Fuzzy Approximations of a Functional Equation in Digital Spatial Image Crypto Techniques System

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ABSTRACT

In this paper, authors analyze the stabilities of a generalized additive functional equation in Fuzzy Banach space using direct and fixed point method and particularly application in digital spatial image crypto techniques system using MATLAB.

Keywords—additive functional equation, Ulam-Hyers stability, Fuzzy Banach space, Fixed point

I. INTRODUCTION

The study of stability problems for functional equations is related to a question of Ulam [47] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [24]. It was further generalized and excellent results obtained by number of authors [3,23,38,41]

The solution and stability of the following additive functional equations

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

$$f(2x-y) + f(x-2y) = 3f(x) - 3f(y) \tag{1.2}$$

$$f(x+y-2z) + f(2x+2y-z) = 3f(x) + 3f(y) - 3f(z) \tag{1.3}$$

$$f(x) + f(y+z) = f(x+y) + f(z) \tag{1.4}$$

$$f(2x \pm y \pm z) + f(x \pm y) + f(x \pm z) \tag{1.5}$$

$$rf\big(s(x-y)\big) + sf\big(r(y-x)\big) + (r+s)f(rx+sy) = (r+s)\big(rf(x) + sf(y)\big) \tag{1.6}$$

$$f(rx+sy) = \frac{r+s}{2}f(x+y) + \frac{r-s}{2}f(x-y) \tag{1.7}$$

$$f\left(\sum_{i=1}^{n}(-1)^{i+1}x_i\right) = \sum_{i=1}^{n}(-1)^{i+1}f(x_i) \tag{1.8}$$

$$kf(x+ky) - f(kx+y) = \frac{k(k^2-1)}{2}[f(x+y) + f(x-y)] + (k-k^3)f(x) + (k^2-1)f(y) \tag{1.9}$$

were discussed in [1, 5, 6, 14, 30, 35, 37, 41, 48].

In this paper, authors analyze the stabilities of a generalized additive functional equation

$$J\{p(Jm+n) + p(Jm-n)\} + p(m+Jn) + p(m-Jn) = p(m+n) + p(m-n) + 2J^2\,p(m) \tag{1.10}$$

in Fuzzy Banach space using direct and fixed point method. Also, an application in digital spatial image crypto techniques system using MATLAB of (1.10) is analyzed.

II. BASIC DEFINITIONS ABOUT FUZZY NORMED SPACE

In this section, the authors provide basic definitions about fuzzy normed space.

Definition 2.1 Let X be a real linear space. A function $N: X \times R \to [0,1]$ (so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in R$,

(F1) $N(x,t) = 0$ for $t \leq 0$;

(F2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;

(F3) $t > 0$ $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;

(F4) $N(x+y, s+t) \geq min\{N(x,s), N(y,t)\}$;

(F5) $N(x,\cdot)$ is a non-decreasing function on R and $\lim_{t\to\infty} N(x,t) = 1$;

(F6) For $x \neq 0$, $N(x,\cdot)$ is (upper semi) continuous on R.

The pair (X,N) is called a fuzzy normed linear space. One may regard $N(X,t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t '.

Example 2.2 Let $(X, ||.||)$ be a normed linear space. Then $N(x,t) = \begin{cases} \frac{t}{t+||x||}, & t > 0, x \in X \\ 0, & t \leq 0, x \in X \end{cases}$ is a fuzzy norm on X.

Definition 2.3 Let (X,N) be a fuzzy normed linear space. Let $\{x_n\}$be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all t > 0 .In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim_{n\to\infty} x_n = x$.

Definition 2.4 A sequence $\{x_n\}$ in X is called Cauchy if for each ε > 0 and each t > 0 there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

Definition 2.5 Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Hereafter throughout this chapter, we assume that B_1, (B_1,N) and (B_2,N') are linear space, fuzzy normed space and fuzzy banach space respectively.

III. STABILITY IN FUZZY BANACH SPACE: DIRECT METHOD

In this section, the authors analyze the stability of a generalized additive functional equation (1.10) in Fuzzy Banach space using direct method.

THEOREM: 3.1 *Let* $\mho \in \{1., -1\}$. *Let* $\partial: D^2 \to (0,\infty]$ *be a function with* $0 < \left(\frac{\mathcal{A}}{J}\right)^{\mho} < 1$

$$N'(\partial(J^{\mho k}m, 0), l) \geq N'(\mathcal{A}^{\mho k}\partial(m,0), l) \tag{3.1}$$

for all $m \in B_1$ and all $\mathcal{A} > 0$ and

$$\lim_{k\to\infty} N'(\partial(\mathcal{A}^{\mho Z}m, \mathcal{A}^{\mho Z}n), \mathcal{A}^{\mho Z}l) = 1 \tag{3.2}$$

for all $m,n \in B_1$ and all $l > 0$. Suppose that a function $p: B_1 \to B_2$ satisfying the inequality

$$N(J\{p(Jm+n) + p(Jm-n)\} + p(m+Jn) + p(m-Jn) - p(m+n) - p(m-n) - 2J^2p(m), l) \geq N'(\partial(m,n), l) \tag{3.3}$$

for all $m,n \in B_1$ and all $l > 0$.Then the limit,

$$P(m) = N - \lim_{k\to\infty} \frac{p(J^{\mho k}m)}{J^{\mho k}} \tag{3.4}$$

exists for all $m \in B_1$ and the mapping $P: B_1 \to B_2$ is a unique mapping satisfying (1.10) and

$$N\left((P(m) - p(m)), l\right) \geq N'(\partial(m,0), l.2J|J - \mathcal{A}|) \tag{3.5}$$

for all $m \in B_1$ and all $l > 0$.

