Fuzzy Approximations of a Functional Equation in Digital Spatial Image

Crypto Techniques System

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ABSTRACT

In this paper, authors analyze the stabilities of a generalized additive functional equation in Fuzzy Banach space using direct and fixed point method and particularly application in digital spatial image crypto techniques system using MATLAB.

Keywords—additive functional equation, Ulam-Hyers stability, Fuzzy Banach space, Fixed point

#  INTRODUCTION

The study of stability problems for functional equations is related to a question of Ulam [48] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [24]. It was further generalized and excellent results obtained by number of authors [3,23,39,42]

The solution and stability of the following additive functional equations

 (1.1)

 (1.2)

 (1.3)

 (1.4) (1.5)

 (1.6)

 (1.7)

 (1.8)

 (1.9)

were discussed in [1, 5, 6, 14, 30, 34, 36, 38, 42, 49].

In this paper, authors analyze the stabilities of a generalized additive functional equation

 (1.10)

in Fuzzy Banach space using direct and fixed point method. Also, an application in digital spatial image crypto techniques system using MATLAB of (1.10) is analyzed.

# BASIC DEFINITIONS ABOUT FUZZY NORMED SPACE

In this section, the authors provide basic definitions about fuzzy normed space.

**Definition 2.1** Let X be a real linear space. A function (so-called fuzzy subset) is said to be a fuzzy norm on X if for all and all ,

(F1) for ;

(F2) if and only if for all ;

(F3) if ;

(F4)

(F5) is a non-decreasing function on R and ;

(F6) For ,is (upper semi) continuous on R.

The pair is called a fuzzy normed linear space. One may regard as the truth-value of the statement the norm of x is less than or equal to the real number t ’.

**Example 2.2** Let be a normed linear space. Then is a fuzzy norm on X.

**Definition 2.3** Let be a fuzzy normed linear space. Let be a sequence in X. Then is said to be convergent if there exists such that for all t > 0 .In that case, is called the limit of the sequence and we denote it by .

**Definition 2.4** A sequence in is called Cauchy if for each ε > 0 and each t > 0 there exists such that for all and all , we have .

**Definition 2.5** Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Hereafter throughout this chapter, we assume that , and are linear space, fuzzy normed space and fuzzy banach space respectively.

# STABILITY IN FUZZY BANACH SPACE: DIRECT METHOD

In this section, the authors analyze the stability of a generalized additive functional equation (1.10) in Fuzzy Banach space using direct method.

## **THEOREM: 3.1** Let . Let be a function with

 (3.1)

for all and all and

 (3.2)

for all and all . Suppose that a function satisfying the inequality

 (3.3)

for all and all .Then the limit,

 (3.4)

exists for all and the mapping is a unique mapping satisfying (1.10) and

 (3.5)

for all and all .

## **PROOF:** For =1. Change as in (3.3) and using (F3), we get

 (3.6)

 (3.7)

for all and all . Again, change as in the above inequality, it gives

 (3.8)

for all and all .Substitute as in (3.8) and using (F3), we achieve

 (3.9)

for all and all . Again, substitute as in (3.8) and using (F3), we receive

 (3.10)

for all and all .Change as in the above inequality and using (F3), we have

 (3.11)

for all and all . It is easy show that,

 (3.12)

for all . Change as in (3.11), we obtain

 (3.13)

for all and all . From (3.12) & (3.13)

which gives,

≥ (3.14)

for all and all . Change as in above inequality, we get

for all and all .Change as in above inequality, we obtain

 (3.15)

for all and all and . Since and . By applying is a decreasing function on *R* and and Cauchy criterion convergence is a Cauchy sequence in . Since is a fuzzy banach space. This sequence is converges to some point . Let us define the function by

 (3.16)

Let and in (3.15)

 (3.17)

for all and all .

To prove that satisfies (1.10).Replace as m and as in (3.3), we get

for all and all .Change as in the above inequality, we obtain

 (3.18)

(3.19)

From (3.18) & (3.19) and applying limit in (3.19), and also using (3.2), we conclude that

which implies, *P* satisfies the equation (1.10).

To prove is unique. Let us consider be another functional equation satisfying (3.3) and (3.5)

for all and all . Since , we obtain ,

so, . Therefore, is unique. Hence the theorem is holds for .

For . Using (3.6), we get

 (3.20)

for all and all .Replace as in above inequality, we obtain

 (3.21)

for all and all . Change as in (3.14), we obtain

 (3.22)

for all and all . The rest of the proof is similar to that of previous case. This completes the proof.

