Numerical Analysis of Fractional Laplace and Extorial Transform Using Difference Operator with Shift Values

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Abstract. In this paper, we present a comprehensive study on the numerical analysis of the fractional Laplace transform and Extorial transform, leveraging the difference operator with shift values. Fractional calculus has gained significant attention due to its extensive applications in various fields, including physics, engineering, and finance. The fractional Laplace transform, a generalization of the classical Laplace transform, is instrumental in solving fractional differential equations. The Extorial transform, a lesser-known but equally powerful tool, provides an alternative method for analyzing complex systems. By incorporating difference operators with shift values, we propose novel numerical methods to approximate these transforms, enhancing the accuracy and efficiency of solutions. We demonstrate the effectiveness of our approach through several illustrative examples and compare the results with existing numerical methods. This study offers new insights and techniques that could be beneficial for researchers and practitioners working with fractional differential equations and transform methods.

Keywords: $\alpha(h)$-difference operator, fractional difference, extorial function, gamma function and polynomial factorial.

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1. Introduction

Fractional calculus, which extends the concepts of integrals and derivatives to non-integer orders, has emerged as a potent tool in mathematical modeling. Its ability to describe memory and hereditary properties of various materials and processes makes it invaluable in diverse domains such as viscoelasticity, anomalous diffusion, and control theory. Central to fractional calculus is the need for effective numerical methods to solve fractional differential equations, which are often challenging due to their non-local nature.

The fractional Laplace transform is a pivotal instrument in this context, extending the classical Laplace transform to fractional orders and providing a robust framework for analyzing linear time-invariant systems. However, the analytical solutions of fractional differential equations are not always feasible, necessitating the development of numerical methods. The Extorial transform, although not as widely known, offers complementary capabilities for transforming and solving differential equations.

The study of integrals and derivatives of arbitrary order is done through fractional calculus theory, a mathematical analysis tool that unifies and generalizes the notations of n fold integration and integer-order differentiation (El-Ajou, Arqub, Al-Zhour, & Momani (2013), Millar & Ross (1993), Oldham & Spanier (1974), Podlubny (1999)). In 1695, Leibniz is credited with developing the concept of fractional calculus; however, L'Hopital's letter asked them, "What does $\dfrac{\partial^m f(x)}{\partial x^m}$ mean if $m = \dfrac{1}{2}$?" (Diethelm, 2010, Hilfer, 2000, Lazarevic, et al., 2014, Millar & Ross, 1993, Kumar & Saxena, 2016).

Since then, numerous studies on this and similar topics have been brought up by well-known mathematicians from the mid-1900s, including Laplace, Fourier, Abel, Liouville, Riemann, Grunwald, Letnkov, Levy, Marchaud, Erdelyi, and Reiszand. Fractional calculus is a new field of mathematical analysis that emerged from these discussions and inquiries (Oldham & Spanier, 1974). Although fractional calculus is almost as old as regular calculus, its ideas and applications have just recently begun to flourish. The researchers recommend that readers study the works [1, 4, 11, 12, 14, 21, 22, 23] to track the recent advancements of this theory.

Without a doubt, one of the most useful and often utilized instruments in signal processing and analysis is the Laplace transform (LT). A significant breakthrough in the solution of fractional order equations for chaos and stability was made possible in 2009 by the Laplace transform approach, as demonstrated by [15, 16]. The significance of this breakthrough may be found in [9, 10, 17, 18, 19, 24].

This essay is structured as follows: The literature review was presented in the first section of the report. Section 2 delves into the preliminary discussions, which include relevant definitions, characteristics, and theorems pertaining to the fractional alpha difference operator. Results and a discussion of the fractional $\alpha(h)$-difference operator and its features are covered in section 3. Sections 4 and 5 showed the findings using MATLAB coding and graphics, along with the core concepts of fractional alpha Laplace and extorial transform that emerged from study. Additionally, the Laplace transform for the fractional difference equation was obtained. The results will be presented in Section 6.

2. Preliminaries

In this section, we present some basic definitions, notations and results.

Definition 2.1. For $\mathfrak{u}(\mathfrak{t})$ be the real , the forward $\mathfrak{h}$-difference operator ${}_{\mathfrak{a}}\mathfrak{d}_{\mathfrak{h}}$ on $\mathfrak{u}(\mathfrak{k})$ is defined by the relation

$$\mathfrak{d}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{t}) = \frac{\mathfrak{u}(\mathfrak{t}+\mathfrak{h}) - \alpha\mathfrak{u}(\mathfrak{t})}{\mathfrak{h}}, \tag{1}$$

while the $\alpha(\mathfrak{h})$-difference infinite sum for inverse $\alpha(\mathfrak{h})$-difference operator is defined by

$$\mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{t})\big|_{\mathfrak{a}}^{\infty} = \mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}\mathfrak{u}(\infty) - \mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{a}) = -\mathfrak{h}\sum_{\mathfrak{r}=0}^{\infty}\alpha^{-(\mathfrak{r}+1)}\mathfrak{u}(\mathfrak{a}+\mathfrak{r}\mathfrak{h}). \tag{2}$$

When $\mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{t})$ at ∞ is 0, it is obvious we get $\mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{t}) = \sum_{\mathfrak{r}=0}^{\infty}\alpha^{-(\mathfrak{r}+1)}\mathfrak{u}(\mathfrak{t}+\mathfrak{r}\mathfrak{h})$.