PROOF: *For* $\mho$*=1.* Change (m,n) as $(m,0)$ in (3.3) and using (F3), we get

$$N(2Jp(Jm) - 2J^2p(m), l) \geq N'(\partial(m,0), l) \tag{3.6}$$

$$N\left(\left(\frac{p(Jm)}{J} - p(m)\right), \frac{l}{2J^2}\right) \geq N'(\partial(m,0), l) \tag{3.7}$$

for all $m \in B_1$ and all $l > 0$. Again, change l as $l.2J^2$ in the above inequality, it gives

$$N\left(\left(\frac{p(Jm)}{J} - p(m)\right), l\right) \geq N'(\partial(m,0), l.2J^2) \tag{3.8}$$

for all $m \in B_1$ and all $l > 0$.Substitute m as Jm in (3.8) and using (F3), we achieve

$$N\left(\left(\frac{p(J^2m)}{J^2} - \frac{p(Jm)}{J}\right), \frac{l}{J}\right) \geq N'(\partial(Jm,0), l.2J^2) \tag{3.9}$$

for all $m \in B_1$ and all $l > 0$. Again, substitute m as J^km in (3.8) and using (F3), we receive

$$N\left(\left(\frac{p(J^{k+1}m)}{J^{k+1}} - \frac{p(J^km)}{J^k}\right), \frac{l}{J^k}\right) \geq N'(\mathcal{A}^k\partial(m,0), l.2J^2) \tag{3.10}$$

for all $m \in B_1$ and all $l > 0$.Change l as $l.J^k$ in the above inequality and using (F3), we have

$$N\left(\left(\frac{p(J^{k+1}m)}{J^{k+1}} - \frac{p(J^km)}{J^k}\right), l\right) \geq N'\left(\partial(m,0), l.2J^2\left(\frac{J}{\mathcal{A}}\right)^k\right) \tag{3.11}$$

for all $m \in B_1$ and all $l > 0$. It is easy show that,

$$\sum_{i=0}^{k-1}\left(\frac{p(J^{i+1}m)}{J^{i+1}}-\frac{p(J^{i}m)}{J^{i}}\right)=\left(\frac{p(J^{k}m)}{J^{k}}-p(m)\right) \tag{3.12}$$

for all $m\in B_1$. Change l as $\frac{l}{2J^2\left(\frac{J}{\mathcal{A}}\right)^k}$ in (3.11), we obtain

$$N\left(\left(\frac{p(J^{k+1}m)}{J^{k+1}}-\frac{p(J^{k}m)}{J^{k}}\right),\frac{l}{2J^2\left(\frac{J}{\mathcal{A}}\right)^k}\right)\geq N'(\partial(m,0),l) \tag{3.13}$$

for all $m\in B_1$ and all $l>0$. From (3.12) & (3.13)

$$N\left(\sum_{i=o}^{k-1}\left(\left(\frac{p(J^{i+1}m)}{J^{i+1}}-\frac{p(J^{i}m)}{J^{i}}\right)\right),\frac{l}{2J^2}\sum_{i=o}^{k-1}\left(\frac{\mathcal{A}}{J}\right)^i\right)$$
$$\geq \min\bigcup_{i=0}^{k-1}\left\{N\left(\left(\frac{p(J^{i+1}m)}{J^{i+1}}-\frac{p(J^{i}m)}{J^{i}}\right),\frac{l}{2J^2}\left(\frac{\mathcal{A}}{J}\right)^i\right)\right\}\geq N'(\partial(m,0),l)$$

which gives,

$$N\left(\frac{p(J^{k}m)}{J^{k}}-p(m),\frac{l}{2J^2}\sum_{i=o}^{k-1}\left(\frac{\mathcal{A}}{J}\right)^i\right)\geq N'(\partial(m,0),l) \tag{3.14}$$

for all $m\in B_1$ and all $l>0$. Change m as $J^u m$ in above inequality, we get

$$N\left(\left(\frac{p(J^{k+u}m)}{J^{k+u}}-\frac{p(J^{u}m)}{J^{u}}\right),\frac{l}{2J^2}\sum_{i=o}^{k-1}\left(\frac{\mathcal{A}}{J}\right)^i\frac{1}{J^u}\right)\geq N'(\mathcal{A}^u\partial(m,0),l)$$

for all $m\in B_1$ and all $l>0$.Change l as $\mathcal{A}^u l$ in above inequality, we obtain

$$N\left(\left(\frac{p(J^{k+u}m)}{J^{k+u}}-\frac{p(J^{u}m)}{J^{u}}\right),l\right)\geq N'\left(\partial(m,0),\frac{l}{\frac{l}{2J^2}\sum_{i=o}^{k+u-1}\left(\frac{\mathcal{A}}{J}\right)^i\left(\frac{\mathcal{A}}{J}\right)^u}\right) \tag{3.15}$$

