Fuzzy Approximations of a Functional Equation within Digital Spatial Image Encryption Schemes

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ABSTRACT

In this study, the authors comprehensively investigate and robustly prove the stability of a generalized additive functional equation in the framework of fuzzy Banach spaces, employing sophisticated direct and fixed point techniques. Furthermore, the paper delivers a powerful application of these theoretical results to cutting-edge digital spatial image cryptography systems, implemented and validated through MATLAB.

Keywords— Linear-type functional equation, Generalized stability in the sense of Ulam-Hyers, Fuzzy Banach space, Banach's Contraction principle.

#  INTRODUCTION

 **The exploration of stability phenomena in functional equations finds its roots in a pivotal question posed by Ulam [48] concerning the robustness of group homomorphisms. This question was decisively addressed by Hyers [24] within the framework of Banach spaces, laying the groundwork for a rich theory of functional stability. Building upon this foundation, numerous** authors **[3, 23, 39, 42] have significantly advanced the field through profound generalizations and impactful results.**

 **In this context, particular emphasis has been placed on the rigorous analysis of both the solutions and the stability characteristics of additive functional equations.**

 (1.1)

 (1.2)

 (1.3)

 (1.4) (1.5)

 (1.6)

 (1.7)

 (1.8)

 (1.9)

were discussed in [1, 5, 6, 14, 30, 34, 36, 38, 42, 49].

In this paper, authors presents a comprehensive examination of the stability properties of the following generalized additive functional equation

 (1.10)

in Fuzzy Banach space by employing both the direct analytical technique and the fixed point framework. Additionally, the practical implementation and relevance of equation (1.10) are explored through its application to digital spatial image encryption methods, developed using MATLAB.

# ESSENTIAL TERMINOLOGIES AND FRAMEWORK OF FUZZY NORMED SPACE

In this section, the authors provide foundational terms and properties associated with fuzzy normed spaces are formally introduced.

**Definition 2.1** Let X be a real vector space. A function referred to as a fuzzy subset, is defined as a fuzzy norm on X if for all and all , the following conditions hold:

(F1) for ;

(F2) if and only if for all ;

(F3) if ;

(F4)

(F5) is a non-decreasing function on R and ;

(F6) For ,is (upper semi) continuous on R.

The pair is called a fuzzy normed linear space. Intuitively can be interpreted as the degree of truth to the assertion that the norm of the vector x is at most the real number t ’.

**Example 2.2** Let be a normed linear space. Then is a fuzzy norm on X.

**Definition 2.3** Let be a fuzzy normed linear space. Let be a sequence in X. Then is said to be convergent if there exists such that for all t > 0 .In that case, is called the limit of the sequence and we denote it by .

**Definition 2.4** A sequence in is called Cauchy if for each ε > 0 and each t > 0 there exists such that for all and all , we have .

**Definition 2.5** Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Hereafter throughout this chapter, we assume that , and are linear space, fuzzy normed space and fuzzy banach space respectively.

# STABILITY CONSIDERATIONS IN FUZZY BANACH SPACE: DIRECT METHOD

In this section, the authors analyze the stability of a generalized additive functional equation (1.10) in Fuzzy Banach space using direct method.

## **THEOREM: 3.1** Let . Let be a function with

 (3.1)

for all and all and

 (3.2)

for all and all . Suppose that a mapping validating the inequality

 (3.3)

for all and all .Then the limit,

 (3.4)

exists for all and the mapping is a unique mapping satisfying (1.10) and

 (3.5)

for all and all .

## **PROOF:** For =1. Change as in (3.3) and using (F3), we get

 (3.6)

 (3.7)

for all and all . Again, change as in the above inequality, it gives

 (3.8)

for all and all .Substitute as in (3.8) and using (F3), we achieve

 (3.9)

for all and all . Again, substitute as in (3.8) and using (F3), we receive

 (3.10)

for all and all .Change as in the above inequality and using (F3), we have

 (3.11)

for all and all . It is easy show that,

 (3.12)

for all . Change as in (3.11), we obtain

 (3.13)

for all and all . From (3.12) & (3.13)

which gives,

≥ (3.14)

for all and all . Change as in above inequality, we get

for all and all .Change as in above inequality, we obtain

 (3.15)

for each and every and . Since and . By applying is a declining function on *R* with and Cauchy criterion convergence is a Cauchy sequence in . Since is a fuzzy banach space. This sequence is converges to some point . Let us define the function by

 (3.16)

Let and in (3.15)

 (3.17)

for all and all .

To prove that satisfies (1.10).Replace as m and as in (3.3), we get

for all and all .Change as in the above inequality, we obtain

 (3.18)

(3.19)

From (3.18) & (3.19) and applying limit in (3.19), and also using (3.2) accordingly, we establish that

which implies, *P* meets the condition of the equation (1.10).

In order to demonstrate the uniqueness of , let us consider an alternative function that also satisfies the given functional equation (3.3) and (3.5)

for all and all . Since , we obtain ,

so, . Therefore, is unique. Thus, the theorem is proven for .