Example 2.2. Since $\mathfrak{d}_{\alpha(\mathfrak{h})}\frac{1}{2^{\mathfrak{t}}} = \frac{1}{\mathfrak{h}}\left[\frac{1}{2^{\mathfrak{t}+\mathfrak{h}}} - \alpha\frac{1}{2^{\mathfrak{t}}}\right] = \frac{1}{\mathfrak{h}}\frac{1}{2^{\mathfrak{t}}}\left(\frac{1}{2^{\mathfrak{h}}} - \alpha\right)$, it is clear that $\mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}\frac{1}{2^{\mathfrak{t}}} = \frac{\mathfrak{h}2^{\mathfrak{h}}}{2^{\mathfrak{k}}\left(1-\alpha 2^{\mathfrak{h}}\right)}$ and $\mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}\frac{1}{2^{\infty}} = 0$. Taking $\mathfrak{u}(\mathfrak{t}) = \frac{1}{2^{\mathfrak{t}}}$ and $\mathfrak{a} = 0$ in (2) gives

$$\mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}\frac{1}{2^{\mathfrak{t}}}\Big|_{0}^{\infty}=\mathfrak{h}\sum_{\mathfrak{r}=0}^{\infty}\alpha^{-(\mathfrak{r}+1)}\frac{1}{2^{\mathfrak{r}\mathfrak{h}}}\Rightarrow\mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}\frac{1}{2^{\infty}}-\mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}\frac{1}{2^{0}}=\mathfrak{h}\sum_{\mathfrak{r}=0}^{\infty}\alpha^{-(\mathfrak{r}+1)}\frac{1}{2^{\mathfrak{r}\mathfrak{h}}}$$

$$\Rightarrow\frac{\mathfrak{h}2^{\mathfrak{h}}}{1-\alpha 2^{\mathfrak{h}}}=\mathfrak{h}\sum_{\mathfrak{r}=0}^{\infty}\alpha^{-(\mathfrak{r}+1)}\frac{1}{2^{\mathfrak{r}\mathfrak{h}}},\quad \mathfrak{h}>0. \tag{3}$$

Which is verified by MATLAB for the particular values $\alpha = 0.5$ and $\mathfrak{h} = 3$ and the coding as follows

$\mathfrak{symsum}(3.*(0.5).\wedge(-(\mathfrak{r}+1))./2.\wedge(\mathfrak{r}.*3),\mathfrak{r},0,\mathfrak{inf}) = 3.*2.\wedge(3)./((0.5.*2.\wedge 3-1))$.

Definition 2.3. [5] For $\mathfrak{h} > 0$ and $\nu \in \mathfrak{R}$, the rising $\mathfrak{h}$-polynomial factorial function is defined by

$$\mathfrak{t}_{\mathfrak{h}}^{[\nu]}=\mathfrak{h}^{\nu}\frac{\mathfrak{d}(\frac{\mathfrak{t}}{\mathfrak{h}}+\nu)}{\mathfrak{d}(\frac{\mathfrak{t}}{\mathfrak{h}})}, \tag{4}$$

where $\mathfrak{t}_{\mathfrak{h}}^{[0]} = 1$, $\mathfrak{d}$ is the Euler gamma function and $\frac{\mathfrak{t}}{\mathfrak{h}}+\nu, \frac{\mathfrak{t}}{\mathfrak{h}}, \notin \{0,-1,-2,-3,...\}$. Similarly, the falling h-polynomial factorial function is defined by

$$\mathfrak{t}_{\mathfrak{h}}^{(\nu)}=\mathfrak{h}^{\nu}\frac{\mathfrak{d}(\frac{\mathfrak{t}}{\mathfrak{h}}+1)}{\mathfrak{d}(\frac{\mathfrak{t}}{\mathfrak{h}}+1-\nu)}, \tag{5}$$

where $\mathfrak{t}_{\mathfrak{h}}^{(0)} = 1$ and $\frac{\mathfrak{t}}{\mathfrak{h}}+1, \frac{\mathfrak{t}}{\mathfrak{h}+1}-\nu, \notin \{0,-1,-2,-3,...\}$, since the division at a pole yields zero.

Lemma 2.1. *Let* $\mathfrak{t} \in (0,\infty)$ *and* $\mathfrak{s}, \mathfrak{h} > 0$, *then we have*

$${}_{\mathfrak{a}}\mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}\frac{1}{\mathfrak{e}^{\mathfrak{s}^{1/\nu}\mathfrak{t}}}\Big|_{\mathfrak{a}}^{\infty}=\frac{\mathfrak{h}}{\mathfrak{e}^{\mathfrak{s}^{1/\nu}\mathfrak{t}}}\left(\alpha-\frac{1}{\mathfrak{e}^{\mathfrak{s}^{1/\nu}\mathfrak{h}}}\right)^{-\nu}. \tag{6}$$

Proof. The proof is similar to the Example 2.2 for the function $\mathfrak{u}(\mathfrak{t}) = \frac{1}{\mathfrak{e}^{\mathfrak{s}^{1/\nu}\mathfrak{t}}}$. □

Remark 2.4. [8] The Euler can be expressed as an infinite product Gamma function is given by $\mathfrak{d}(\mathfrak{z}) = \frac{1}{\mathfrak{z}}\prod_{\mathfrak{n}=1}^{\infty}\frac{(1+\frac{1}{\mathfrak{n}})^{\mathfrak{z}}}{(1+\frac{\mathfrak{z}}{\mathfrak{n}})}, \mathfrak{z} \notin \{0-1,-2,-3,...\}$.

Lemma 2.2. *The polynomial factorials satisfy the following identities;*

$$\text{(i)}\quad \mathfrak{d}_{\mathfrak{h}}\mathfrak{t}_{\mathfrak{h}}^{[\nu]}=\nu(\mathfrak{t}+\mathfrak{h})_{\mathfrak{h}}^{[\nu-1]}\quad \textit{and}\quad \text{(ii)}\quad \mathfrak{d}_{\mathfrak{h}}\mathfrak{t}_{\mathfrak{t}}^{(\nu)}=\nu\mathfrak{t}_{\mathfrak{h}}^{(\nu-1)}. \tag{7}$$

Proof. The identities (iv) and (v) are obtained by applying $\mathfrak{d}_{\mathfrak{h}}$ on (4) and (5) respectively. □