for all $m\in B_1$ and all $l>0$ and $k,u\geq 0$. Since $0<\mathcal{A}<J$ and $\sum_{i=0}^{k}\left(\frac{\mathcal{A}}{J}\right)^i$. By applying $N(x.)$ is a decreasing function on R and $\lim_{t\to\infty}N(x,t)=1$ and Cauchy criterion convergence $\left\{\frac{p(J^{k}m)}{J^{k}}\right\}$ is a Cauchy sequence in (B_2,N'). Since (B_2,N') is a fuzzy banach space. This sequence is converges to some point $P(m)\in B_2$. Let us define the function $P:B_1\to B_2$ by

$$\lim_{k\to\infty}N\left(\left(p(m)-\frac{p(J^{k}m)}{J^{k}}\right),l\right)=1 \tag{3.16}$$

Let $u=0$ and $k\to\infty$ in (3.15)

$$N\left(\left(P(m)-p(m)\right),l\right)\geq N'\left(\partial(m,0),l.2J(J-\mathcal{A})\right) \tag{3.17}$$

for all $m\in B_1$ and all $l>0$.

To prove that P satisfies (1.10).Replace m as J^km and n as $J^k n$ in (3.3), we get

$$N\left(\left[J\left\{\frac{p(J^k(Jm+n))}{J^k}+\frac{p(J^k(Jm-n))}{J^k}\right\}+\frac{p(J^k(m+Jn))}{J^k}+\frac{p(J^k(m-Jn))}{J^k}-\frac{p(J^k(m+n))}{J^k}\right.\right.$$
$$\left.\left.-\frac{p(J^k(m-n))}{J^k}-2J^2\frac{p(J^k m)}{J^k}\right],\frac{l}{J^k}\right)\geq N'(\partial(J^k\mathrm{m},J^k n),l)$$

for all $m,n\in B_1$ and all $l>0$.Change l as $l.J^k$ in the above inequality, we obtain

$$N\left(\left[J\left\{\frac{p(J^k(Jm+n))}{J^k}+\frac{p(J^k(Jm-n))}{J^k}\right\}+\frac{p(J^k(m+Jn))}{J^k}+\frac{p(J^k(m-Jn))}{J^k}-\frac{p(J^k(m+n))}{J^k}\right.\right.$$
$$\left.\left.-\frac{p(J^k(m-n))}{J^k}-2J^2\frac{p(J^k m)}{J^k}\right],l\right)$$
$$\geq N'(\partial(J^k\mathrm{m},J^k n),l.J^k) \tag{3.18}$$

$$N(J\{P(Jm+n)+P(Jm-n)\}+P(m+Jn)+P(m-Jn)-P(m+n)-P(m-n)-2J^2P(m),l)$$

$$= N\Bigg(J\{P(Jm+n)+P(Jm-n)\}+P(m+Jn)+P(m-Jn)-P(m+n)-P(m-n)-2J^2P(m)$$
$$+J\frac{p(J^k(Jm+n))}{J^k}-J\frac{p(J^k(Jm+n))}{J^k}+J\frac{p(J^k(Jm-n))}{J^k}-J\frac{p(J^k(Jm-n))}{J^k}$$
$$+\frac{p(J^k(m+Jn))}{J^k}-\frac{p(J^k(m+Jn))}{J^k}+\frac{p(J^k(m-Jn))}{J^k}-\frac{p(J^k(m-Jn))}{J^k}+\frac{p(J^k(m+n))}{J^k}$$
$$-\frac{p(J^k(m+n))}{J^k}+\frac{p(J^k(m-n))}{J^k}-\frac{p(J^k(m-n))}{J^k}+2J^2\frac{p(J^km)}{J^k}-2J^2\frac{p(J^km)}{J^k},l\Bigg)$$

$$\geq N\Bigg(J\left(P(Jm+n)-J\frac{p(J^k(Jm+n))}{J^k}\right)+J\left(P(Jm-n)-J\frac{p(J^k(Jm-n))}{J^k}\right)$$
$$+\left(P(m+Jn)-\frac{p(J^k(m+Jn))}{J^k}\right)+\left(P(m-Jn)-\frac{p(J^k(m-Jn))}{J^k}\right)$$
$$+\left(-P(m+n)+\frac{p(J^k(m+n))}{J^k}\right)+\left(-P(m-n)+\frac{p(J^k(m-n))}{J^k}\right)$$
$$+\left(-2J^2P(m)+2J^2\frac{p(J^km)}{J^k}\right)$$
$$+\Bigg(J\left\{\frac{p(J^k(Jm+n))}{J^k}+\frac{p(J^k(Jm-n))}{J^k}\right\}+\frac{p(J^k(m+Jn))}{J^k}+\frac{p(J^k(m-Jn))}{J^k}$$
$$-\frac{p(J^k(m+n))}{J^k}-\frac{p(J^k(m-n))}{J^k}-2J^2\frac{p(J^km)}{J^k}\Bigg),\frac{l}{8}+\frac{l}{8}+\frac{l}{8}+\frac{l}{8}+\frac{l}{8}+\frac{l}{8}+\frac{l}{8}+\frac{l}{8}\Bigg)$$

