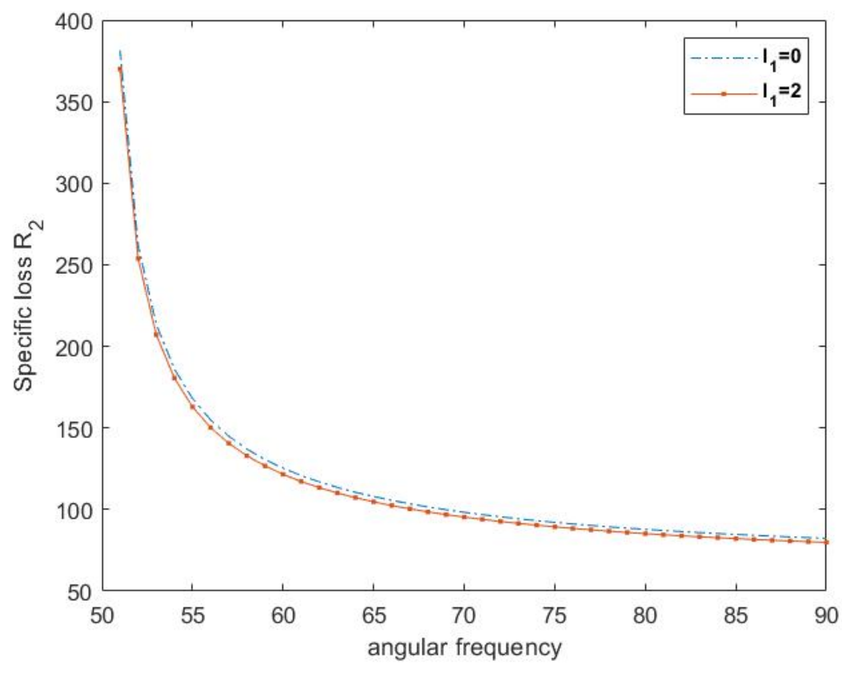

Figure 8: Distinction of magnitude of specific loss R_2 with angular frequency

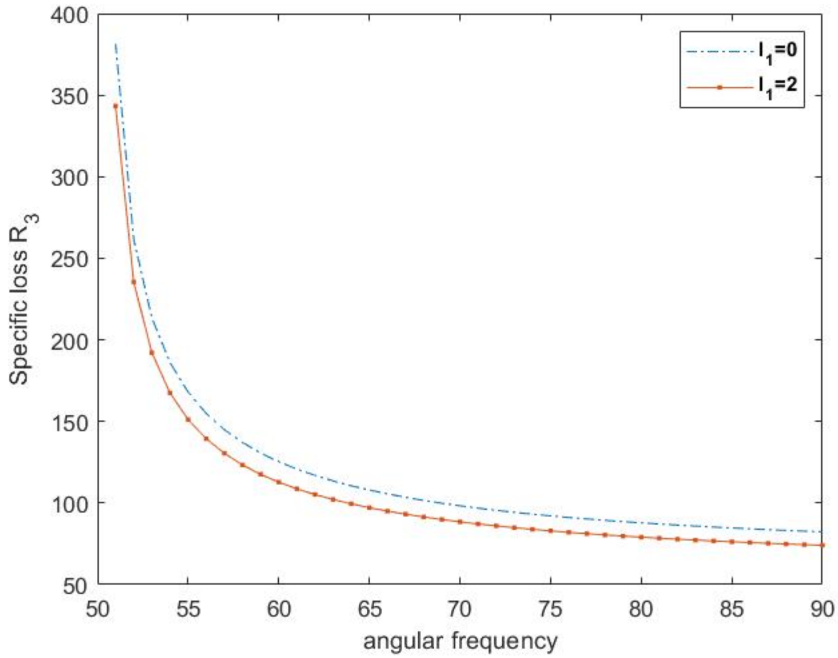

Figure 9: Distinction of magnitude of specific loss R_3 with angular frequency

Figure 1-3 depict the impact of gradient parameter on the phase velocities $S_i(i = 1, 2, 3)$ with the angular frequency ω.
On comparing the curves of phase velocities, we conclude that
1. The phase velocity S_1 decrease smoothly with the increase in value of ω. While the phase velocities S_2 and S_3 first decrease sharply; thereafter move constantly and after that increase is reported for the range $85° \leq \theta_0 \leq 90°$.
2. The phase velocities $S_i(i = 1, 2, 3)$ without gradient parameter i.e. $l_1 = 0$ dominates over the phase velocities $S_i(i = 1, 2, 3)$ with gradient parameter i.e. $l_1 \neq 0$.
3. The phase velocity S_1 obtained its maximal value at $\omega = 0°$ while S_2 and S_3 obtain their maximal value nearly at $\omega = 52°$.
4. The pattern of all the curves of the phase velocities $S_i(i = 1, 2, 3)$ without gradient parameter and with gradient parameter is similar.

Figure 4-6 show the impact of gradient parameter on the attenuation coefficients $Q_i(i = 1, 2, 3)$ with the angular frequency ω.
On comparing the curves of attenuation coefficients, we conclude that
1. The attenuation coefficient Q_1 moves constantly for the range $0° \leq \theta_0 \leq 10°$ and after that increase continuously with the increase in value of angular frequency. The attenuation coefficient Q_2 first increase sharply at $0° \leq \theta_0 \leq 2°$ then increase continuously with the in-

crease in value of angular frequency. The attenuation coefficient Q_3 increases smoothly with the increase in value of ω.

2. The attenuation coefficients $Q_i(i = 1, 2, 3)$ without gradient parameter i.e. $l_1 = 0$ dominates over the attenuation coefficients $Q_i(i = 1, 2, 3)$ with gradient parameter i.e. $l_1 \neq 0$.

3. The attenuation coefficients $Q_i(i = 1, 2, 3)$ obtained their maximal value at normal incidence.