Theorem 2.3. *Let $\mathfrak{u}(\mathfrak{t})$ and $\mathfrak{v}(\mathfrak{t})$ be two real valued functions. Then*

$$\mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}[\mathfrak{u}(\mathfrak{t})\mathfrak{v}(\mathfrak{t})] = \mathfrak{u}(\mathfrak{t})\mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}\mathfrak{v}(\mathfrak{t}) - \mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}[\mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}\mathfrak{v}(\mathfrak{t}+\mathfrak{h})\mathfrak{d}_{\mathfrak{h}}\mathfrak{u}(\mathfrak{t})]. \tag{8}$$

Proof. From the Definition 2.1, we have

$$\mathfrak{d}_{\alpha(\mathfrak{h})}[\mathfrak{u}(\mathfrak{t})\mathfrak{w}(\mathfrak{t})] = \mathfrak{u}(\mathfrak{t})\mathfrak{d}_{\alpha(\mathfrak{h})}\mathfrak{w}(\mathfrak{t}) + \mathfrak{w}(\mathfrak{t}+\mathfrak{h})\mathfrak{d}_{\mathfrak{h}}\mathfrak{u}(\mathfrak{t}). \tag{9}$$

Taking $\mathfrak{d}_{\alpha(\mathfrak{h})}\mathfrak{w}(\mathfrak{t}) = \mathfrak{v}(\mathfrak{t})$ and $\mathfrak{w}(\mathfrak{t}) = \mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}\mathfrak{v}(\mathfrak{t})$ in equation (9), we obtain (8). □

Lemma 2.4. *Let $\mathfrak{t} \in (\mathfrak{o}, \infty)$ and $\mathfrak{s}, \nu > \mathfrak{o}$, then we have*

$$\mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}(\mathfrak{t}^{(\mu)}_{\mathfrak{h}} \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{t}}) = \sum_{\mathfrak{r}=1}^{\mu+1} \frac{\mu^{(\mathfrak{r}-1)}(\mathfrak{r}-1)^{(\mathfrak{r}-1)}}{(\mathfrak{r}-1)!} \frac{\mathfrak{h}^{\mathfrak{r}}\mathfrak{t}^{(\mu+1-\mathfrak{r})}_{\mathfrak{h}} \mathfrak{e}^{-\mathfrak{s}^{1/\nu}(\mathfrak{t}+(\mathfrak{r}-1)\mathfrak{h})}}{(\alpha - \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{h}})^{\mathfrak{r}}}. \tag{10}$$

Proof. Taking $\mathfrak{u}(\mathfrak{t}) = \mathfrak{t}^{(\mu)}_{\mathfrak{h}}$ and $\mathfrak{v}(\mathfrak{t}) = \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{t}}$ in (8), we obtain

$$\mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}(\mathfrak{t}^{(\mu)}_{\mathfrak{h}} \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{t}}) = \mathfrak{t}^{(\nu)}_{\mathfrak{h}} \frac{\mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{t}}}{(\alpha - \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{h}})} - \mathfrak{d}^{-1}_{\alpha(\mathfrak{h})}\Big(\frac{\mathfrak{e}^{-\mathfrak{s}^{1/\nu}(\mathfrak{t}+\mathfrak{h})}}{(\alpha - \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{h}})}\{\mu\mathfrak{h}\mathfrak{t}^{(\mu-1)}_{\mathfrak{h}}\}\Big).$$

Using (6), (7) and applying (9) for $\mathfrak{t}^{(\mu-1)}_{\mathfrak{h}}\mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{t}}, \mathfrak{t}^{(\mu-2)}_{\mathfrak{h}}\mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{t}}, \cdots, \mathfrak{t}^{(1)}_{\mathfrak{h}}\mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{t}}$, we get (10). □

3. Fractional $\alpha(\mathfrak{h})$-difference Operator and its Properties

Definition 3.1. Let $\nu > \mathfrak{o}$ and $\mathfrak{u}(\mathfrak{t}) \in \mathfrak{h}^{\nu}(\infty)$. The negative alpha fractional $\mathfrak{h}$-difference operator, denoted as $\mathfrak{d}^{-\nu}_{\alpha(\mathfrak{h})}$,is defined by

$$\mathfrak{d}^{-\nu}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{t})\Big|_{\mathfrak{a}}^{\infty} = \mathfrak{h}^{\nu}\sum_{\mathfrak{r}=\mathfrak{o}}^{\infty} \alpha^{-(\mathfrak{r}+\nu)} \frac{\mathfrak{d}(\nu+\mathfrak{r})}{\mathfrak{d}(\mathfrak{r}+1)\mathfrak{d}(\nu)}\mathfrak{u}(\mathfrak{a}+\mathfrak{r}\mathfrak{h}). \tag{11}$$

Definition 3.2. For a fraction $\mathfrak{o} < \nu < 1$, the Caputo $\alpha(\mathfrak{h})$-difference operator ${}_{\mathfrak{a}}\mathfrak{d}^{\nu}_{\alpha(\mathfrak{h})}$ on $\mathfrak{u}(\mathfrak{t})$ is defined by

$${}_{\mathfrak{a}}\mathfrak{d}^{\nu}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{t}) = \mathfrak{d}_{\alpha(\mathfrak{h})}({}_{\mathfrak{a}}\mathfrak{d}^{-(1-\nu)}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{t})). \tag{12}$$

Following theorem gives a closed form function for fractional difference of $\frac{1}{\mathfrak{c}^{\mathfrak{t}}}$.