$$\geq min\Bigg\{N\left(J\left(P(Jm+n)-J\frac{p(J^k(Jm+n))}{J^k}\right),\frac{l}{8}\right)+N\left(J\left(P(Jm-n)-J\frac{p(J^k(Jm-n))}{J^k}\right),\frac{l}{8}\right)$$
$$+N\left(P(m+Jn)-\frac{p(J^k(m+Jn))}{J^k},\frac{l}{8}\right)+N\left(P(m-Jn)-\frac{p(J^k(m-Jn))}{J^k},\frac{l}{8}\right)$$
$$+N\left(-P(m+n)+\frac{p(J^k(m+n))}{J^k},\frac{l}{8}\right)+N\left(-P(m-n)+\frac{p(J^k(m-n))}{J^k},\frac{l}{8}\right)$$
$$+N\left(-2J^2P(m)+2J^2\frac{p(J^km)}{J^k},\frac{l}{8}\right)$$
$$+N\Bigg(J\left\{\frac{p(J^k(Jm+n))}{J^k}+\frac{p(J^k(Jm-n))}{J^k}\right\}+\frac{p(J^k(m+Jn))}{J^k}+\frac{p(J^k(m-Jn))}{J^k}$$
$$-\frac{p(J^k(m+n))}{J^k}-\frac{p(J^k(m-n))}{J^k}-2J^2\frac{p(J^km)}{J^k},\frac{l}{8}\Bigg)\Bigg\}$$

(3.19)

From (3.18) & (3.19) and applying limit in (3.19), and also using (3.2), we conclude that

$$N(J\{P(Jm+n)+P(Jm-n)\}+P(m+Jn)+P(m-Jn)-P(m+n)-P(m-n)-2J^2P(m),l) \geq min\{1,1,1,1,1,1,1,1\}$$

which implies, P satisfies the equation (1.10).

To prove $P(m)$ is unique. Let us consider $P'(m)$ be another functional equation satisfying (3.3) and (3.5)

$$N\big((P(m)-P'(m)),l\big)=N\left(\left(\frac{P(J^km)}{J^k}-\frac{P'(J^km)}{J^k}\right),l\right)$$
$$=N\left(\left(\frac{P(J^km)}{J^k}-\frac{p(J^km)}{J^k}\right)+\left(\frac{P'(J^km)}{J^k}+\frac{p(J^km)}{J^k}\right),\frac{l}{2}+\frac{l}{2}\right)$$
$$\geq min\left\{N\left(\left(\frac{P(J^km)}{J^k}-\frac{p(J^km)}{J^k}\right),\frac{l}{2}\right),\left(\left(-\frac{P'(J^km)}{J^k}+\frac{p(J^km)}{J^k}\right),\frac{l}{2}\right)\right\}$$
$$\geq min\left\{N'\left(\partial(m,0),\frac{2lJ(J-\mathcal{A})}{2}\left(\frac{J}{\mathcal{A}}\right)^k\right),N'\left(\partial(m,0),\frac{2lJ(J-\mathcal{A})}{2}\left(\frac{J}{\mathcal{A}}\right)^k\right)\right\}$$
$$\geq N'\left(\partial(m,0),\frac{2lJ(J-\mathcal{A})}{2}\left(\frac{J}{\mathcal{A}}\right)^k\right)$$

for all $m \in B_1$ and all $l > 0$. Since $\lim_{k\to\infty} \frac{2lJ(J-\mathcal{A})}{2}\left(\frac{J}{\mathcal{A}}\right)^k = \infty$, we obtain $\lim_{k\to\infty} N'\left(\partial(m,0), \frac{2lJ(J-\mathcal{A})}{2}\left(\frac{J}{\mathcal{A}}\right)^k\right) = 1$, so, $P(m) = P'(m)$. Therefore, $P(m)$ is unique. Hence the theorem is holds for $\mho = 1$.

For $\mho = -1$. Using (3.6), we get

$$N(p(Jm) - Jp(m), l) \geq N'(\partial(m,0), l.2J) \tag{3.20}$$

for all $m \in B_1$ and all $l > 0$.Replace m as $\frac{m}{J}$ in above inequality, we obtain

$$N\left(p(m) - Jp\left(\frac{m}{J}\right), l\right) \geq N'\left(\partial\left(\frac{m}{J}, 0\right), l.2J\right) \tag{3.21}$$

for all $m \in B_1$ and all $l > 0$. Change m as $\frac{m}{J^{k+1}}$ in (3.14), we obtain

$$N\left(J^k p\left(\frac{m}{J^k}\right) - J^{k+1} p\left(\frac{m}{J^{k+1}}\right), lJ^k\right) \geq N'(\partial(m,0), l.2J\mathcal{A}^{k+1}) \tag{3.22}$$

for all $m \in B_1$ and all $l > 0$. The rest of the proof is similar to that of previous case. This completes the proof.