**Corollary 3.2** Suppose a function statisfies the inequality

 (3.23)

for all and all . where τ and c are constants with .Then there exists a unique mapping satisfying the functional equation (3.5) and

 (3.24)

for all and all .

# STABILITY IN FUZZY BANACH SPACE: FIXED POINT METHOD

 In this section, we analyze the stability of a generalized additive functional equation (1.10) in Fuzzy Banach space using fixed point method.

## **THEOREM: 4.1 [33]** Let be a complete generalized metric space and Let be a strictly contractive mapping with Lipschitz constant .Then for each given element , either

for all non negative integers n**.**

or there exists positive integers such that

1. for all .
2. The sequence converges to a fixed point of J.
3. is the unique fixed point of J in the set .
4. for all y ϵX.

## **THEOREM: 4.2** Let be a function for which there exist a function satisfying the inequality

 (4.1)

with the condition

 (4.2)

for all and all .If there exists such that

 (4.3)

has the property

 (4.4)

 (4.5)

for all and all .The mapping is a unique mapping satisfying (1.10) and

 (4.6)

for all and all .

***PROOF:*** Consider the set and introduce the generalized metric on by,

 (4.7)

It is easy to see that is complete. Define by

 for all (4.8)

Now for ,we have

which implies

for all and all .Replace as in above inequality, we get

 (4.9)

for all and all . D is strictly contractive mapping on with Lipschtiz constant .

It follows from (3.7), that

 (4.10)

for all and all .It follows from (3.15), that

 (4.11)

for all and all .From (4.10) & (4.11), we conclude,

 (4.12)

which [FP1] holds. Now from the fixed point alternative [FP2] in both cases, it gives that there exists a fixed point P of D in such that

 for all

Hence P satisfies the functional equation (1.10). By [FP3], since is unique fixed point of D in the set

Therefore P is unique function such that

Finally by [FP4], We obtain

for all and all .This completes the proof of the theorem.

**Corollary 4.3:**

Suppose a function statisfies the inequality

 (4.13)

for all and all . where τ and c are constants with .Then there exists a unique mapping satisfying the functional equation (3.5) and

 (4.14)

for all and all .

**PROOF*:***

Here,

Using (4.4) and (4.5).Let

Now,

**Case (i):**  for

Similarly, for , we get

Therefore,

Let

Now,

**Case (ii):**  for

Similarly, for , we get

Therefore,

This completes the proof of the corollary.

**VI. FUNCTIONAL EQUATIONS BASED SPATIAL IMAGE CRYPTO TECHNIQUE**

The term remote sensing takes on a specific implication dealing with space-borne imaging systems used to remotely sense the surface. Remote sensing is defined as data collected from a distance without visiting or interacting directly. When the distance between the object and viewer is large, or rather small, remote sensing approach suggests the use of spatial image. In modern days, the image based cryptographic techniques have advocated new and efficient ways to develop secure spatial image encryption techniques, see [2], [6].

In this research work, functional equations are used to improve the level of security in spatial image encryption. We apply functional equation (1.10) in digital spatial image crypto techniques system using MATLAB. An elementary idea is to encrypt the digital spatial image by applying the left hand side of (1.10). As the result, the intricate cypher image is obtained. See figures 6.1 and 6.2.

### Positioning Figures and Tables: Place figures and tables at the top and bottom of columns. Avoid placing them in the middle of columns. Large figures and tables may span across both columns. Figure captions should be below the figures; table heads should appear above the tables. Insert figures and tables after they are cited in the text. Make them bold (figure and table title).



***Figure 6.1. Encryption***



***Figure 6.2. Image Encryption***

When cypher image reaches the receiver, he must use right hand side of (1.1) as a key. On entering the accurate key, the MATLAB code decrypts the entire image and provides original image to the receiver. See

figures 6.3 and 6.4.



***Figure 6.3. Decryption*** 

***Figure 6.4. Image Decryption***

4.1. Security Analysis. The distinctive approach in applying functional equations on spatial image crypto technique is, we use two different keys with same solutions that are LHS of functional equations for encrypting and RHS of functional equations for decrypting, whereas, traditional systems like DES, Triple- DES, RSA and IDEA use single key for both encryption and decryption. This uniqueness of functional equation progresses the security level of transmitting spatial image and overwhelmed traditional techniques limitations. A statistical analysis shows that the tactic for image crypto technique provides an effective and secure way for real time spatial image encryption and transmission from the cryptographic viewpoint.

# CONCLUSION

We introduced a generalized additive functional equation, obtained its general solution and stabilities in modular space by using fixed point theory. Also, we applied (1.10) in digital spatial image crypto techniques system using MATLAB.

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