Theorem 3.3. Let $\nu > \mathfrak{o}$ be a fraction, $\mathfrak{t} \in [\mathfrak{o}, \infty)$ and $\mathfrak{c} \neq \mathfrak{o}$. Then, we have

$${}_{\mathfrak{a}}\mathfrak{d}^{-\nu}_{\alpha(\mathfrak{h})}\frac{1}{\mathfrak{c}^{\mathfrak{t}}}\Big|_{\mathfrak{a}}^{\infty} = \frac{\mathfrak{h}^{\nu}}{\mathfrak{c}^{\mathfrak{t}}}\left(\alpha - \frac{1}{\mathfrak{c}^{\mathfrak{h}}}\right)^{-\nu}\Big|_{\mathfrak{a}}^{\infty} = -\mathfrak{h}^{\nu}\sum_{\mathfrak{r}=\mathfrak{o}}^{\infty}\alpha^{-(\mathfrak{r}+\nu)}\frac{\mathfrak{d}(\nu+\mathfrak{r})}{\mathfrak{d}(\nu)\mathfrak{d}(\mathfrak{r}+1)}\frac{1}{\mathfrak{c}^{\mathfrak{a}+\mathfrak{r}\mathfrak{h}}}. \tag{13}$$

Proof. The binomial expansion for rational index gives ${}_{\mathfrak{a}}\mathfrak{d}^{\nu}_{\alpha(\mathfrak{h})}\frac{1}{\mathfrak{c}^{\mathfrak{t}}} = \frac{\mathfrak{h}^{-\nu}}{\mathfrak{c}^{\mathfrak{t}}}\left(\alpha - \frac{1}{\mathfrak{c}^{\mathfrak{h}}}\right)^{\nu}$. Now (13) follows by taking ${}_{\mathfrak{a}}\mathfrak{d}^{-\nu}_{\mathfrak{h}}$ on both sides and equating with (11). □

In the following illustrative example, the identity (13) is verified by MATLAB.

Example 3.4. Taking $\mathfrak{c} = 3$, $\mathfrak{h} = 2, \mathfrak{a} = 4, \alpha = 3$ and $\nu = 0.3$ in (13) gives

$$ {}_{4}\mathfrak{d}^{-0.3}_{3(2)}\frac{1}{3^{\mathfrak{t}}}\Big|_{4}^{\infty} = \frac{2^{0.3}}{3^{\mathfrak{t}}}\left(3 - \frac{1}{3^{2}}\right)^{-0.3}\Big|_{4}^{\infty} = -2^{0.3}\sum_{\mathfrak{r}=0}^{\infty} 3^{-(\mathfrak{r}+0.3)}\frac{\mathfrak{d}(0.3+\mathfrak{r})}{\mathfrak{d}(0.3)\mathfrak{d}(\mathfrak{r}+1)}\frac{1}{3^{4+2\mathfrak{r}}}. $$

The values of the above relations are verified by MATLAB with the coding as given below:

$2.\wedge(0.3).*\mathfrak{symsum}(3.\wedge(-(\mathfrak{r}+0.3)).*\mathfrak{gamma}(0.3+\mathfrak{r})./(\mathfrak{gamma}(0.3).*\mathfrak{gamma}(\mathfrak{r}+1).* 3.\wedge(4+\mathfrak{r}.*2)),\mathfrak{r},0,\mathfrak{inf}) = 2.\wedge(0.3)./((3.\wedge 4).*(3-(1./3.\wedge 2)).\wedge(0.3))$.

Corollary 3.5. For $\nu > 0$, $\mathfrak{t} \in [0,\infty), \mathfrak{s} > 0$ and $\mathfrak{h} > 0$, we have

$$ {}_{\mathfrak{a}}\mathfrak{d}^{-\nu}_{\alpha(\mathfrak{h})}\frac{1}{\mathfrak{e}^{\mathfrak{s}^{1/\nu}\mathfrak{t}}}\Big|_{\mathfrak{a}}^{\infty} = \frac{\mathfrak{h}^{\nu}}{\mathfrak{e}^{\mathfrak{s}^{1/\nu}\mathfrak{t}}}\left(\alpha - \frac{1}{\mathfrak{e}^{\mathfrak{s}^{1/\nu}\mathfrak{h}}}\right)^{-\nu}\Big|_{\mathfrak{a}}^{\infty} = -\mathfrak{h}^{\nu}\sum_{\mathfrak{r}=0}^{\infty}\alpha^{-(\mathfrak{r}+\nu)}\frac{\mathfrak{d}(\nu+\mathfrak{r})}{\mathfrak{d}(\nu)\mathfrak{d}(\mathfrak{r}+1)}\frac{1}{\mathfrak{e}^{\mathfrak{s}^{1/\nu}\mathfrak{a}+\mathfrak{r}\mathfrak{h}}}. \tag{14} $$

Proof. The proof follows by replacing $\frac{1}{\mathfrak{c}^{\mathfrak{t}}}$ by $\frac{1}{\mathfrak{e}^{\mathfrak{s}^{1/\nu}\mathfrak{t}}}$ in Theorem 3.3. □

4. Fractional Alpha Laplace Transform by $\alpha(\mathfrak{h})$-difference Operator

In this section, we develop a new type fractional Laplace transform with examples. Our findings and the outcomes obtained by applying on $\mathfrak{h}$-factorial functions are analyed and verified by MATLAB with diagrams.

Definition 4.1. Let $\mathfrak{u}(\mathfrak{t})$ be the real valued function. Then the Fractional Alpha Laplace Transform(FALT) is defined as

$$ \mathfrak{L}^{\nu}_{\alpha(\mathfrak{h})}[\mathfrak{u}(\mathfrak{t})] = \mathfrak{d}^{-\nu}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{t})\mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{t}}\Big|_{0}^{\infty} = -\mathfrak{h}^{\nu}\sum_{\mathfrak{r}=0}^{\infty}\alpha^{-(\mathfrak{r}+\nu)}\frac{\mathfrak{d}(\nu+\mathfrak{r})}{\mathfrak{d}(\nu)\mathfrak{d}(\mathfrak{r}+1)}\mathfrak{u}(\mathfrak{r}\mathfrak{h})\mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{r}\mathfrak{h}}. \tag{15} $$