Corollary 3.2 Suppose a function $p: B_1 \to B_2$ statisfies the inequality

$$N(J\{p(Jm+n) + p(Jm-n)\} + p(m+Jn) + p(m-Jn) - p(m+n) - p(m-n) - 2J^2 p(m), l)$$
$$\geq \begin{cases} \tau \\ \tau\{||v||^c + ||w||^c\} \end{cases} \tag{3.23}$$

for all $m, n \in B_1$ and all $l > 0$. where τ and c are constants with $\tau > 0$.Then there exists a unique mapping $P: B_1 \to B_2$ satisfying the functional equation (3.5) and

$$N\left((p(m) - P(m)), l\right) \geq \begin{cases} N'(\tau, l2J|J-1|),\ J \neq 1 \\ N'\left(\tau||m||^c, l2J|J-J^c|\right),\ J^c \neq J, c \neq 1 \end{cases} \tag{3.24}$$

for all $m \in B_1$ and all $l > 0$.

IV. STABILITY IN FUZZY BANACH SPACE: FIXED POINT METHOD

In this section, we analyze the stability of a generalized additive functional equation (1.10) in Fuzzy Banach space using fixed point method.

THEOREM: 4.1 [33] Let (X, d) be a complete generalized metric space and Let $J: X \to Y$ be a strictly contractive mapping with Lipschitz constant $L < 1$.Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all non negative integers n**.**

or there exists positive integers n_0 such that

[AFP1] $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$.

[AFP2] The sequence $\{J^n x\}$ converges to a fixed point y^* of J.

[AFP3] y^* is the unique fixed point of J in the set $Y = \{y \in X / d(J^{n_0} x, y < \infty)\}$.

[AFP4] $d(y, y^*) \leq \left(\frac{1}{1-L}\right) d(y, Jy)$ for all y ϵX.

THEOREM: 4.2 Let $p: B_1 \to B_2$ be a function for which there exist a function $\partial: D^2 \to (0, \infty]$ satisfying the inequality

$$N(J\{p(Jm+n) + p(Jm-n)\} + p(m+Jn) + p(m-Jn) - p(m+n) - p(m-n) - 2J^2 p(m), l)$$
$$\geq N'(\partial(m,n), l) \tag{4.1}$$

with the condition

$$\lim_{k\to\infty} N'(\partial(\mathcal{F}_r^k m, \mathcal{F}_r^k n), \mathcal{F}_r^k l) = 1 \tag{4.2}$$

for all $m, n \in B_1$ and all $l > 0$.If there exists $\mathcal{F} = \mathcal{F}(r)$ such that

$$\mathcal{F}_i = \begin{cases} J, & if\ r = 0 \\ \frac{1}{J}, & if\ r = 1 \end{cases} \tag{4.3}$$

has the property

$$m \to \mathcal{Z}(m) = \frac{1}{2J}\partial\left(\frac{m}{J}, 0\right) \tag{4.4}$$

$$N'\left(L\frac{\mathcal{Z}(\mathcal{F}_r m)}{\mathcal{F}_r}, l\right) = N'(\mathcal{Z}(m), l) \tag{4.5}$$

for all $m \in B_1$ and all $l > 0$.The mapping $P: B_1 \to B_2$ is a unique mapping satisfying (1.10) and

$$N\left((P(m) - p(m)), l\right) \geq N'\left(\frac{L^{1-i}}{1-L}\mathcal{Z}(m), l\right) \tag{4.6}$$

for all $m \in B_1$ and all $l > 0$.

PROOF: Consider the set ∇= $\{p|\ p: B_1 \to B_2, p(0) = 0\}$ and introduce the generalized metric on ∇ by,

$$d(p,q) = inf\{\mathcal{E} \in (0,\infty) /\ N(p(m) - q(m), l) \geq N'(\mathcal{Z}(m), \mathcal{E}l)\} \tag{4.7}$$

It is easy to see that (∇, d) is complete. Define $D: \nabla \to \nabla$ by

$$D\, p(m) = \frac{p(\mathcal{F}_r m)}{\mathcal{F}_r} \qquad \text{for all } m \in B_1 \tag{4.8}$$

Now for $p, q \in \nabla$,we have

$$d(p.q) = \mathcal{E}$$

which implies
$$N(p(m) - q(m), l) \geq N'(\mathcal{Z}(m), \mathcal{E}l)$$

$$N\left\{\mathcal{F}_r\left(\frac{p(\mathcal{F}_r m)}{\mathcal{F}_r} - \frac{q(\mathcal{F}_r m)}{\mathcal{F}_r}\right), l\right\} \geq N'(\mathcal{Z}(\mathcal{F}_r m), \mathcal{E}l)$$

$$N\left\{\left(\frac{p(\mathcal{F}_r m)}{\mathcal{F}_r} - \frac{q(\mathcal{F}_r m)}{\mathcal{F}_r}\right), \frac{l}{\mathcal{F}_r}\right\} \geq N'(\mathcal{Z}(\mathcal{F}_r m), \mathcal{E}l)$$

for all $m \in B_1$ and all $l > 0$.Replace l as $\mathcal{F}_r l$ in above inequality, we get

$$N\{(D\, p(m) - D\, q(m)), l\} \geq N'(\mathcal{Z}(\mathcal{F}_r m), \mathcal{E}\mathcal{F}_r l)$$

$$N\{(D\, p(m) - D\, q(m)), l\} \geq \mathcal{E}L$$

$$d(Dp, Dq) \leq \mathrm{L} d(p, q). \tag{4.9}$$

for all $m \in B_1$ and all $l > 0$. D is strictly contractive mapping on ∇ with Lipschtiz constant L.