For . Using (3.6), we get

 (3.20)

for all and all .Replace as in above inequality, we obtain

 (3.21)

for all and all . Change as in (3.14), we obtain

 (3.22)

for all and all . The rest of the reasoning is identical to that employed in the earlier scenario. This completes the proof.

**Corollary 3.2** Suppose a function adheres to the inequality

 (3.23)

for all and all . where τ and c are constants with .Then there is a uniquely determined mapping which adheres to the functional equation (3.5) and

 (3.24)

for all and all .

# STABILITY CONSIDERATIONS IN FUZZY BANACH SPACE USING FIXED POINT PRINCIPLES

 This section is devoted to examining the stability of the generalized additive functional equation (1.10) in a fuzzy Banach space by means of the fixed point approach.

## **THEOREM: 4.1 [33]** Let be a complete generalized metric space and Let be a strictly contractive mapping with Lipschitz constant .Then for each given element , either

for all non negative integers n**.**

or there exists positive integers such that

1. for all .
2. The sequence converges to a fixed point of J.
3. is the unique fixed point of J in the set .
4. for all y ϵX.

## **THEOREM: 4.2** Let be a function where one can find a function subject to the inequality

 (4.1)

with the condition

 (4.2)

for every and all . If there exists for which

 (4.3)

has the property

 (4.4)

 (4.5)

for each and every .The mapping is the uniquely determined mapping fulfilling (1.10) and

 (4.6)

for all and all .

***PROOF:*** Let us examine the set and set forth the generalized metric on by,

 (4.7)

It can be readily seen that is complete. Delimit by

 for all (4.8)

Now for ,we have

which implies

for all and all .Replace as in above inequality, we get

 (4.9)

for all and all . D is strictly contractive mapping on with Lipschtiz constant .

It follows from (3.7), that

 (4.10)

for all and all .It follows from (3.15), that

 (4.11)

for all and all .From (4.10) & (4.11), we conclude,

 (4.12)

which [FP1] holds. Invoking the fixed point property [FP2] in each instance, it gives that there exists a fixed point P of D in such that

 for all

Hence P obeys the functional equation (1.10). By [FP3], since is unique fixed point of D in the set

Therefore P is unique mapping such that

Finally by [FP4], We obtain

for all and all . This concludes the proof of the theorem.

**Corollary 4.3:**

Suppose a function which adheres to the inequality

 (4.13)

for all and all . where τ and c are constants with . One can assert the existence of a unique mapping which adheres to the functional equation (3.5) and

 (4.14)

for all and all .

**PROOF*:***

Here,

Using (4.4) and (4.5).Let

Now,

**Case (i):**  for

Similarly, for , we get

Therefore,

Let

Now,

**Case (ii):**  for

Similarly, for , we get

Therefore,

Therefore, the corollary has been established.

**VI. FUNCTIONAL EQUATIONS BASED SPATIAL IMAGE CRYPTO TECHNIQUE**

The term remote sensing takes on a specific implication dealing with space-borne imaging systems used to remotely sense the surface. Remote sensing is defined as data collected from a distance without visiting or interacting directly. When the distance between the object and viewer is large, or rather small, remote sensing approach suggests the use of spatial image. In modern days, the image based cryptographic techniques have advocated new and efficient ways to develop secure spatial image encryption techniques, see [2], [6].

In this research work, functional equations are used to improve the level of security in spatial image encryption. We apply functional equation (1.10) in digital spatial image crypto techniques system using MATLAB. An elementary idea is to encrypt the digital spatial image by applying the left hand side of (1.10). As the result, the intricate cypher image is obtained. See figures 6.1 and 6.2.

### Positioning Figures and Tables: Place figures and tables at the top and bottom of columns. Avoid placing them in the middle of columns. Large figures and tables may span across both columns. Figure captions should be below the figures; table heads should appear above the tables. Insert figures and tables after they are cited in the text. Make them bold (figure and table title).



***Figure 6.1. Encryption***



***Figure 6.2. Image Encryption***

When cypher image reaches the receiver, he must use right hand side of (1.1) as a key. On entering the accurate key, the MATLAB code decrypts the entire image and provides original image to the receiver. See

figures 6.3 and 6.4.



***Figure 6.3. Decryption*** 

***Figure 6.4. Image Decryption***

4.1. Security Analysis. The distinctive approach in applying functional equations on spatial image crypto technique is, we use two different keys with same solutions that are LHS of functional equations for encrypting and RHS of functional equations for decrypting, whereas, traditional systems like DES, Triple- DES, RSA and IDEA use single key for both encryption and decryption. This uniqueness of functional equation progresses the security level of transmitting spatial image and overwhelmed traditional techniques limitations. A statistical analysis shows that the tactic for image crypto technique provides an effective and secure way for real time spatial image encryption and transmission from the cryptographic viewpoint.

# CONCLUSION

We introduced a generalized additive functional equation, obtained its general solution and stabilities in modular space by using fixed point theory. Also, we applied (1.10) in digital spatial image crypto techniques system using MATLAB.

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