Theorem 4.2. For any fraction $\nu > 0$, $\mathfrak{h} > 0, \mu \in \mathfrak{N}(1)$, and $\mathfrak{k} \in [0,\infty)$, we have

$$ \mathfrak{d}^{-\nu}_{\alpha(\mathfrak{h})}[\mathfrak{t}^{(\mu)}_{\mathfrak{h}}\mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{t}}] = \sum_{\mathfrak{r}=1}^{\mu+1}\frac{\mathfrak{h}^{\nu}\mu^{(\mathfrak{r}-1)}}{(\mathfrak{r}-1)!}\frac{(\nu+\mathfrak{r}-2)^{(\mathfrak{r}-1)}\mathfrak{h}^{\mathfrak{r}-1}}{(\alpha-\mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{h}})^{\mathfrak{r}+\nu-1}}\frac{\mathfrak{t}^{(\mu+1-\mathfrak{r})}_{\mathfrak{h}}}{\mathfrak{e}^{\mathfrak{s}^{1/\nu}(\mathfrak{t}+(\mathfrak{r}-1)\mathfrak{h})}}. \tag{16} $$

Proof. Since the order is fraction, we shall prove (16) by showing $\mathfrak{d}_{\alpha(\mathfrak{h})}[\mathfrak{h}(\mathfrak{t})] = \mathfrak{d}_{\alpha(\mathfrak{h})}[\mathfrak{g}(\mathfrak{t})]$, where $\mathfrak{h}(\mathfrak{t}) = \mathfrak{d}^{-\nu}_{\alpha(\mathfrak{h})}[\mathfrak{t}^{(\mu)}_{\mathfrak{h}} \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{t}}]$ and

$$\mathfrak{g}(\mathfrak{t}) = \sum_{\mathfrak{r}=1}^{\mu+1} \frac{\mathfrak{h}^{\nu}\mu^{(\mathfrak{r}-1)}}{(\mathfrak{r}-1)!} \frac{(\nu+\mathfrak{r}-2)^{(\mathfrak{r}-1)}\mathfrak{h}^{\mathfrak{r}-1}}{(\alpha - \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{h}})^{\mathfrak{r}+\nu-1}} \frac{\mathfrak{t}^{(\mu+1-\mathfrak{r})}_{\mathfrak{h}}}{\mathfrak{e}^{\mathfrak{s}^{1/\nu}(\mathfrak{t}+(\mathfrak{r}-1)\mathfrak{h})}}.$$

From Lemma 2.4, we arrive

$$\mathfrak{d}_{\alpha(\mathfrak{h})}\mathfrak{g}(\mathfrak{t}) = \sum_{\mathfrak{r}=1}^{\mu+1} \frac{\mathfrak{h}^{\nu}\mu^{(\mathfrak{r}-1)}}{(\mathfrak{r}-1)!} \frac{(\nu+\mathfrak{r}-2)^{(\mathfrak{r}-1)}\mathfrak{h}^{\mathfrak{r}-1}}{(\alpha - \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{h}})^{\mathfrak{r}+\nu-1}} \mathfrak{d}_{\alpha(\mathfrak{h})} \frac{\mathfrak{t}^{(\mu+1-\mathfrak{r})}_{\mathfrak{h}}}{\mathfrak{e}^{\mathfrak{s}^{1/\nu}(\mathfrak{t}+(\mathfrak{r}-1)\mathfrak{h})}}$$

$$= \sum_{\mathfrak{r}=1}^{\mu+1} \frac{\mathfrak{h}^{\nu-1}\mu^{(\mathfrak{r}-1)}}{(\mathfrak{r}-1)!} \frac{(\nu+\mathfrak{r}-3)^{(\mathfrak{r}-1)}\mathfrak{h}^{\mathfrak{r}-1}}{(\alpha - \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{h}})^{\mathfrak{r}+\nu-2}} \frac{\mathfrak{t}^{(\mu+1-\mathfrak{r})}_{\mathfrak{h}}}{\mathfrak{e}^{\mathfrak{s}^{1/\nu}(\mathfrak{t}+(\mathfrak{r}-1)\mathfrak{h})}}. \tag{17}$$

Again from Lemma 2.4, we obtain

$$\mathfrak{d}_{\alpha(\mathfrak{h})}[\mathfrak{h}(\mathfrak{t})] = \mathfrak{d}_{\alpha(\mathfrak{h})}({}_{\mathfrak{a}}\mathfrak{d}^{-\nu}_{\alpha(\mathfrak{h})})[\mathfrak{t}^{(\mu)}_{\mathfrak{h}} \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{t}}] = {}_{\mathfrak{a}}\mathfrak{d}^{-(\nu-1)}_{\alpha(\mathfrak{h})}[\mathfrak{t}^{(\mu)}_{\mathfrak{h}} \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{t}}]$$

$$= \sum_{\mathfrak{r}=1}^{\mu+1} \frac{\mathfrak{h}^{\nu-1}\mu^{(\mathfrak{r}-1)}}{(\mathfrak{r}-1)!} \frac{(\nu+\mathfrak{r}-3)^{(\mathfrak{r}-1)}\mathfrak{h}^{\mathfrak{r}-1}}{(\alpha - \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{h}})^{\mathfrak{r}+\nu-2}} \frac{\mathfrak{t}^{(\mu+1-\mathfrak{r})}_{\mathfrak{h}}}{\mathfrak{e}^{\mathfrak{s}^{1/\nu}(\mathfrak{t}+(\mathfrak{r}-1)\mathfrak{h})}}. \tag{18}$$

The proof follows by comparing (17) and (18). □

Corollary 4.3. Let $\mu \in \mathfrak{N}(1)$, then we have

$$\mathfrak{d}^{-\nu}_{\alpha(\mathfrak{h})}[\mathfrak{t}^{(\mu)}_{\mathfrak{h}} \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{t}}]\Big|_{0}^{\infty} = \frac{\mathfrak{h}^{\nu}\mu^{(\mu)}}{\mu!} \frac{(\nu+\mu-1)^{(\mu)}\mathfrak{h}^{\mu}}{(\alpha - \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{h}})^{\mu+\nu}} \frac{1}{\mathfrak{e}^{\mathfrak{s}^{1/\nu}\mu\mathfrak{h}}}. \tag{19}$$