It follows from (3.7), that

$$N\left(\left(\frac{p(Jm)}{J} - p(m)\right), l\right) \geq N'(\partial(m, 0), l.2J^2)$$

$$N\left(\left(\frac{p(Jm)}{J} - p(m)\right), l\right) \geq N'\left(\frac{1}{2J}\partial(m, 0), l.J\right)$$

$$d(Dp, p) \leq \mathrm{L} = L^{1-r} \tag{4.10}$$

for all $m \in B_1$ and all $l > 0$.It follows from (3.15), that

$$N\left(p(m) - Jp\left(\frac{m}{J}\right), l\right) \geq N'\left(\frac{1}{2J}\partial\left(\frac{m}{J}, 0\right), l\right)$$

$$N\left(p(m) - Jp\left(\frac{m}{J}\right), l\right) \geq N'(\mathcal{Z}(m), l)$$

$$d(p, Dp) \leq 1 = L^{1-r} \tag{4.11}$$

for all $m \in B_1$ and all $l > 0$.From (4.10) & (4.11), we conclude,

$$d(p, Dp) \leq L^{1-r} < \infty \tag{4.12}$$

which [FP1] holds. Now from the fixed point alternative [FP2] in both cases, it gives that there exists a fixed point P of D in ∇ such that

$$P(m) = N - \lim_{k\to\infty} \frac{p\left({\mathcal{F}_r}^k m\right)}{{\mathcal{F}_r}^k} \text{ for all } m \in B_1$$

Hence P satisfies the functional equation (1.10). By [FP3], since P is unique fixed point of D in the set

$$\forall = \{p \in \nabla \mid d(p, P) < \infty\}$$

Therefore P is unique function such that $N\left((p(m) - P(m)), l\right) \geq N'(\mathcal{Z}(m), \mathcal{E}l)$

Finally by [FP4], We obtain

$$d(p, P) \leq \frac{1}{1-L} d(p, Dp)$$

$$N\left((P(m) - p(m)), l\right) \geq N'\left(\frac{L^{1-i}}{1-L}\mathcal{Z}(m), l\right)$$

for all $m \in B_1$ and all $l > 0$.This completes the proof of the theorem.

Corollary 4.3:

Suppose a function $p: B_1 \to B_2$ statisfies the inequality

$$N(J\{p(Jm+n) + p(Jm-n)\} + p(m+Jn) + p(m-Jn) - p(m+n) - p(m-n) - 2J^2 p(m), l) \geq \begin{cases} \tau \\ \tau\{||v||^c + ||w||^c\} \end{cases} \tag{4.13}$$

for all $m, n \in B_1$ and all $l > 0$. where τ and c are constants with $\tau > 0$.Then there exists a unique mapping $P: B_1 \to B_2$ satisfying the functional equation (3.5) and

$$N\left((p(m) - P(m)), l\right) \geq \begin{cases} N'(\tau, l2J|J-1|), & J \neq 1 \\ N'\left(\tau||m||^c, l2J|J - J^c|\right), & J^c \neq J, c \neq 1 \end{cases} \tag{4.14}$$

for all $m \in B_1$ and all $l > 0$.

PROOF*:*

Here, $\partial(m,n) = \begin{cases} \tau \\ \tau\{||v||^c + ||w||^c\} \end{cases}$

$$N'(\partial(\mathcal{F}_r^k m, \mathcal{F}_r^k n), \mathcal{F}_r^k l) = \begin{cases} N'(\tau, \mathcal{F}_r^k l) \\ N'\left(\partial\left(\tau\{||\mathcal{F}_r^k v||^c + ||\mathcal{F}_r^k w||^c\}\right), \mathcal{F}_r^k l\right) \end{cases}$$

$$= \begin{cases} N'(\tau, \mathcal{F}_r^k l) \\ N'\left(\partial\left(\mathcal{F}_r^{kc}\tau\{||v||^c + ||w||^c\}\right), \mathcal{F}_r^k l\right) \end{cases}$$

$$= \begin{cases} N'(\tau, \mathcal{F}_r^k l) \\ N'\left(\partial\left(\tau\{||v||^c + ||w||^c\}\right), l\dfrac{\mathcal{F}_r^k}{\mathcal{F}_r^{kc}}\right) \end{cases}$$

$$= \begin{cases} \to 1 \; as \; k \to \infty \\ \to 1 \; as \; k \to \infty \end{cases}$$

Using (4.4) and (4.5).Let $\partial(m,n) = \tau$

$$N'(\mathcal{Z}(m), r) = N'\left(\frac{1}{2J}\partial\left(\frac{m}{J}, 0\right), l\right)$$

$$N'(\mathcal{Z}(m), r) = N'\left(\frac{\tau}{2J}, l\right)$$

Now,
$$N'\left(\frac{\mathcal{Z}(\mathcal{F}_r m)}{\mathcal{F}_r}, r\right) = N'\left(\frac{\tau}{\mathcal{F}_r 2J}, l\right)$$

$$N'\left(\frac{\mathcal{Z}(\mathcal{F}_r m)}{\mathcal{F}_r}, r\right) = N'\left(\mathcal{F}_r^{-1}\frac{\tau}{2J}, l\right)$$

$$N'\left(\frac{\mathcal{Z}(\mathcal{F}_r m)}{\mathcal{F}_r}, r\right) = N'(\mathcal{F}_r^{-1}\mathcal{Z}(m), l)$$