4. The pattern of all the curves of the attenuation coefficients $Q_i(i = 1, 2, 3)$ without gradient parameter and with gradient parameter is similar.

Figure 7-9 show the impact of gradient parameter on the specific loss $R_i(i = 1, 2, 3)$ with the angular frequency ω.

On comparing the curves of specific loss, we conclude that

1. The specific loss R_1 increases smoothly with the increase in value of ω. While the specific loss R_2 and R_3 first decrease sharply and after that move smoothly with the increase in value of ω.

2. The specific loss $R_i(i = 1, 2, 3)$ without gradient parameter i.e. $l_1 = 0$ dominates over the specific loss $R_i(i = 1, 2, 3)$ with gradient parameter i.e. $l_1 \neq 0$.

3. The specific loss R_1 obtained its maximal value at normal incidence while R_2 and R_3 obtain their maximal value nearly at $\omega = 52°$.

4. The pattern of all the curves of the specific loss $R_i(i = 1, 2, 3)$ without gradient parameter and with gradient parameter is similar.

4 Conclusion

In this chapter, the study of wave propagation due to incidence of longitudinal displacement wave in functionally graded non local micropolar couple stress elastic medium has been undertaken. The impact of gradient parameter has been examined. Further, we found there are four waves, viz. a longitudinal displacement wave, a longitudinal microrotational wave and a set of two coupled waves propagating with different speeds.

The major consequences are as follows:

1. The phase speeds, attenuation coefficients and specific loss with respect to angle of incidence are calculated numerically and graphically.

2. The impact of gradient parameter are not seen significantly at grazing incidence but seen significantly at normal incidence.

3. The impact of gradient parameter on the phase speeds, attenuation coefficients and specific loss are maximum at normal incidence.

4. The curves of all phase speeds, attenuation coefficients and specific loss without gradient parameter i.e. $l_1 = 0$ dominates the curves of all phase speeds, attenuation coefficients and specific loss with gradient parameter i.e. $l_1 \neq 0$ respectively.