Proof. The proof follows by applying the limits 0 to ∞ in (16). □

Corollary 4.4. The following identity hold for $\mu \in N(1)$,

$$\mathfrak{L}^{\nu}_{\alpha(\mathfrak{h})}[\mathfrak{t}^{(\mu)}_{\mathfrak{h}}] = \frac{\mathfrak{h}^{\mu+\nu}\mu^{(\mu)}}{\mu!} \frac{(\nu+\mu-1)^{(\mu)}}{(\alpha - \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{h}})^{\mu+\nu}} \frac{1}{\mathfrak{e}^{\mathfrak{s}^{1/\nu}\mu\mathfrak{h}}} = \mathfrak{h}^{\nu} \sum_{\mathfrak{r}=0}^{\infty} \alpha^{-(\mathfrak{r}+\nu)} \frac{\mathfrak{d}(\nu+\mathfrak{r})(\mathfrak{r}\mathfrak{h})^{(\mu)}_{\mathfrak{h}} \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{r}\mathfrak{h}}}{\mathfrak{d}(\nu)\mathfrak{d}(\mathfrak{r}+1)}. \tag{20}$$

Proof. Since, we have

$$\mathfrak{L}^{\nu}_{\alpha(\mathfrak{h})}[\mathfrak{t}^{(\mu)}_{\mathfrak{h}}] = \mathfrak{d}^{-\nu}_{\alpha(\mathfrak{h})}[\mathfrak{t}^{(\mu)}_{\mathfrak{h}} \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{t}}]\Big|_{0}^{\infty} = \frac{\mathfrak{h}^{\nu}\mu^{(\mu)}}{\mu!} \frac{(\nu+\mu-1)^{(\mu)}\mathfrak{h}^{\mu}}{(\alpha - \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{h}})^{\mu+\nu}} \frac{1}{\mathfrak{e}^{\mathfrak{s}^{1/\nu}\mu\mathfrak{h}}}. \tag{21}$$

Now (20) follows from (15). □

Example 4.5. By taking $\mu = 3$ in (20), we get fractional alpha Laplace transform of $\mathfrak{t}^{(3)}_{\mathfrak{h}}$ as

$$\mathfrak{L}^{\nu}_{\alpha(\mathfrak{h})}[\mathfrak{t}^{(3)}_{\mathfrak{h}}] = \frac{\mathfrak{h}^{\nu+3}(\nu)(\nu+1)(\nu+2)}{(\alpha - \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{h}})^{3+\nu}\mathfrak{e}^{3\mathfrak{s}^{1/\nu}\mathfrak{h}}} = \mathfrak{h}^{\nu} \sum_{\mathfrak{r}=0}^{\infty} \alpha^{-(\mathfrak{r}+\nu)} \frac{\mathfrak{d}(\nu+\mathfrak{r})}{\mathfrak{d}(\nu)\mathfrak{d}(\mathfrak{r}+1)} (\mathfrak{r}\mathfrak{h})^{(3)}_{\mathfrak{h}} \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{r}\mathfrak{h}}. \tag{22}$$

Now for $\nu = \mathfrak{0.4}, \mathfrak{s} = \mathfrak{5}, \alpha = \mathfrak{0.2}$ and $\mathfrak{h} = \mathfrak{2}$ (22) is verified by MATLAB.

The following diagrams shows that the FALT for the function(signal) $\mathfrak{u}(\mathfrak{t}) = \mathfrak{t}_{\mathfrak{2}}^{(\mathfrak{3})}$ in time domain and the in the frequency(s) domain the nature of outcomes are generated by varying the factors α and ν simultaneously using MATLAB are shown below.