Case (i): $L = \mathcal{F}_r^{-1} = J^{-1}$ for $r = 0$

$$N\left((P(m) - p(m)), l\right) \geq N'\left(\frac{L^{1-i}}{1-L}\mathcal{Z}(m), l\right)$$

$$N\left((P(m) - p(m)), l\right) \geq N'\left(\frac{J^{-1}}{1-J^{-1}}\frac{\tau}{2J}, l\right)$$

$$N\left((P(m) - p(m)), l\right) \geq N'(\tau, l2J(1-J))$$

Similarly, $L = \mathcal{F}_r^{-1} = \left(\frac{1}{J}\right)^{-1}$ for $r = 0$, we get

$$N\left((P(m) - p(m)), l\right) \geq N'(\tau, l2J(J-1))$$

Therefore,
$$N\left((P(m) - p(m)), l\right) \geq N'(\tau, l2J|J-1|)$$

Let $\partial(m,n) = \tau\{||v||^c + ||w||^c\}$

$$N'(\mathcal{Z}(m), r) = N'\left(\frac{1}{2J}\partial\left(\frac{m}{J}, 0\right), l\right)$$

$$N'(\mathcal{Z}(m), r) = N'\left(\frac{\tau}{2J}\left\|\frac{m}{J}\right\|^c, l\right)$$

Now,
$$N'\left(\frac{\mathcal{Z}(\mathcal{F}_r m)}{\mathcal{F}_r}, r\right) = N'\left(\frac{\tau}{2J\mathcal{F}_r}\left\|\frac{\mathcal{F}_r m}{J}\right\|^c, l\right)$$

$$N'\left(\frac{\mathcal{Z}(\mathcal{F}_r m)}{\mathcal{F}_r}, r\right) = N'(\mathcal{F}_r^{c-1}\mathcal{Z}(m), l)$$

Case (ii): $L = \mathcal{F}_r^{c-1} = J^{c-1}$ for $r = 0$

$$N\left((P(m) - p(m)), l\right) \geq N'\left(\frac{L^{1-i}}{1-L}\mathcal{Z}(m), l\right)$$

$$N\left((P(m) - p(m)), l\right) \geq N'\left(\frac{J^{c-1}}{1-J^{c-1}}\frac{\tau\|m\|^c}{J^c 2J}, l\right)$$

$$N\left((P(m) - p(m)), l\right) \geq N'\left(\tau||m||^c, l2J(J^c - J)\right)$$

Similarly, $L = \mathcal{F}_r^{c-1} = \left(\frac{1}{J}\right)^{c-1}$ for $r = 0$, we get

$$N\left((P(m) - p(m)), l\right) \geq N'(\tau, l2J(J - J^c))$$

Therefore, $$N\left(\left(P(m)-p(m)\right),l\right) \geq N'\left(\tau||m||^{c}, l2J|J-J^{c}|\right)$$

This completes the proof of the corollary.

VI. FUNCTIONAL EQUATIONS BASED SPATIAL IMAGE CRYPTO TECHNIQUE

The term remote sensing takes on a specific implication dealing with space-borne imaging systems used to remotely sense the surface. Remote sensing is defined as data collected from a distance without visiting or interacting directly. When the distance between the object and viewer is large, or rather small, remote sensing approach suggests the use of spatial image. In modern days, the image based cryptographic techniques have advocated new and efficient ways to develop secure spatial image encryption techniques, see [2], [6].

In this research work, functional equations are used to improve the level of security in spatial image encryption. We apply functional equation (1.10) in digital spatial image crypto techniques system using MATLAB. An elementary idea is to encrypt the digital spatial image by applying the left hand side of (1.10). As the result, the intricate cypher image is obtained. See figures 6.1 and 6.2.

Positioning Figures and Tables: Place figures and tables at the top and bottom of columns. Avoid placing them in the middle of columns. Large figures and tables may span across both columns. Figure captions should be below the figures; table heads should appear above the tables. Insert figures and tables after they are cited in the text. Make them bold (figure and table title).