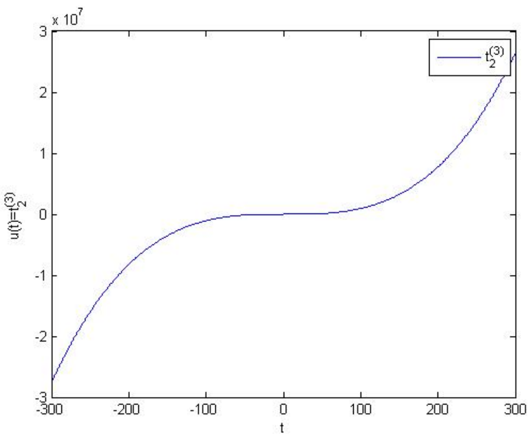

FIGURE 1. Time(t) Domain

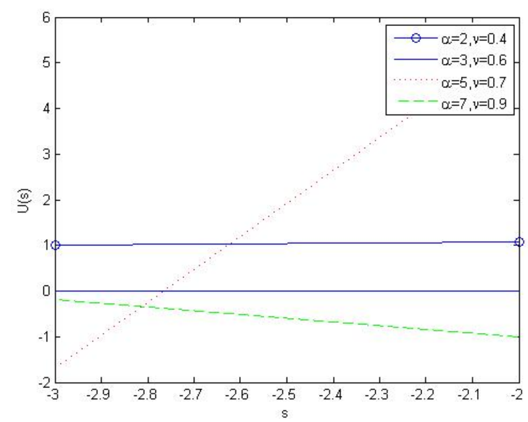

FIGURE 2. Frequency(s) Domain

4.1. **Laplace Transform of Fractional Difference Equations.** Since the Leibniz rule for the product of two functions like $\mathfrak{u}(\mathfrak{t})$ and $\mathfrak{v}(\mathfrak{t})$ for the $\mathfrak{Z}$ domain as given by $\mathfrak{d}^{\nu}[\mathfrak{u}(\mathfrak{t})\mathfrak{v}(\mathfrak{t})] = \sum_{\mathfrak{r}=\mathfrak{0}}^{\infty} \binom{\nu}{\mathfrak{r}} \mathfrak{d}^{\nu-\mathfrak{r}}\mathfrak{u}(\mathfrak{t})\mathfrak{d}^{\mathfrak{r}}\mathfrak{v}(\mathfrak{t}+\nu-\mathfrak{r})$. Now in this section, we present the product formula on fractional difference operator $\mathfrak{d}^{\nu}_{\alpha(\mathfrak{h})}$ on $\mathfrak{h}\mathfrak{Z}$ domain such as $\mathfrak{d}^{\nu}_{\alpha(\mathfrak{h})}[\mathfrak{u}(\mathfrak{t})\mathfrak{v}(\mathfrak{t})] = \sum_{\mathfrak{r}=\mathfrak{0}}^{\infty} \binom{\nu}{\mathfrak{r}} \mathfrak{d}^{\nu-\mathfrak{r}}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{t})\mathfrak{d}^{\mathfrak{r}}_{\alpha(\mathfrak{h})}\mathfrak{v}(\mathfrak{t}+(\nu-\mathfrak{r})\mathfrak{h})$.

The following theorem gives the important role on solving fractional difference equation by Laplace transform.

Theorem 4.1. *Let $\mathfrak{u}(\mathfrak{t})$ be the real valued function and $\alpha, \mathfrak{s}, \mathfrak{h}, \nu > \mathfrak{0}$. Then we have*

$$\mathfrak{L}_{\alpha(\mathfrak{h})}[\mathfrak{d}^{\nu}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{t})] = \frac{(\alpha - \mathfrak{e}^{-\mathfrak{s}^{\mathfrak{1}/\nu}\mathfrak{h}})^{\nu}}{\mathfrak{h}^{\nu}}\mathfrak{L}_{\alpha(\mathfrak{h})}[\mathfrak{u}(\mathfrak{t}+\nu\mathfrak{h})] - \sum_{\mathfrak{r}=\mathfrak{1}}^{\infty} \frac{(\alpha - \mathfrak{e}^{-\mathfrak{s}^{\mathfrak{1}/\nu}\mathfrak{h}})^{\nu-\mathfrak{r}}}{\mathfrak{h}^{\nu-\mathfrak{r}}}\mathfrak{d}^{\mathfrak{r}-\mathfrak{1}}_{\alpha(\mathfrak{h})}\mathfrak{u}((\nu-\mathfrak{r})\mathfrak{h}). \tag{23}$$

Proof. Taking $\mathfrak{u}(\mathfrak{t}) = \mathfrak{d}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{t})$ in Definition 15, we get
$\mathfrak{L}^{\nu}_{\alpha(\mathfrak{h})}[\mathfrak{d}^{\nu}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{t})] = \mathfrak{d}^{-\nu}_{\alpha(\mathfrak{h})}[\mathfrak{d}^{\nu}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{t})\mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{t}}]\Big|_{0}^{\infty}$. Now applying (8) and solving we get

$$\mathfrak{L}^{\nu}_{\alpha(\mathfrak{h})}[\mathfrak{d}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{t})] = \frac{(\alpha - \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{h}})}{\mathfrak{h}}\mathfrak{L}_{\alpha(\mathfrak{h})}[\mathfrak{u}(\mathfrak{t}+\mathfrak{h})] - \mathfrak{u}(0) \tag{24}$$

Again taking $\mathfrak{u}(\mathfrak{t}) = \mathfrak{d}^{2}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{t})$ and using (8),(15), applying (24), gives

$$\mathfrak{L}^{\nu}_{\alpha(\mathfrak{h})}[\mathfrak{d}^{2}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{t})] = \frac{(\alpha - \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{h}})^2}{\mathfrak{h}^2}\mathfrak{L}_{\alpha(\mathfrak{h})}[\mathfrak{u}(\mathfrak{t}+2\mathfrak{h})] - \frac{(\alpha - \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{h}})}{\mathfrak{h}}\mathfrak{u}(\mathfrak{h}) - \mathfrak{d}_{\alpha(\mathfrak{h})}\mathfrak{u}(0) \tag{25}$$

Continuing this process for integer $\mathfrak{n}$, arrives

$$\mathfrak{L}_{\alpha(\mathfrak{h})}[\mathfrak{d}^{\mathfrak{n}}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{t})] = \frac{(\alpha - \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{h}})^{\mathfrak{n}}}{\mathfrak{h}^{\mathfrak{n}}}\mathfrak{L}_{\alpha(\mathfrak{h})}[\mathfrak{u}(\mathfrak{t}+\mathfrak{n}\mathfrak{h})] - \sum_{\mathfrak{r}=1}^{\mathfrak{n}} \frac{(\alpha - \mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{h}})^{\mathfrak{n}-\mathfrak{r}}}{\mathfrak{h}^{\mathfrak{n}-\mathfrak{r}}}\mathfrak{d}^{\mathfrak{r}-1}_{\alpha(\mathfrak{h})}\mathfrak{u}((\mathfrak{n}-\mathfrak{r})\mathfrak{h}). \tag{26}$$

Since the order is fraction, so we consider the equation (26) for the fraction ν as mentioned in (23). □

5. Fractional Alpha Extorial Transform by $\alpha(\mathfrak{h})$-difference Operator

The newly created extorial function obtained from exponential expression by replacing polynomials into polynomial factorial is used to develop fractional alpha extorial transform for factorial functions.

Definition 5.1. For $\mathfrak{h} > 0$ and $\nu > 0$, an $\mathfrak{h}$-extorial function is defined by

$$\mathfrak{e}^{\mathfrak{t}^{[\nu]}_{\mathfrak{h}}} = 1 + \frac{\mathfrak{t}^{[\nu]}_{\mathfrak{h}}}{1!} + \frac{\mathfrak{t}^{[2\nu]}_{\mathfrak{h}}}{2!} + \frac{\mathfrak{t}^{[3\nu]}_{\mathfrak{h}}}{3!} + \cdots \tag{27}$$

and the $\mathfrak{h}$-extorial function for negative index is defined by

$$\mathfrak{e}^{\mathfrak{t}^{[-\nu]}_{\mathfrak{h}}} = 1 + \frac{1}{1!}\frac{1}{\mathfrak{t}^{[\nu]}_{\mathfrak{h}}} + \frac{1}{2!}\frac{1}{\mathfrak{t}^{[2\nu]}_{\mathfrak{h}}} + \frac{1}{3!}\frac{1}{\mathfrak{t}^{[3\nu]}_{\mathfrak{h}}} + \cdots \tag{28}$$

Remark 5.2. (i) In particular, when $\nu = 1$, (27) becomes

$$\mathfrak{e}^{\mathfrak{t}^{[1]}_{\mathfrak{h}}} = 1 + \frac{\mathfrak{t}^{[1]}_{\mathfrak{h}}}{1!} + \frac{\mathfrak{t}^{[2]}_{\mathfrak{h}}}{2!} + \frac{\mathfrak{t}^{[3]}_{\mathfrak{h}}}{3!} + \cdots \tag{29}$$

(ii) Since $\mathfrak{e}^{\mathfrak{t}} \neq \mathfrak{e}^{\mathfrak{t}^{[1]}_{\mathfrak{h}}}$, do not take $\mathfrak{t}^{[1]}_{\mathfrak{h}} = \mathfrak{t}$ in the LHS of (29).
(iii) For $\mathfrak{h} > 0$, $\mathfrak{e}^{(\mathfrak{t}_1+\mathfrak{t}_2)^{[1]}_{\mathfrak{h}}} = \mathfrak{e}^{\mathfrak{t}^{[1]}_{1\mathfrak{h}}}\mathfrak{e}^{\mathfrak{t}^{[1]}_{2\mathfrak{h}}}$.

We create fractional extorial transform method for real valued function $\mathfrak{u}(\mathfrak{t})$.

Definition 5.3. Fractional alpha extorial transform of $\mathfrak{u}(\mathfrak{t}), \mathfrak{t} > 0$, is defined as

$$\mathscr{E}^{\nu}_{\alpha(\mathfrak{h})}[\mathfrak{u}(\mathfrak{t})] = \mathfrak{d}^{-\nu}_{\alpha(\mathfrak{h})}\mathfrak{u}(\mathfrak{t})\mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{t}^{[\nu]}_{\mathfrak{h}}}\Big|_{0}^{\infty} = -\mathfrak{h}^{\nu}\sum_{\mathfrak{r}=0}^{\infty}\alpha^{-(\mathfrak{r}+\nu)}\frac{\mathfrak{d}(\nu+\mathfrak{r})}{\mathfrak{d}(\nu)\mathfrak{d}(\mathfrak{r}+1)}\mathfrak{u}(\mathfrak{r}\mathfrak{h})\mathfrak{e}^{-\mathfrak{s}^{1/\nu}\mathfrak{r}\mathfrak{h}^{[\nu]}_{\mathfrak{h}}}. \quad (30)$$

Following theorems show the fractional extorial transform of factorial and logarithmic functions.

Theorem 5.4. For fraction $\nu > 0$, $\alpha, \mathfrak{h} > 0$, and $\mathfrak{t} \in [0, \infty)$, we have

$$\mathscr{E}^{\nu}_{\alpha(\mathfrak{h})}[\mathfrak{t}^{(\mu)}_{\mathfrak{h}}] = -\mathfrak{h}^{\nu}\sum_{\mathfrak{r}=0}^{\infty}\alpha^{-(\mathfrak{r}+\nu)}\frac{\mathfrak{d}(\nu+\mathfrak{r})}{\mathfrak{d}(\nu)\mathfrak{d}(\mathfrak{r}+1)}(\mathfrak{r}\mathfrak{h})^{(\mu)}_{\mathfrak{h}}\mathfrak{e}^{-\mathfrak{s}^{1/\nu}(\mathfrak{r}\mathfrak{h})^{[\nu]}_{\mathfrak{h}}}. \quad (31)$$

Proof. The proof follows directly from (30) by taking $\mathfrak{u}(\mathfrak{t}) = \mathfrak{t}^{(\mu)}_{\mathfrak{h}}$. □

Theorem 5.5. For fraction $\nu > 0$, $\alpha, \mathfrak{h} > 0$, and $\mathfrak{t} \in [0, \infty)$, we have

$$\mathscr{E}^{\nu}_{\alpha(\mathfrak{h})}[\log \mathfrak{a}\mathfrak{t}] = -\mathfrak{h}^{\nu}\sum_{\mathfrak{r}=0}^{\infty}\alpha^{-(\mathfrak{r}+\nu)}\frac{\mathfrak{d}(\nu+\mathfrak{r})}{\mathfrak{d}(\nu)\mathfrak{d}(\mathfrak{r}+1)}\log \mathfrak{a}(\mathfrak{r}\mathfrak{h})\mathfrak{e}^{-\mathfrak{s}^{1/\nu}(\mathfrak{r}\mathfrak{h})^{[\nu]}_{\mathfrak{h}}}, \quad (32)$$

Proof. Taking $\mathfrak{u}(\mathfrak{t}) = \log \mathfrak{a}\mathfrak{t}$ in (30) gives (32). □

Remark 5.6. When $\mathfrak{h} \to 0$ and $\alpha = \nu = 1$, fractional alpha Laplace transform becomes classical Laplace transform existing in the literature.

6. Conclusion

This paper defines the fundamental concept of the alpha difference operator and highlights the qualities and outcomes that are obtained. With the aid of the alpha fractional difference operator, the researcher has also established a new kind of alpha extorial transform. The Laplace transform has been successfully transformed into the fractional alpha Laplace transform (FALT). The equations and characteristics of the fractional alpha Laplace transform, as well as the outcomes of its transform for specific functions, have been effectively deduced by the researchers. As a result, the FALT outcomes evaluation is a novel method based on an established one from the literature. Consequently, the researchers deduce that when $\nu = \alpha = \mathfrak{h} = 1$, the FALT transforms into a standard Laplace transform.

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