Figure 6.1. Encryption

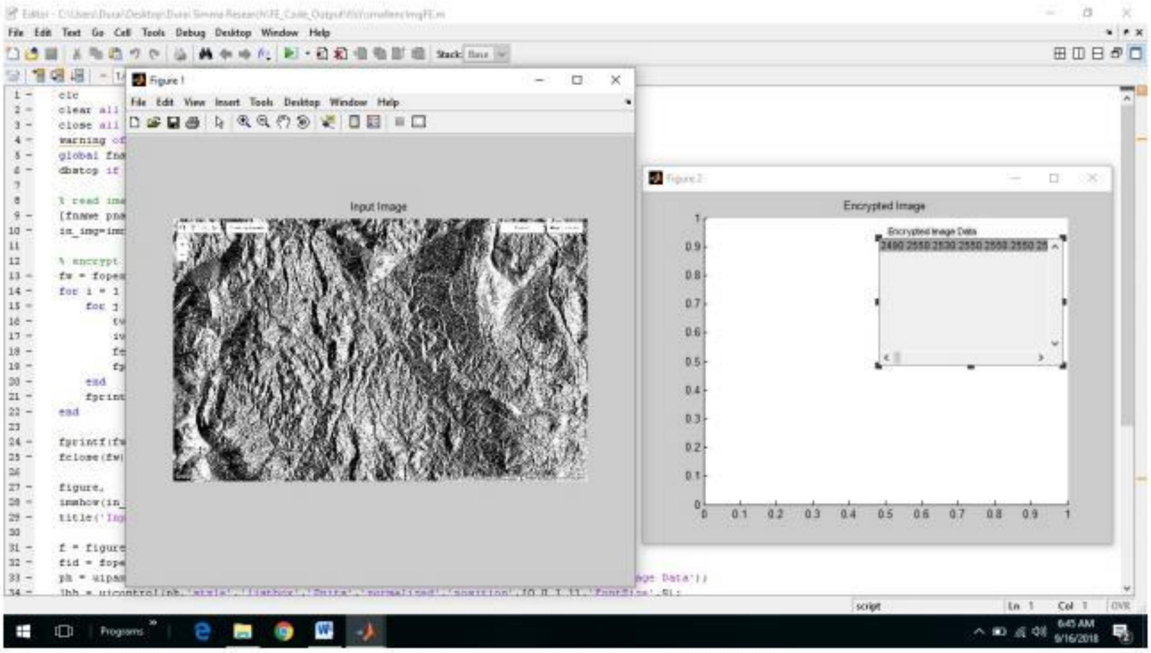

Figure 6.2. Image Encryption

When cypher image reaches the receiver, he must use right hand side of (1.1) as a key. On entering the accurate key, the MATLAB code decrypts the entire image and provides original image to the receiver. See figures 6.3 and 6.4.

Figure 6.3. Decryption

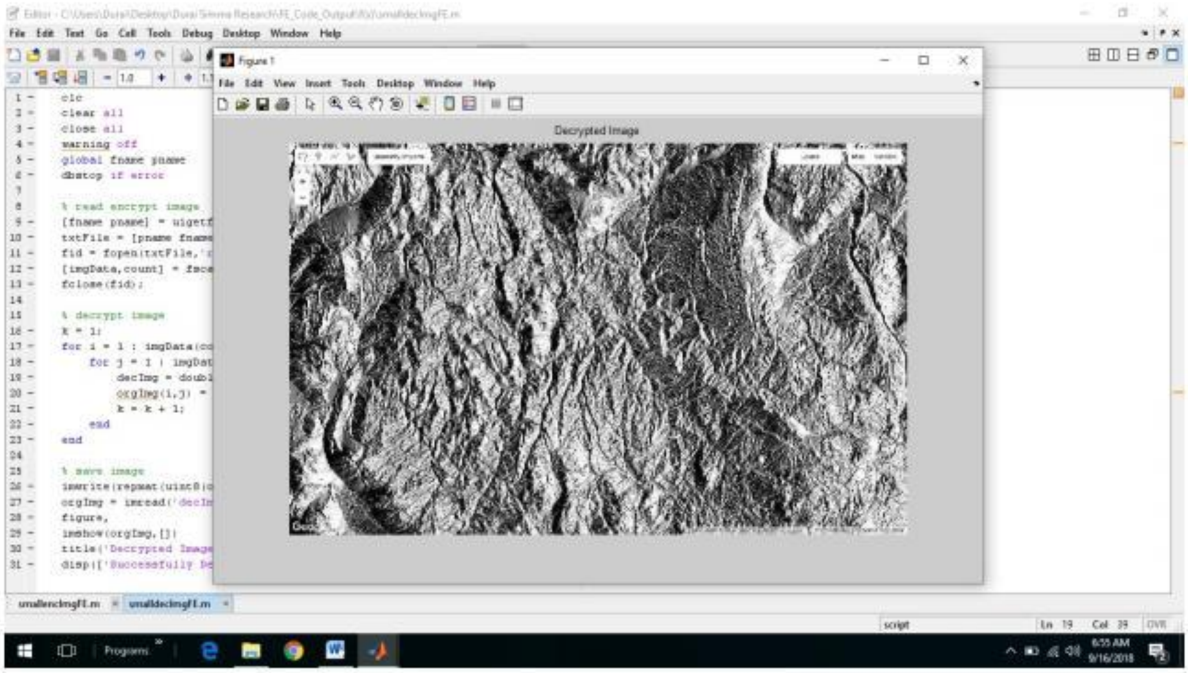

Figure 6.4. Image Decryption

4.1. Security Analysis. The distinctive approach in applying functional equations on spatial image crypto technique is, we use two different keys with same solutions that are LHS of functional equations for encrypting and RHS of functional equations for decrypting, whereas, traditional systems like DES, Triple- DES, RSA and IDEA use single key for both encryption and decryption. This uniqueness of functional equation progresses the security level of transmitting spatial image and overwhelmed traditional techniques limitations. A statistical analysis shows that the tactic for image crypto technique provides an effective and secure way for real time spatial image encryption and transmission from the cryptographic viewpoint.

V. CONCLUSION

We introduced a generalized additive functional equation, obtained its general solution and stabilities in modular space by using fixed point theory. Also, we applied (1.10) in digital spatial image crypto techniques system using MATLAB.

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